SOLUTIONS TO FINAL EXAM

Note Title 5/18/2009

MATH 443/543 - SPRING 2009

Exercise 1
Consider the linear differential equation

$$x' = -x + 2\sin t$$

(a) Find the general solution explicitly.
(b) Show that there is a unique periodic solution.
(c) Plot the phase portrait in the (t, x) plane. Explain what does it mean to say that the periodic solution is 'asymptotically stable' and then prove it.
(d) [Extra credit: 10 points] Can you redo ANY of the above for the nonlinear equation

$$x' = -\sin x + 2\sin t$$

instead?

(a) This is a 1^{st} order linear diff eq.

Seeking a particular solution of the form

$x_p(t) = A\cos t + B\sin t$ leads to

$$2\sin t = x_p' + x_p = A(-\sin t) + B\cos t + A\cos t + B\sin t$$

$$= (A+B)\cos t + (B-A)\sin t$$

Identifying the coeff of $\cos t$ and $\sin t$

$$\begin{cases} A+B=0 \\ B-A=2 \end{cases} \Rightarrow B=1,\ A=-1$$

$\Rightarrow x_p(t) = \sin t - \cos t$

General solution is $x(t) = x_p(t) + x_h(t)$

where x_h solves the homogeneous eq $x' = -x$

$\Rightarrow x_h(t) = ce^{-t}$

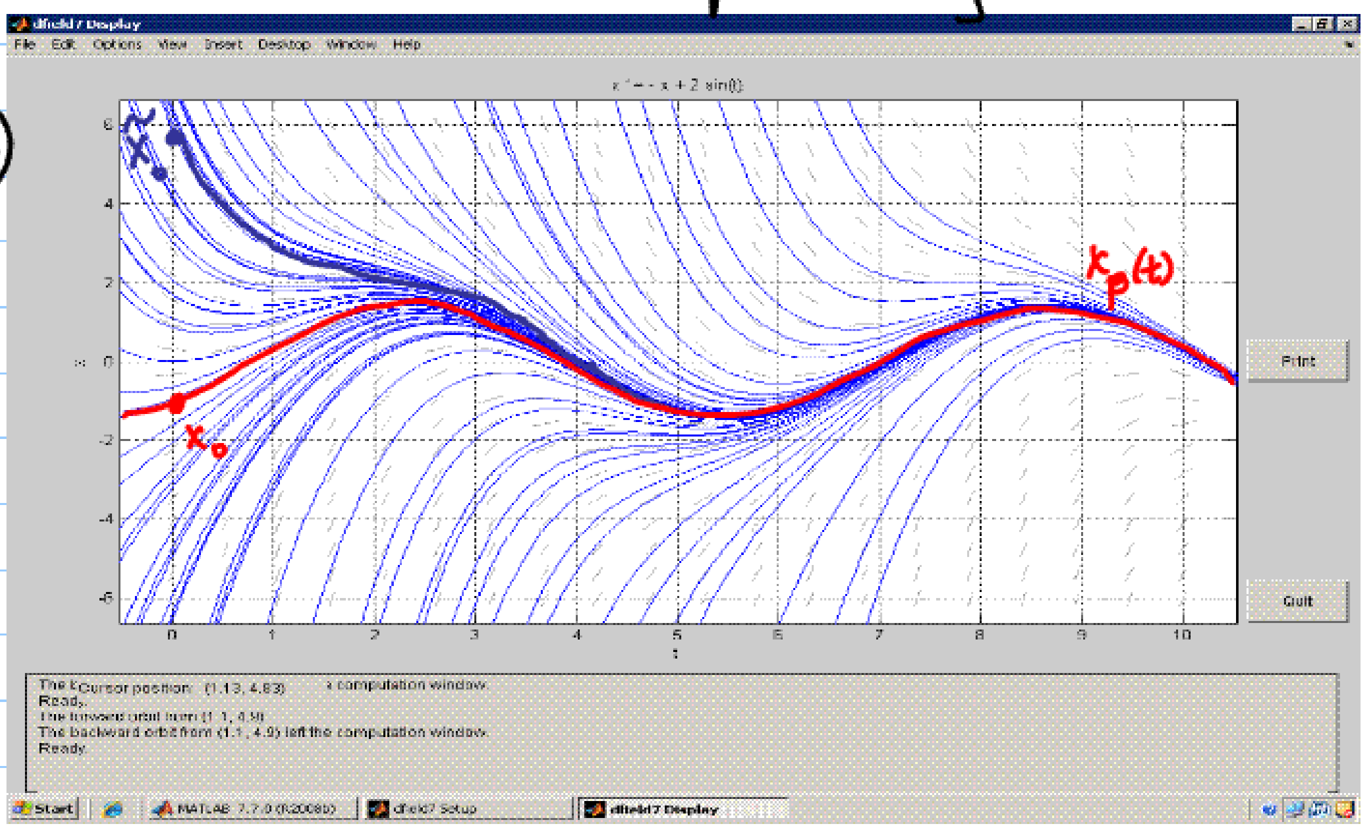
$\Rightarrow \boxed{x(t) = \sin t - \cos t + ce^{-t}}$, c = arbitrary constant

b) Clearly, among <u>all</u> solutions (general sol) there is only one which is periodic

$\boxed{x_p(t) = \sin t - \cos t}$, corresponding to $c = 0$

[You can see that the presence of e^{-t} in the solution makes $x(t)$ non-periodic]

c)



We say that the solution $x_p(t) = \phi(t, x_0)$ is asymptotically stable if there exists a neighborhood $\mathcal{N}$ of $x_0 = x_p(0)$, such that

$$\lim_{t \to \infty} \left| \phi(t, \tilde{x}_0) - \phi(t, x_0) \right| = 0$$

for all $\tilde{x}_0 \in \mathcal{N}$.

From the picture, it appears that this property holds for all $\tilde{x}_0 \in \mathbb{R}$ (so the "basin of attraction" of $x_p = x_p(t)$ is the entire $\mathbb{R}$.)

