SOLUTIONS

Wed, Mar 4, 2009 ~~STUDENT NAME~~

EXAM I – MATH 443, SPRING 2009

READ EACH PROBLEM CAREFULLY! To get full credit, you must show all work! The exam has 5 problems on 5 pages. Make sure you turn in all pages! **NO GRAPHING CALCULATORS ALLOWED!**

- **Exercise 1** ***[20pts]*** Show that the differential system

$$X' = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} X,$$

has periodic (in time) solutions. Describe all initial conditions X_0 which correspond to periodic solutions $X(t)$, with $X(0) = X_0$.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}; \quad 0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & -1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(1-\lambda)(-1-\lambda)+2\right] = (1-\lambda)(\lambda^2-1+2)$$

$$= (1-\lambda)(\lambda^2+1)$$

$\Rightarrow$ eigenvalues are $\lambda_1 = 1$, $\lambda_{2,3} = \pm i$

$\Rightarrow$ canonical form of A is $J = T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

where T is $\left[X^{(1)} \; U^{(2)} \; V^{(2)} \right]$.

$$\lambda_1 = 1 \Rightarrow \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & -1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_3 = 0 \\ x_1 = x_2 \end{matrix} \Rightarrow X^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\lambda_2 = i \Rightarrow \begin{pmatrix} 1-i & 0 & -2 \\ 0 & 1-i & 0 \\ 1 & -1 & -1-i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_2 = 0 \\ (1-i)x_1 - 2x_3 = 0 \end{matrix} \Rightarrow X^{(2)} = \begin{pmatrix} 2 \\ 0 \\ 1-i \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$\Rightarrow T = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$. Since $X' = AX$ is conjugate to $Y' = (T^{-1}AT)Y = JY$ via the conjugacy $X = TY$, it means that periodic solutions $X(t)$ correspond to periodic solutions $Y(t)$, which are in the $y_1 = 0$ plane $Y_0 = \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix} \in \text{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$$\Rightarrow X_0 = TY_0 \in \text{span}\left\{T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\right\}.$$

- **Exercise 2** ***[20pts]*** Given the (nonlinear) differential equation

$$x' = x^2 - 2ax + b$$

(a) Determine the equilibria (in terms of the parameters a and b).
(b) Sketch the phase lines and direction fields for the values $(a, b) = (0, 4)$, $(2, 4)$ and $(4, 4)$.
(c) Identify the regions in the parameter plane (a, b) where the system has distinct behavior (that is, the systems corresponding to different regions are not conjugate). Explain your conclusion.

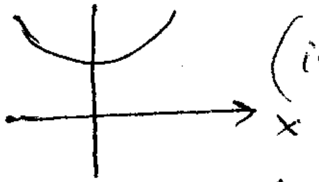
(a) Equilibria are: $0 = x^2 - 2ax + b \Rightarrow x_{1,2} = a \pm \sqrt{a^2 - b}$, if $a^2 - b >$

So, if $b < a^2 \Rightarrow$ 2 distinct equilibria $x_1^* = a - \sqrt{a^2-b}$, $x_2^* = a + \sqrt{a^2-b}$

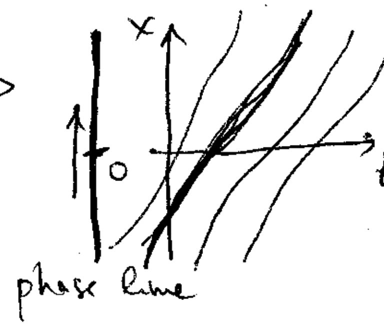
$b = a^2 \Rightarrow$ one equilibrium $x^* = a$

$b > a^2 \Rightarrow$ no equilibrium

(b) $(a,b) = (0,4) \Rightarrow x' = x^2 + 4$

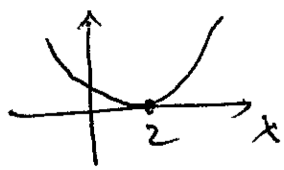
RHS 

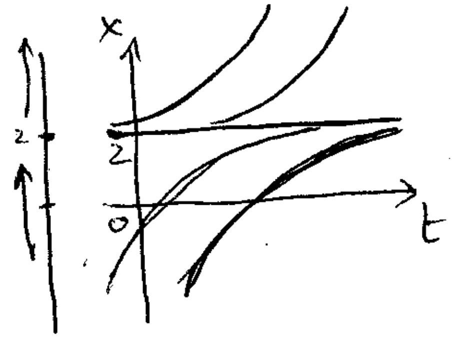
(is positive for all x)

