Mon, May 11, 2009 **STUDENT NAME:** ____________________

FINAL EXAM – MATH 4/543, SPRING 2009

Prof. Radu C. Cascaval

Due Mon, May 18, 5pm

YOU MAY NOT USE ANY BREATHING SOURCE OF HELP

(e.g. HUMANS, which includes colleagues, friends, parents, grandparents, cousins, strangers)!!!

NO TEAM WORK OF ANY KIND WILL BE TOLERATED!!!!

Any attempt (voluntary or involuntary) to consult with others will be severely punished!!! No kidding! All parties involved will receive ZERO on the Exam and, in addition, disciplinary actions may be taken.

Please read each problem CAREFULLY. You may consult the lecture notes, homework solutions, textbook etc and also use computer to help in solving (parts) of the problems.

Each exercise is worth 30 points (except #4 and #5 - worth 25 points each). Turn in the solutions for each problem attached to the individual page with the statement of the problem, staple them in correct order. Make sure you have included all problems!!!

Exercise 1

Consider the linear differential equation

$$x' = -x + 2\sin t$$

(a) Find the general solution **explicitly**.
(b) **Show** that there is a unique periodic solution.
(c) Plot the phase portrait in the (t, x) plane. Explain what does it mean to say that the periodic solution is 'asymptotically stable' and then **prove it**.
(d) [Extra credit: 10 points] Can you redo ANY of the above for the nonlinear equation

$$x' = -\sin x + 2\sin t$$

instead?

Exercise 2

Consider the linear system $X' = AX$, where

$$A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}.$$

(a) Find a matrix T that puts A in **canonical form** $B = T^{-1}AT$.
(b) Find the **general solution** of both $X' = AX$ and $Y' = BY$ and plot the phase portraits of both system.
(c) **Show** that $L(x, y) = x^2 + y^2$ is a Liapunov function for the system $Y' = BY$.
(d) **Construct** a Liapunov function for the original system $X' = AX$.

Exercise 3
Consider the system of differential equations

$$x' = x(-x - y + 1)$$
$$y' = y(-ax - y + b)$$

where $a, b > 0$ are parameters. Suppose this system is only defined for $x, y \geq 0$.
(a) Use the nullclines to sketch the **phase portrait** for this system for various values of a and b.
(b) Determine the values of a and b at which a **bifurcation** occurs.
(c) **Sketch** the regions in the (a, b) plane where this system has qualitatively similar phase portraits, and describe the bifurcations that occur as parameters cross the boundaries of these regions.

Exercise 4

Find all the **equilibria** of the system:

$$x' = 1 - xy$$
$$y' = x - y^3$$
$$z' = x - z$$

and determine whether they are stable or unstable.

Exercise 5

Prove that *all* solutions of the equations

$$(a)\, x'' + x^3 = 0 \qquad\qquad (b)\, x'' + \frac{x}{x^2+1} = 0$$

are **periodic**.

Exercise 6

Consider the nonlinear system

$$x' = x - 4y^3$$
$$y' = -y$$

(a) Show that the origin $(0, 0)$ is a **nonlinear saddle** and plot the phase portrait, including the nullclines.

(b) *Explicitly* solve the nonlinear system for $X(t) = [x(t), y(t)]^T$ and compute the **flow** $\phi_t(X_0) = X(t)$ for all $X_0 = [x_0, y_0]^T$.

(c) Determine the stable $W^s(0, 0)$ and unstable $W^u(0, 0)$ curves of the saddle point $(0, 0)$ and then compare your conclusion with the phase portrait.

[$W^s(0, 0) = \{X_0|\phi_t(X_0) \to (0, 0) \text{ as } t \to +\infty\}$, $W^u(0, 0) = \{X_0|\phi_t(X_0) \to (0, 0) \text{ as } t \to -\infty\}$]

Exercise 7
Consider the system

$$x' = y$$
$$y' = x(x-1) + cy$$

where c is a parameter.
(a) Find all **equilibria** and discuss their stability in terms of c.
(b) Show that for $c > 2$ there exists a **heteroclinic** orbit $\gamma(t) = (x(t), y(t))$ such that $x'(t) > 0$ for all t.
[Hint: Show that there exists a triangular region bounded by $y = 0$, $x = 1$ and $y = \mu x$, (with μ to be determined) which is negatively invariant for the dynamical system]