To prove this fact, recall that

$\phi(t, x_0) = x_p(t) = \sin t - \cos t$, with $x_0 = x_p(0) = -1$

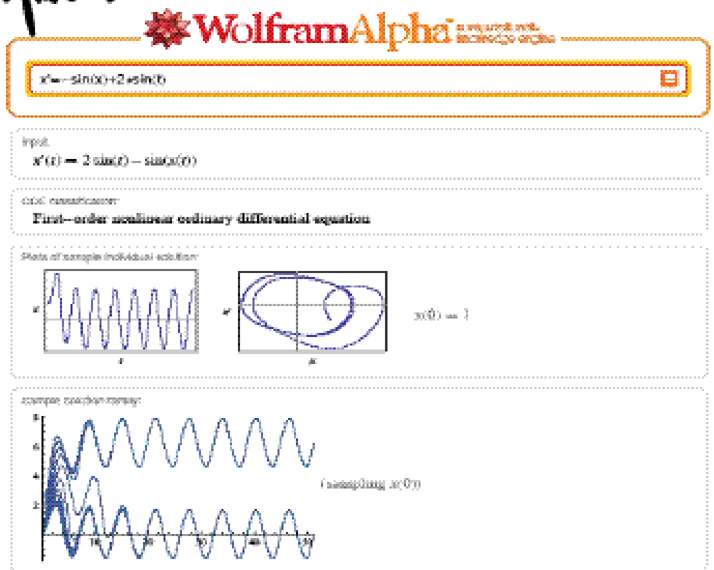
$\phi(t, \tilde{x}_0) = x(t) = \sin t - \cos t + c e^{-t}$, with $c = \tilde{x}_0 + 1$

$\Rightarrow \left| \phi(t, \tilde{x}_0) - \phi(t, x_0) \right| = |\tilde{x}_0 + 1| e^{-t} \to 0$ as $t \to +\infty$ for all $\tilde{x}_0 \in \mathbb{R}$.

(d) $x' = -\sin x + 2 \sin t$ is nonlinear, not separable so there are few other methods that might apply to give explicit solutions.

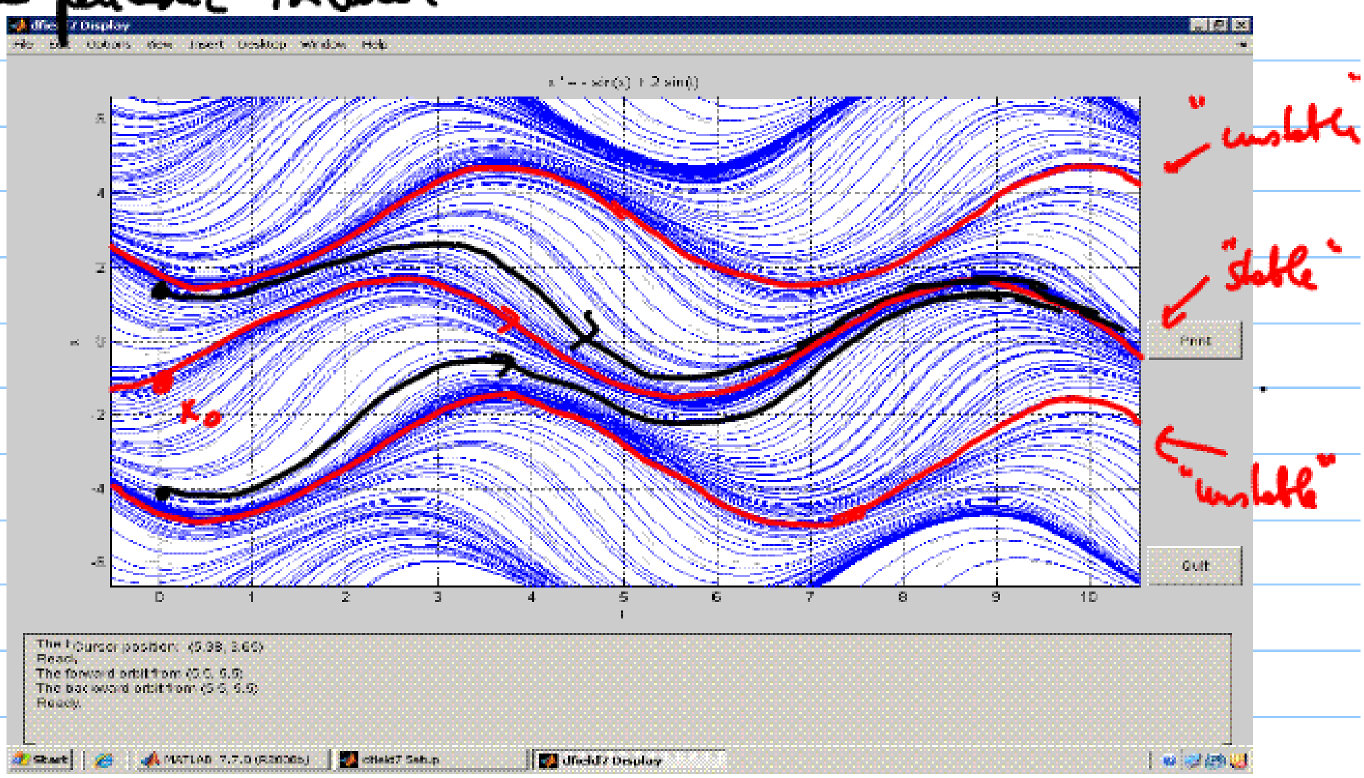
I've even tried the new Wolfram Alpha, which (as of today 5/18/09) gives the following output.

So (a) is pretty hopeless.



[as a comparison, try
$x' = -x + 2\sin t$
to see explicit solutions in Wolfram Alpha]

Using our dfield7.m code, we can see that (b) is false, in that there appear to be more than one periodic solution



How to prove multiple periodic solutions exist? And how to prove stability? These questions seem to require tools beyond our textbook, but they surely are intriguing! :)

Exercise 2
Consider the linear system $X' = AX$, where

$$A = \begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}.$$

(a) Find a matrix T that puts A in canonical form $B = T^{-1}AT$.
(b) Find the general solution of both $X' = AX$ and $Y' = BY$ and plot the phase portraits of both system.
(c) Show that $L(x, y) = x^2 + y^2$ is a Liapunov function for the system $Y' = BY$.
(d) Construct a Liapunov function for the original system $X' = AX$.