$\Rightarrow$ 

$x(t)$ can be computed explicitly to see shape is $\tan^{-1}$. (but this was not required)

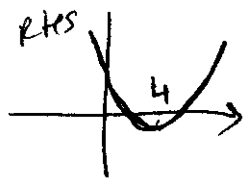
$(a,b) = (2,4) \Rightarrow x' = x^2 - 4x + 4 = (x-2)^2$

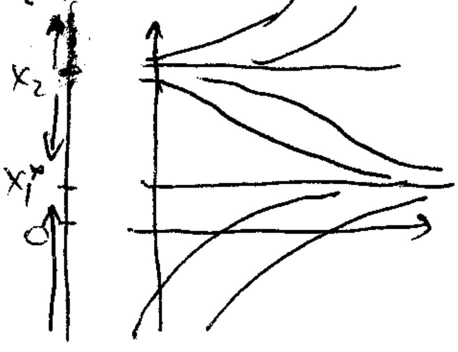


$\Rightarrow$ 

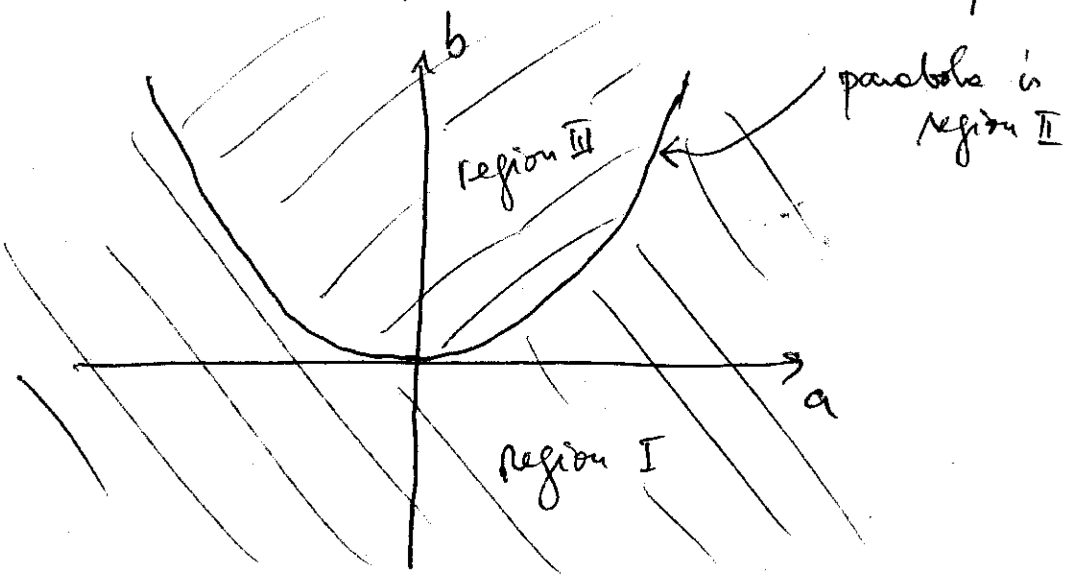
$x(t)$ can be computed explicitly again, to see shape is $\frac{1}{x-2} = t + \dots$ (hyperbolas)

$(a,b) = (4,4) \Rightarrow x' = x^2 - 8x + 4 = (x - x_1)(x - x_2)$, $x_{1,2} = 4 \pm \sqrt{12} = 2(2 \pm \sqrt{3})$

RHS 



(c) Clearly, the conditions $b - a^2 < 0$, > 0 or $= 0$ determine systems which are not conjugate (since the # of equilibrium points changes the conjugacy type)



region I $(b < a^2)$ $\Rightarrow$ two equilibrium points
region II $(b = a^2)$ $\Rightarrow$ one equilibrium point
region III $(b > a^2)$ $\Rightarrow$ no equilibrium points

- **Exercise 3** *[20pts]*

(a) *Determine whether each of the following statements is TRUE or FALSE, (giving a brief explanation if true and or a counterexample if false*

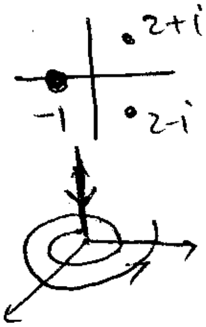
Two 2×2 linear differential systems are conjugate if and only if they have (counting multiplicity):
(i) the same number of eigenvalues with negative real part ; TRUE FALSE
(ii) the same number of eigenvalues with positive real part ; TRUE FALSE
(iii) the same number of pure imaginary eigenvalues. TRUE FALSE

