Sample Exam 1

Math 443/543 - Spring 2009 - Instructor: Radu C. Cascaval

Exercise 1

Suppose that two $n \times n$ systems $X' = AX$ and $X' = BX$ are conjugate, that is there exists an homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\phi^B(t, h(X_0)) = h(\phi^A(t, X_0)).$$

(a) Show that if one system (say for A) has a periodic orbit with period T, that is

$$\phi^A(t + T, X_0) = \phi^A(t, X_0)$$

for some initial point X_0, then the other system has a corresponding periodic orbit *with the same period.*

(b) Use part (a) to show that two systems with imaginary eigenvalues $\pm i\beta$ and $\pm i\gamma$, with $\beta > 0, \gamma > 0$ and $\beta \neq \gamma$ are not conjugate.

Exercise 2

For autonomous differential equations (1-dim system), a similar concept of conjugacy exists. We say that two differential equations $x' = f(x)$ and $x' = g(x)$ are conjugate if there exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ (that is h a continuous function which is invertible) such as it maps the flow $\phi^f(t, x_0)$ of the first equation into the flow $\phi^g(t, h(x_0))$ of the second equation. In this case h *is either monotonically increasing or monotonically decreasing on the entire axis.*

Decide whether the following (1-dim) equations are conjugate or not. Explain.

$$x' = x(1 - x) \quad \text{and} \quad x' = x(1 - x)^2$$

Exercise 3

List all possible canonical forms for a 5×5 real matrix whose eigenvalues are $\lambda_1 = 2$ and $\lambda_{2,3} = 3 \pm i$.

Exercise 4

Find the general (real-valued) solution of the linear system

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -9 & 0 \end{pmatrix}$$

Exercise 5

Suppose A is a $n \times n$ matrix.

(a) Show that $\exp(A)$ is an invertible matrix and find its inverse.

[Hint: $\exp(A + B) = \exp(A)\exp(B)$ for any matrices A, B which commute: $AB = BA$]

(b) Show that if $B = T^{-1}AT$, for T invertible, then

$$\exp(B) = T^{-1}\exp(A)T.$$

(c) Find $\exp(tA)$ if

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$