Solution: eigenvalues of A: $\begin{vmatrix} 1-\lambda & 2 \\ -4 & -3-\lambda \end{vmatrix} = 0$

$(\lambda-1)(\lambda+3)+8=0 \Rightarrow \lambda^2+2\lambda-3+8=0 \Rightarrow \lambda^2+2\lambda+5=0$

$\Rightarrow (\lambda+1)^2+4=0$

$\Rightarrow \boxed{\lambda = -1 \pm 2i}$

eigenvectors for A

$\lambda_1 = -1+2i \Rightarrow (A-\lambda I)\mathbb{X} = 0 \Rightarrow \begin{bmatrix} 1-(-1+2i) & 2 \\ -4 & -3-(-1+2i) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow (2-2i)x + 2y = 0 \Rightarrow (1-i)x + y = 0 \Rightarrow \mathbb{X}_1 = \begin{bmatrix} 1 \\ -1+i \end{bmatrix}$

$\Rightarrow \mathbb{X}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}$

$$B = T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

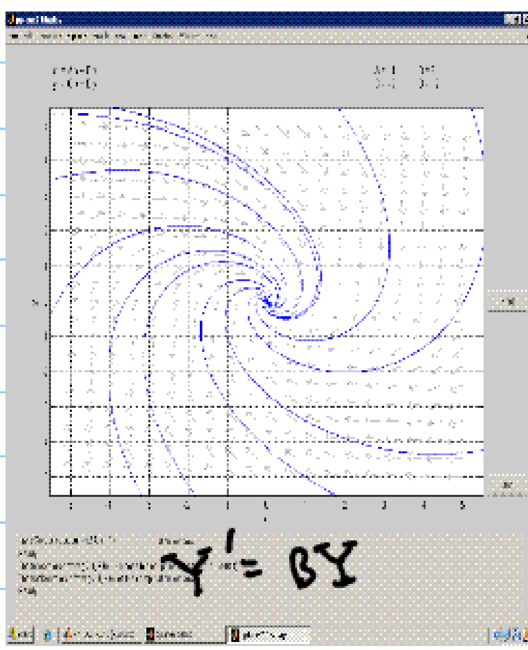
Canonical form $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

General sol of $Y' = BY = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix} Y$

is given by
$$\boxed{Y(t) = c_1 e^{-t}\begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}}$$

or
$$Y(t) = \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -c_1 \sin 2t + c_2 \cos 2t \end{bmatrix} e^{-t}$$

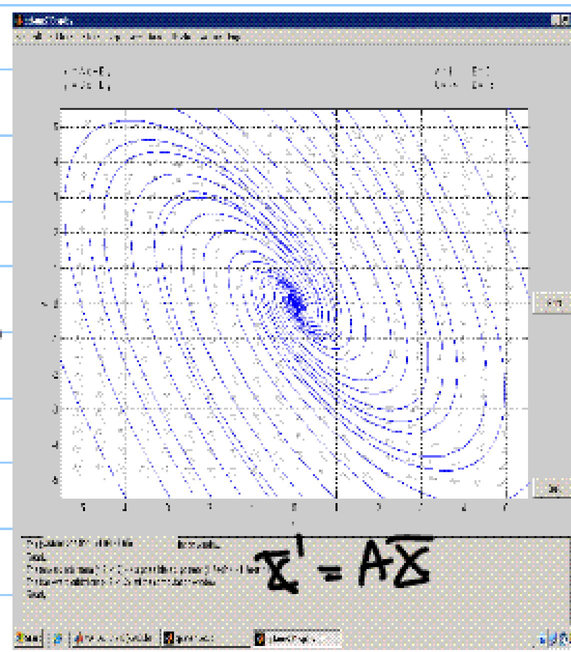
$$\Rightarrow X(t) = TY(t) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} Y(t) =$$
$$= c_1 e^{-t}\begin{bmatrix} \cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} \sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}.$$



$Y' = BY$

$\xrightarrow{T}$

$X = TY$



$X' = AX$

$$Y' = BY \iff \begin{cases} x' = -x + 2y \\ y' = -2x - y \end{cases}$$

(c) $L(x,y) = x^2 + y^2$

$$\dot{L}(x,y) = \frac{d}{dt} L(x(t), y(t))\Big|_{t=0} = 2xx' + 2yy'$$

$$= 2x(-x+2y) + 2y(-2x-y) = -2x^2 - 2y^2 \leq 0$$

with $\dot{L} < 0$ for $X \neq 0$

$\Rightarrow$ L is a strict Liapunov function

(and $(0,0)$ is asymptotically stable)

(d) We use c) to find a Liapunov function for $X' = AX$

indeed $\tilde{L}(X) := L(T^{-1}X)$ must be such a Liapunov function

$$X = \begin{pmatrix} x \\ y \end{pmatrix} : \tilde{L}\begin{pmatrix} x \\ y \end{pmatrix} = L\left(T^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right) = L\begin{pmatrix} x \\ x+y \end{pmatrix}$$

So $\tilde{L}(x,y) = x^2 + (x+y)^2 = 2x^2 + 2xy + y^2$

Indeed $\frac{d}{dt}\tilde{L}(X(t)) = \frac{d}{dt} L(T^{-1}X(t)) = \frac{d}{dt} L(Y(t)) = \dot{L} \leq 0$

Exercise 3

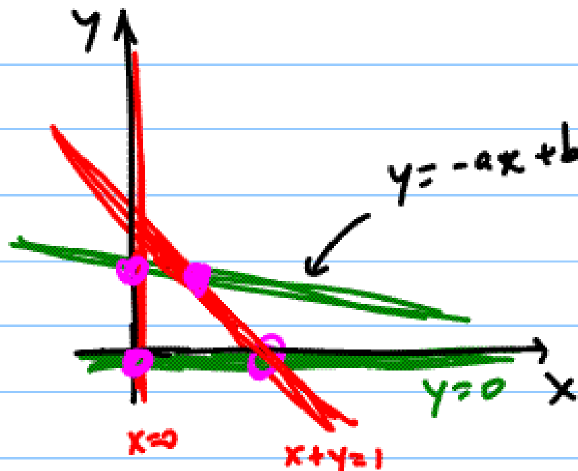
Consider the system of differential equations

$$x' = x(-x - y + 1)$$
$$y' = y(-ax - y + b)$$

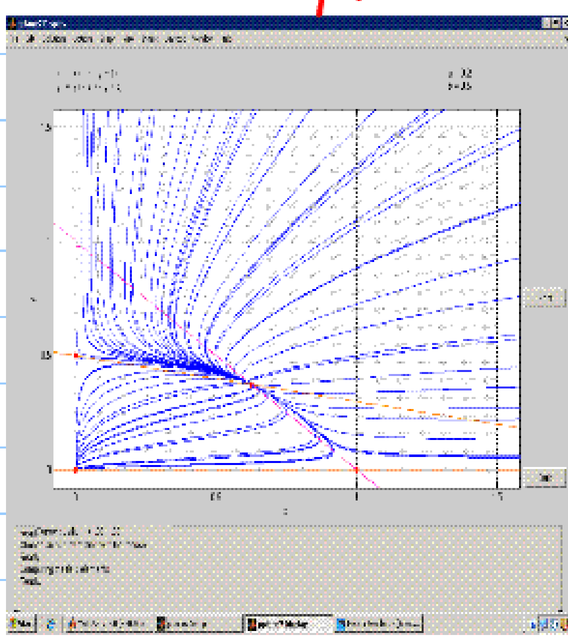
where $a, b > 0$ are parameters. Suppose this system is only defined for $x, y \geq 0$.
(a) Use the nullclines to sketch the **phase portrait** for this system for various values of a and b.
(b) Determine the values of a and b at which a **bifurcation** occurs.
(c) Sketch the regions in the (a, b) plane where this system has qualitatively similar phase portraits, and describe the bifurcations that occur as parameters cross the boundaries of these regions.