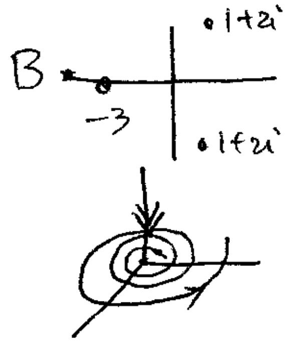
In (i),(ii) as long as there are no pure imaginary eigenvalues, then the # of negative evalues is either 0, 1 or 2, and in each case, the possible Jordan canonical forms are conjugate

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \sim \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \sim \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \sim \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

(iii) is false, since systems with distinct pure imag eigenvalues are not conjugate, since they don't have periodic sol. with same period

(b) Let A be a 3×3 matrix with eigenvalues -1 and $2 \pm i$ and B a 3×3 matrix with eigenvalues -3 and $1 \pm 2i$. Assume both are in canonical form. Sketch a typical solution for each of the systems $X' = AX$ and $Y' = BY$. Determine whether these systems are conjugate or not.

values of A 

, B 

$$\Rightarrow A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

(c) Let A be a 4×4 matrix with eigenvalues -1, $2 \pm i$ and B a 4×4 matrix with eigenvalues -3, $1 \pm 2i$. List all possible combinations of canonical forms for A and B. Does the same conclusion, as in (b), hold?
[Hint: Using the homeomorphisms $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ which exist between conjugate 2×2 systems, such as in part (a), write down a homeomorphism $h : \mathbb{R}^4 \to \mathbb{R}^4$ which acts as a conjugacy between the two 4×4 systems.]

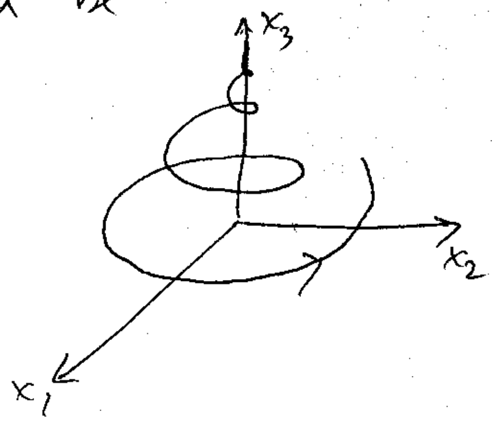
$$A = \begin{pmatrix} \boxed{-1} & & & \\ & \boxed{-1} & & \\ & & 2 & 1 \\ & & -1 & 2 \end{pmatrix} \text{ or } A = \begin{pmatrix} -1 & 1 & & \\ & -1 & & \\ & & 2 & 1 \\ & & -1 & 2 \end{pmatrix}$$

no other combinations possible for A

$$B = \begin{pmatrix} \boxed{-3} & & & \\ & \boxed{-3} & & \\ & & 1 & 2 \\ & & -2 & 1 \end{pmatrix} \text{ or } B = \begin{pmatrix} -3 & 1 & & \\ & -3 & & \\ & & 1 & 2 \\ & & -2 & 1 \end{pmatrix}$$

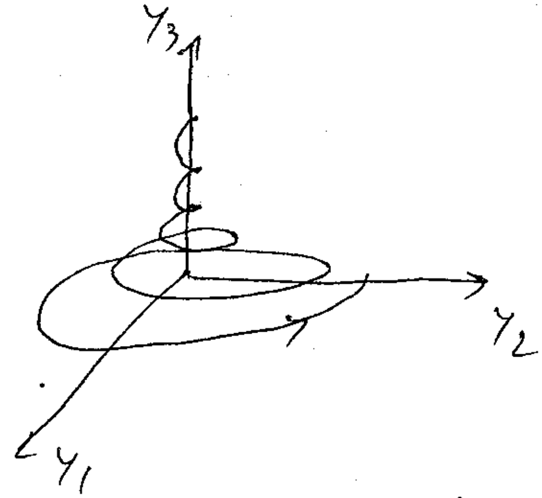
(same for B)

#3b) A typical solution for $X = AX$ and $Y = BY$ would be



(approaches the (x_1, x_2) plane) but spirals out

(spiral saddle)



Similar behavior, but approaches the (y_1, y_2) plane faster and spirals out slower.