Solution Nullclines are $x' = 0 \Rightarrow x = 0$ or $x + y = 1$

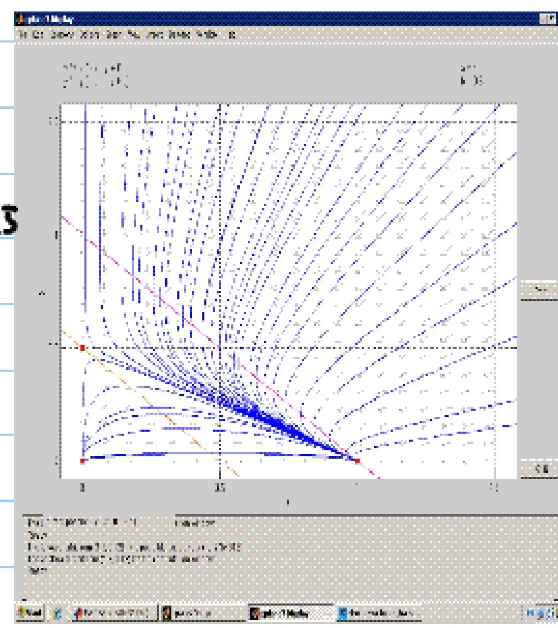
$y' = 0 \Rightarrow y = 0$ or $ax + y = b$
or
$y = -ax + b$



$a = 0.2$
$b = 0.5$



$a = 1$
$b = 0.5$



etc. To determine for which values of a,b we have bifurcations, we need to find all equilibria:

$$\begin{cases} x(-x-y+1)=0 \\ y(-ax-y+b)=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}, \begin{cases} x=0 \\ -ax-y+b=0 \end{cases}, \begin{cases} -x-y+1=0 \\ y=0 \end{cases} \text{ or } \begin{cases} -x-y+1=0 \\ -ax-y+b=0 \end{cases}$$

Equilibria are:

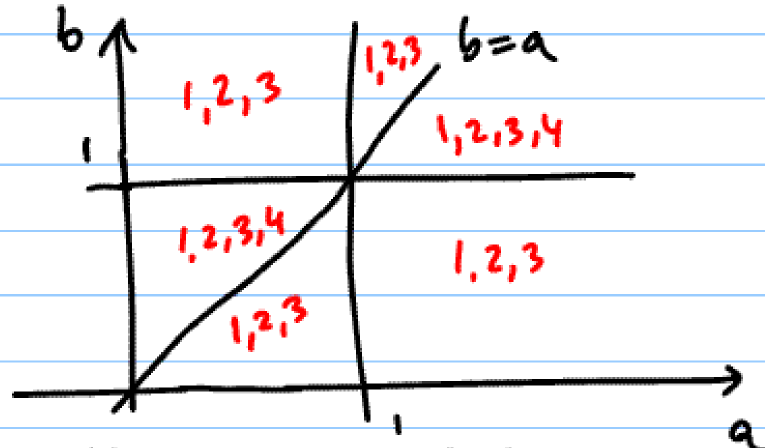
$$\underbrace{(0,0)}_{\#1}, \underbrace{(0,b)}_{\#2}, \underbrace{(1,0)}_{\#3}, \underbrace{\left(\frac{1-b}{1-a}, \frac{b-a}{1-a}\right)}_{\#4}$$

$$\left(\text{Since } \begin{cases} x+y=1 \\ ax+y=b \end{cases} \Rightarrow x = \frac{\begin{vmatrix} 1 & 1 \\ b & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix}} = \frac{1-b}{1-a}, \; y = \frac{\begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ a & 1 \end{vmatrix}} = \frac{b-a}{1-a} \right)$$

The first three equilibria are in the first quadrant $x,y \geq 0$ and the fourth one $\left(\frac{1-b}{1-a}, \frac{b-a}{1-a}\right)$ is in the first quadrant only when

$$\frac{1-b}{1-a} \geq 0$$

$$\frac{b-a}{1-a} \geq 0$$



For #4 equilibrium to be in the positive quadrant:

If $a>1 \Rightarrow b>1$ and $b<a$

If $a<1 \Rightarrow b<1$ and $b>a$

Stability of equilibria:

Linearization $DF\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-2x-y+1 & -x \\ -ay & -ax-2y+b\end{pmatrix}$

At $\begin{pmatrix}0\\0\end{pmatrix}$ $DF\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & b\end{pmatrix}$

$b>0 \Rightarrow \left.\begin{matrix}\det > 0\\ \text{trace} > 0\end{matrix}\right\} \Rightarrow$ source!

At $\begin{pmatrix}0\\b\end{pmatrix}$ $DF\begin{pmatrix}0\\b\end{pmatrix} = \begin{pmatrix}-b+1 & 0 \\ -ab & -b\end{pmatrix}$

$\det = b(b-1)$
$\text{trace} = -2b+1$

If $b \in (0,1) \to \det < 0 \Rightarrow$ saddle
If $b > 1 \Rightarrow \text{trace} < 0 \Rightarrow$ sink

At $\begin{pmatrix}1\\0\end{pmatrix}$ $DF\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1 & -1 \\ 0 & -a\end{pmatrix}$

$\left.\begin{matrix}\det = a > 0\\ \text{trace} = -1-a < 0\end{matrix}\right\} \Rightarrow$ sink!

Finally,
at $\left(\frac{1-b}{1-a}, \frac{b-a}{1-a}\right)$ $DF = \begin{pmatrix}\frac{b-1}{1-a} & \frac{b-1}{1-a} \\ -a\frac{b-a}{1-a} & -\frac{b-a}{1-a}\end{pmatrix}$

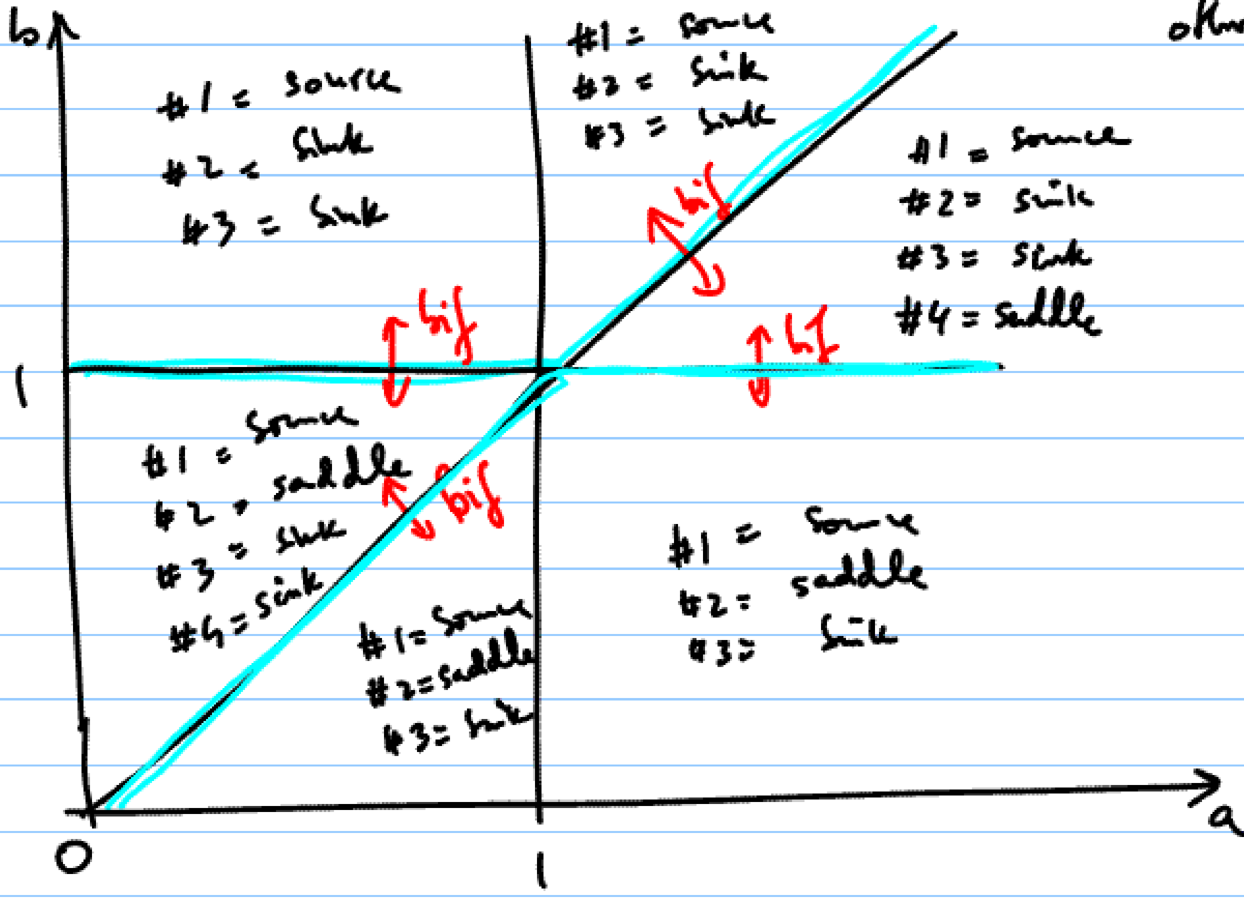
$$\det = \frac{b-1}{1-a} \cdot \frac{b-a}{1-a} \begin{vmatrix} 1 & 1 \\ -a & -1 \end{vmatrix} = -\frac{(b-1)(b-a)}{1-a}$$

$$\text{trace} = \frac{b-1}{1-a} - \frac{b-a}{1-a} = -1 < 0$$

So if $\frac{(1-b)(b-a)}{1-a} < 0 \Rightarrow \det < 0 \Rightarrow$ saddle

otherwise $\det > 0 \Rightarrow$ sink!

So bifurcation diagram for the 4th equilibrium (similarly for all other)



Note that in none of the cases discussed below can a periodic orbit exist in the first quadrant! Can you prove this?

Exercise 4
Find all the equilibria of the system:

$$x' = 1 - xy$$
$$y' = x - y^3$$
$$z' = x - z$$

and determine whether they are stable or unstable.

It is easy to see that $(1,1,1)$ and $(-1,-1,-1)$ are the only equilibria.

Linear stability: $(1,1,1)$ is a sink (all eigenvalues of $DF\begin{pmatrix}1\\1\\1\end{pmatrix}$ are negative)

and $(-1,-1,-1)$ is a saddle (2 eigenvalues of $DF\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$ are negative, one is positive

Exercise 5
Prove that *all* solutions of the equations

$$(a)\ x'' + x^3 = 0 \qquad (b)\ x'' + \frac{x}{x^2+1} = 0$$

are periodic.

<u>Solution</u>: Both are conservative systems of the form $x'' + F(x) = 0$

or $\begin{cases} x' = y \\ y' = -F(x) \end{cases}$ which can be written in

Hamiltonian form

$$\begin{cases} x' = \dfrac{\partial H}{\partial y} \\ y' = -\dfrac{\partial H}{\partial x} \end{cases} \quad \text{where } H(x,y) = \frac{y^2}{2} + \int F(x)\,dx$$

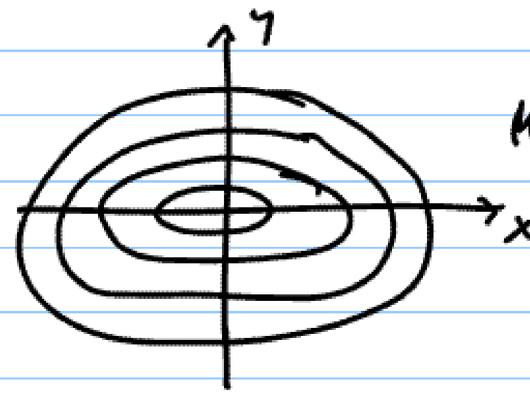
(a) $\begin{cases} x' = y \\ y' = -x^3 \end{cases} \Rightarrow H(x,y) = \dfrac{y^2}{2} + \dfrac{x^4}{4} \Rightarrow \begin{cases} x' = \dfrac{\partial H}{\partial x} \\ y' = -\dfrac{\partial H}{\partial y} \end{cases}$

$\Rightarrow$ Solutions are bound to stay on level curves $H = \text{const}$

Since $H \to \infty$ as $(x,y) \to \infty$ $\Rightarrow$ level curves are bounded, hence by Poincare Bendixson the ω-limit sets are limit cycles (or contain an equilibrium)

But $(0,0)$ is the only equilibrium here $\Rightarrow$

$\Rightarrow$ all solutions that start at $X_0 \neq 0$ must be periodic.