The two systems are conjugate, since one can find $h: \mathbb{R}^3 \to \mathbb{R}^3$, $h\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \tilde{h}_1(x_1) \\ \tilde{h}_2\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

Explicitly $\tilde{h}_1(x_1) = x_1^3$ and $\tilde{h}_2\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}$ makes the conjugacy between $X' = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} X$ and $Y' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} Y$ (not explicit).

#3c) Any of the 4 combinations (A, B) in canonical form give <u>conjugate</u> systems, since the # of negative eigenvalues for $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ & -1 \end{pmatrix}$, $\begin{pmatrix} -3 & \\ & -3 \end{pmatrix}$, $\begin{pmatrix} -3 & 1 \\ & -3 \end{pmatrix}$ are the same and this is also true for the eigenvalues with positive real part.

$h: \mathbb{R}^4 \to \mathbb{R}^4$ $\quad h\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \tilde{h}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \tilde{h}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$.

... (real valued) solution of the system

$$X' = AX, \quad \text{where } A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$$

This is already in canonical form, so it can be decoupled to the corresponding blocks

$$X_{12} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad X_{12}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} X_{12} \Rightarrow X_{12}(t) = \exp\left(t\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\right) X_{12}(0)$$

$$X_{34} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \quad X_{34}' = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} X_{34} \rightarrow X_{34}(t) = \exp\left(t\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\right) X_{34}(0)$$

Since $\exp\left(t\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\right) = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$, $\exp\left(t\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}\right) = \begin{pmatrix} e^{3t}\cos t & e^{3t}\sin t \\ -e^{3t}\sin t & e^{3t}\cos t \end{pmatrix}$

$$\Rightarrow X(t) = \begin{pmatrix} c_1 e^{2t} + c_2 t e^{2t} \\ c_2 e^{2t} \\ c_3 e^{3t}\cos t + c_4 e^{3t}\sin t \\ -c_3 e^{3t}\sin t + c_4 e^{3t}\cos t \end{pmatrix}.$$

Note: One can get the same form by following the procedure described earlier in the course, using eigenvalues & eigenvectors:

$$X(t) = c_1 X_1(t) + c_2 X_2(t) + c_3 \cancel{V}_3(t) + c_4 V_3(t)$$

$$= c_1 e^{2t} X^{(1)} + c_2\left(e^{2t} X^{(2)} + t e^{2t} X^{(1)}\right) + c_3 \,\mathrm{Real}\left(e^{(3+i)t} X^{(3)}\right) + c_4 \,\mathrm{Im}\left(e^{(3+i)t} X^{(3)}\right)$$

where $X^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $X^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $X^{(3)}$ = complex e.v for $\lambda = 3+i$

e.v for $\lambda = 2$ generalized e.v. for $\lambda = 2$

- **Exercise 5** *[20pts]*

(a) Show that the matrix $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is such that $B^k = 0$ for some $k > 0$.

(b) Compute $\exp(tB)$.

(c) For $\lambda \neq 0$, let $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \lambda I_3 + B.$

Show that

$$\exp(tA) = e^{t\lambda} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

(d) Write down $\exp(tC)$ (no need to show computations, just briefly explain your answer) for

$$C = \left(\begin{array}{ccc|cc} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right) \Rightarrow \exp(tC) = \left(\begin{array}{ccc|cc} e^{2t} & te^{2t} & \frac{t^2}{2}e^{2t} & & \\ 0 & e^{2t} & te^{2t} & & \\ 0 & 0 & e^{2t} & & \\ \hline & & & e^{2t} & te^{2t} \\ & & & 0 & e^{2t} \end{array}\right)$$

(a) $B^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$B^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \Rightarrow k = 3.$

(b) $\exp(tB) = I + tB + \frac{1}{2}(tB)^2 + \underbrace{\frac{1}{6}(tB)^3}_{0} + \underbrace{\cdots}_{0} \Rightarrow$

$\Rightarrow \exp(tB) = I + tB + \frac{t^2}{2}B^2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 0 & t & \\ & 0 & t \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{t^2}{2} \\ & 0 & 0 \\ & & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{pmatrix}.$

(c) Since λI_3 and B commute

$\exp(tA) = \exp(\lambda t I_3 + tB) = \exp(\lambda t I_3)\exp(tB) = e^{\lambda t}\exp(tB)$

$= e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2 \\ & 1 & t \\ & & 1 \end{pmatrix}$

see above

(d) C is in canonical form, with a 3x3 block and a 2x2 block.