$H(x,y) = \frac{x^2}{2} + \frac{y^4}{4} = c$

(b)

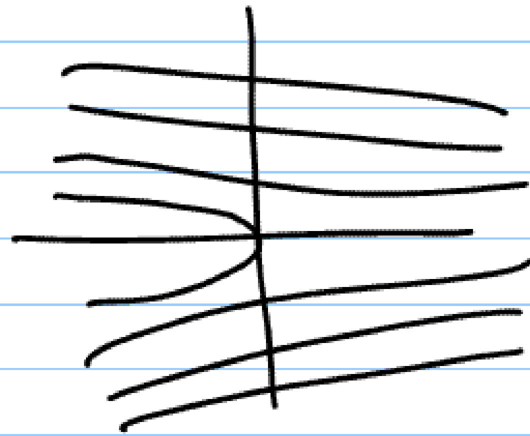
Similarly for $\begin{cases} x' = y \\ y' = -\dfrac{x}{x^2+1} \end{cases} \Rightarrow H(x,y) = \dfrac{y^2}{2} + \dfrac{1}{2}\ln(x^2+1)$

Note: The argument breaks down for something like

$$x'' + \frac{1}{x^2+1} = 0 \to \quad H(x,y) = \frac{y^2}{2} + \arctan x$$

where level curves are no longer bounded (because $\lim_{(x,y)\to\infty} H(x,y) \neq +\infty$)

So solutions are **not** periodic!!



Exercise 6

Consider the nonlinear system

$$x' = x - 4y^3$$
$$y' = -y$$

(a) Show that the origin $(0,0)$ is a nonlinear saddle and plot the phase portrait, including the nullclines.

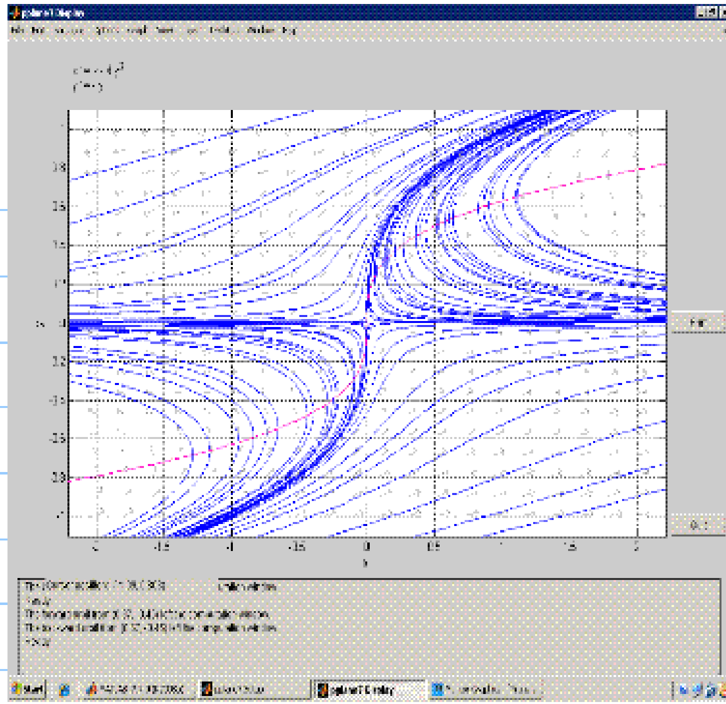
(b) *Explicitly* solve the nonlinear system for $X(t) = [x(t), y(t)]^T$ and compute the flow $\phi_t(X_0) = X(t)$ for all $X_0 = [x_0, y_0]^T$.

(c) Determine the stable $W^s(0,0)$ and unstable $W^u(0,0)$ curves of the saddle point $(0,0)$ and then compare your conclusion with the phase portrait.

[$W^s(0,0) = \{X_0 | \phi_t(X_0) \to (0,0) \text{ as } t \to +\infty\}$, $W^u(0,0) = \{X_0 | \phi_t(X_0) \to (0,0) \text{ as } t \to -\infty\}$]

(a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an equilibrium and $DF\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so

$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a nonlinear saddle.



(b) Solve first for y: $\quad y' = -y \Rightarrow \quad y = y_0 e^{-t}$

$\Rightarrow \quad x' = x - 4y^3 \Rightarrow x' = x - 4y_0^3 e^{-3t}$ (linear in x)

$$x' - x = -4y_0^3 e^{-3t} \,/\, e^{-t}$$

$$(xe^{-t})' = -4y_0^3 e^{-4t} \Rightarrow xe^{-t} = y_0^3 e^{-4t} + c$$

$$\Rightarrow x = y_0^3 e^{-3t} + ce^{t}$$

$t=0 \rightarrow \quad x_0 = y_0^3 + c \Rightarrow c = x_0 - y_0^3 \Rightarrow$

$$\Rightarrow \begin{cases} x = y_0^3 e^{-3t} + (x_0 - y_0^3)e^{t} \\ y = y_0 e^{-t} \end{cases}$$

So the flow $\quad \phi_t\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} (x_0 - y_0^3)e^{t} + y_0^3 e^{-3t} \\ y_0 e^{-t} \end{pmatrix}$

(c) To have $\phi_t\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to +\infty$

we need $x_0 - y_0^3 = 0 \Rightarrow \boxed{x_0 = y_0^3}$

If $\bar{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ satisfies this $\Rightarrow$ $\phi_t\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0^3 e^{-3t} \\ y_0 e^{-t} \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

and satisfies $x(t) = y(t)^3 \Rightarrow$

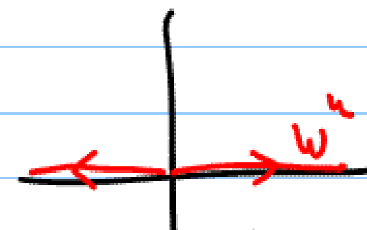
$$W^s(0,0) = \{ (x,y) \mid x = y^3 \}$$

Similarly, to have $\phi_t\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to -\infty$

we need $y_0 = 0$. In this case $\phi_t\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} (x_0 - y_0^3) e^{t} \\ 0 \end{pmatrix}$

So $y(t) = 0$

$\Rightarrow W^u(0,0) = \{ (x,y) \mid y = 0 \}$



which matches with the info on the phase portrait.

Exercise 7

Consider the system

$$x' = y$$
$$y' = x(x-1) + cy$$

where c is a parameter.

(a) Find all equilibria and discuss their stability in terms of c.

(b) Show that for $c > 2$ there exists a heteroclinic orbit $\gamma(t) = (x(t), y(t))$ such that $x'(t) > 0$ for all t.

[Hint: Show that there exists a triangular region bounded by $y = 0$, $x = 1$ and $y = \mu x$, (with μ to be determined) which is negatively invariant for the dynamical system]

(a) Clearly the only equilibria are $(0,0)$, $(1,0)$

$$DF\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\ -1 & c\end{pmatrix} \qquad \left.\begin{array}{l}\det = 1 > 0\\ Tr = c\end{array}\right] \Rightarrow \begin{array}{l}\text{If } c<0 \Rightarrow \text{sink (stable)}\\ \text{If } c>0 \Rightarrow \text{source (unstable)}\end{array}$$

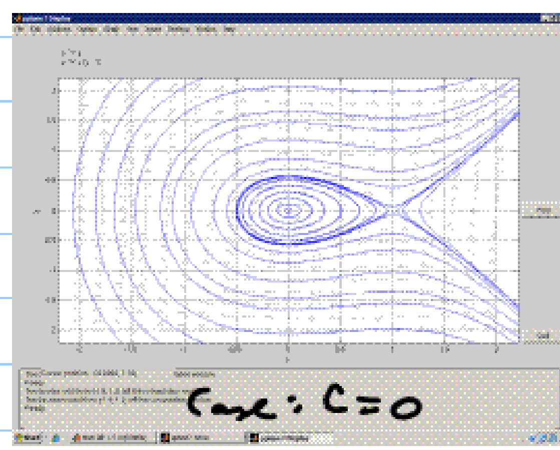
The case $c=0$ cannot be determined from linearization (see below)

$$DF\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & c\end{pmatrix} \Rightarrow \det = -1 < 0 \Rightarrow \text{saddle (unstable)}$$

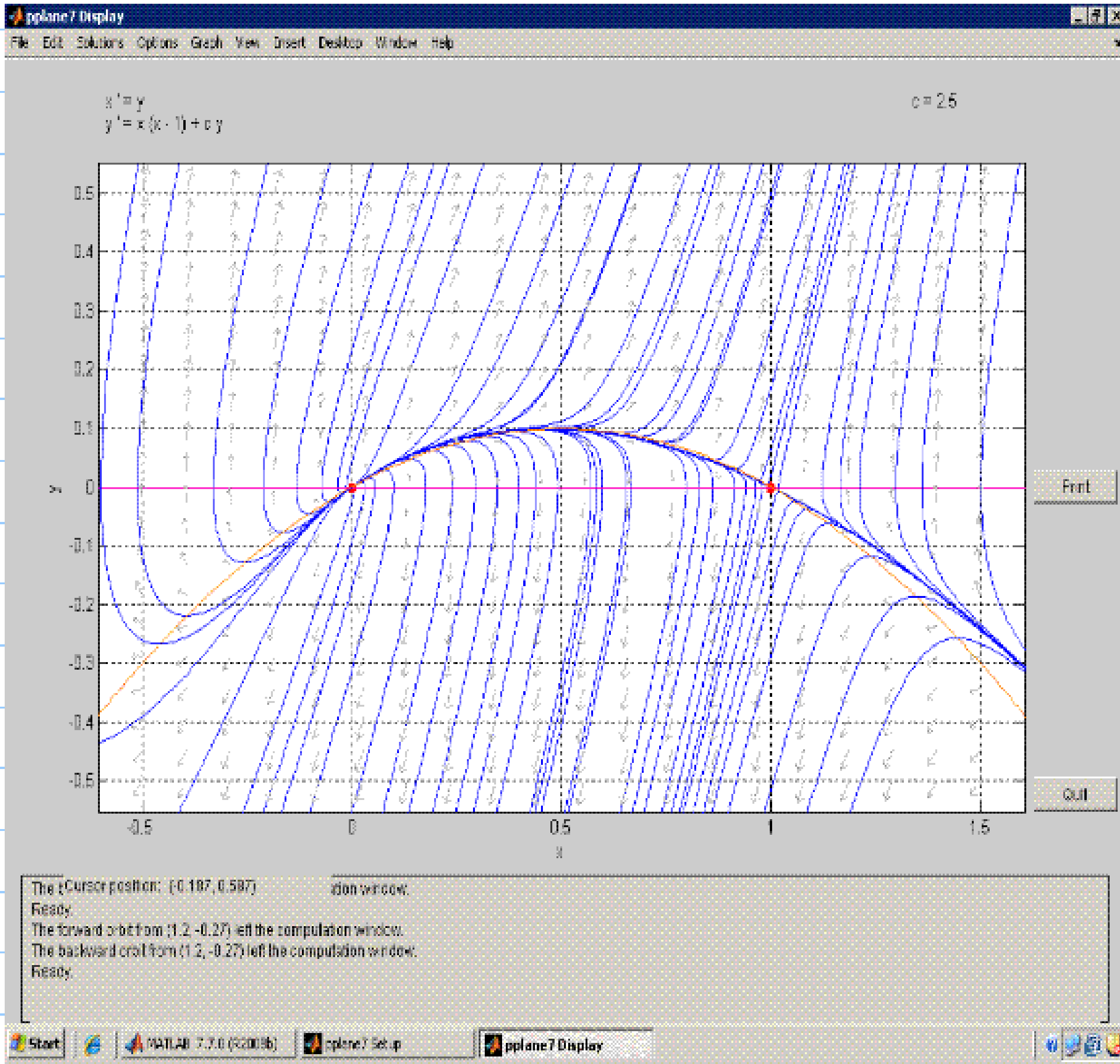
For $c=0$: $\begin{cases} x' = y \\ y' = x(x-1)\end{cases}$ is hamiltonian with

$$H(x,y) = y^2 + \frac{x^2}{2} - \frac{x^3}{3}$$

and in the neighborhood of $(0,0)$ the level curves are bounded hence $\begin{pmatrix}0\\0\end{pmatrix}$ is stable (but not asymptotically)

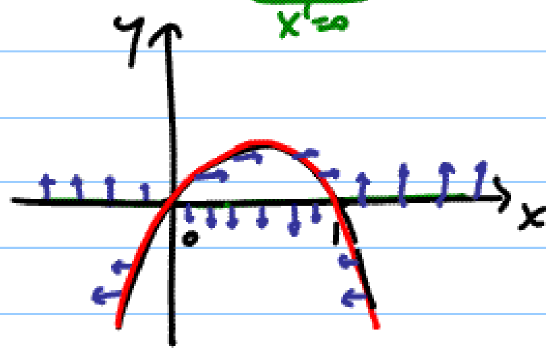


(b) Let $c > 2 \Rightarrow \begin{cases} x' = y \\ y' = x(x-1) + cy \end{cases}$ has a source at $(0,0)$ and a saddle at $(1,0)$

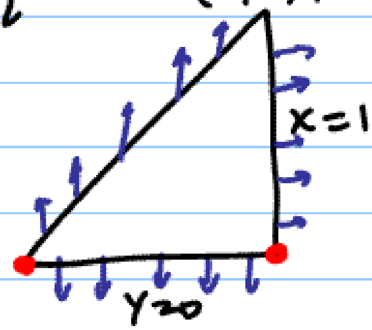


The nullclines are $\boxed{y=0}$ and $\boxed{x(x-1)+cy=0}$

$x'=0$ $\qquad$ $y'=0$



We show that there exists a bounded region (triangle bounded by $y=0$, $x=1$ and $y=\mu x$) which is negatively invariant, that is direction field "leaves" the region at all its boundary points except equilibria $(0,0)$, $(1,0)$.



For $y_0=0$: $\underline{X}_0 = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$, $x_0\in(0,1)$

direction $F(\underline{X}_0) = \begin{pmatrix} 0 \\ x_0(x_0-1) \end{pmatrix}$

points away from the region

since $y_0' = x_0(x_0-1) < 0 \Rightarrow y(t)<0$ for $t>0$

For $x_0 = 1 \Rightarrow \underline{X}_0 = \begin{pmatrix} 1 \\ y_0 \end{pmatrix}$, $y_0 \in (0,1)$

$F(\underline{X}_0) = \begin{pmatrix} y_0 \\ cy_0 \end{pmatrix} \Rightarrow x_0' = y_0 > 0 \Rightarrow x(t) > 1$ for $t>0$

Finally $y_0 = \mu x_0 \Rightarrow F(\underline{X}_0) = \begin{pmatrix} y_0 \\ x_0(x_0-1)+cy_0 \end{pmatrix}$

$$\left(\frac{y(t)}{x(t)}\right)' = \frac{y'x - yx'}{x^2} = \frac{(x_0(x_0-1)+cy_0)x_0 - y_0 y_0}{x_0^2}$$

$$= x_0 - 1 + c\mu - \mu^2$$

To have (x,y) leave the region $\to$ $\frac{y(t)}{x(t)} > \mu$ for $t>0$ $\Rightarrow$

$\Rightarrow \left(\frac{y}{x}\right)' > 0$ for all $x_0 \in (0,1)$

$$\Rightarrow \quad -1 + c\mu - \mu^2 > 0 \quad \text{or}$$

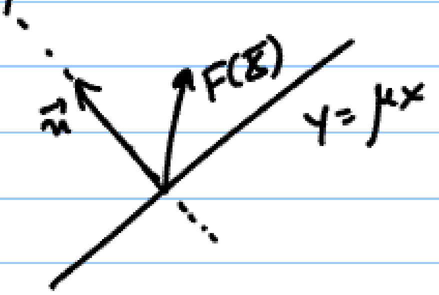
$$\mu^2 - c\mu + 1 < 0$$

$$\mu = \frac{c \pm \sqrt{c^2-4}}{2} \quad \text{real distinct} \quad (\text{for } c>2)$$

$\Rightarrow$ is enough to pick $\mu \in \left(\frac{c-\sqrt{c^2-4}}{2}, \frac{c+\sqrt{c^2-4}}{2}\right)$

Say $\boxed{\mu = \frac{c}{2}}$

[Another way to see this is



$\to \mu x - y = 0$

$n = \begin{pmatrix} -\mu \\ 1 \end{pmatrix}$ outward

$$\vec{n} = \begin{pmatrix} -\mu \\ 1 \end{pmatrix} \qquad F(\bar{x}) = \begin{pmatrix} y \\ x(x-1) + cy \end{pmatrix}$$

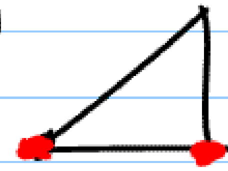
$$\vec{n} \cdot F(\bar{x}) = -\mu y + 1\,[x(x-1) + cy]$$

$$= x(x-1) + (c-\mu)y = x(x-1) + (c-\mu)\mu x$$

F points outwards if $\vec{n} \cdot F(\bar{x}) \geq 0$ for all $x > 0$

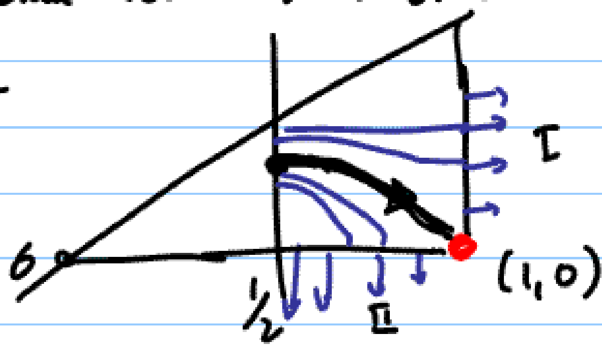
$\Rightarrow x - 1 + (c-\mu)\mu > 0 \Rightarrow \mu^2 - c\mu + 1 < 0 \quad \dots\dots]$

$\Rightarrow$ This shows there is a triangle which is negatively invariant.



Moreover, inside this triangle $x' = y > 0$ (if $y > 0$) so $x(t)$ is decreasing.

We now claim there is a stable curve for $(1,0)$ inside this triangle



Indeed, every point $(\frac{1}{2}, y)$, $y \in (0, \frac{1}{2})$ must exit the region, either through boundary I or II by uniqueness of solutions, $\exists\, y^*$ such that

$$\phi_t\left(\tfrac{1}{2}, y^*\right) \to (1,0) \quad \text{as } t \to \infty$$

By P.B, the α-limit set of $(\frac{1}{2}, y^*)$ must contain $(0,0)$ (no limit cycle can exist due to $x' > 0$) so

$$\phi_t\left(\tfrac{1}{2}, y^*\right) \to (0,0) \text{ as } t \to -\infty$$

$\Rightarrow \left(\frac{1}{2}, y^*\right)$ belongs to the (unique) heteroclinic orbit!

Please note: The heteroclinic orbit is close to (but cannot coincide with) the parabolic nullcline, so an explicit formula for this orbit cannot be obtained!