Wed, April 15, 2009

SOLUTIONS STUDENT NAME:

EXAM II – MATH 4/543, SPRING 2009

Prof. Radu C. Cascaval

Due Mon, Apr 20

YOU MAY NOT USE ANY BREATHING SOURCE OF HELP
(e.g. HUMANS, which includes colleagues, friends, parents, grandparents, cousins, strangers)!!!

NO TEAM WORK OF ANY KIND WILL BE TOLERATED!!!!
Any attempt (voluntary or involuntary) to consult with others will be severely punished!!! No kidding! All parties involved will receive ZERO on the Exam and, in addition, disciplinary actions may be taken.

Please read each problem CAREFULLY. You may consult the lecture notes, homework solutions, textbook etc and also use computer to help in solving (parts) of the problems.

Turn in the solutions for each problem attached to the individual page with the statement of the problem, staple them in correct order (including this page). Make sure you have included all problems!!!

Exercise 1
Consider the 1-dimensional dynamical system governed by the equation

$$x' = f(x)$$

where f is a continuously differentiable function on x. Show that if $y \in \mathbb{R}$ is a ω-limit point for a

Solution 1: Note that $x' = f(x)$ can be written as a gradient system

$$x' = -\text{grad}\, V(x), \quad \text{where } V' = -f \Rightarrow V = -\int f(x)\,dx$$

Clearly, V is decreasing along solutions

$$\dot V(x) = \frac{d}{dt} V(x(t))\Big|_{t=0} = V'(x(t))\cdot x'(t) = -\left[V'(x(t))\right]^2 = -\left(f(x(t))\right)^2 \le 0$$

The conclusion that y is an equilibrium follows the same way as for 2D gradient systems. Let $x(t_n) \to y \in \omega(x_0)$ and
Assume the contrary: $f(y) \neq 0 \Rightarrow$ there is a solution $y(t)$ starting at y such that $V(y(s)) < V(y)$ for $s > 0$. By continuity,

$$V(x(t_n)) \searrow V(y) \quad \text{and} \quad V(x(t_n+s)) \searrow V(y(s)) < V(y)$$

which is a contradiction!

Solution 2: Let $y \in \omega(x_0) \Rightarrow$ there exists a sequence $t_n \to \infty$ such that $x(t_n) = \phi(t_n; x_0) \longrightarrow y$

We can easily choose (possibly a subsequence of) $\{t_n\}$ with

$$|t_{n+1} - t_n| > 1$$

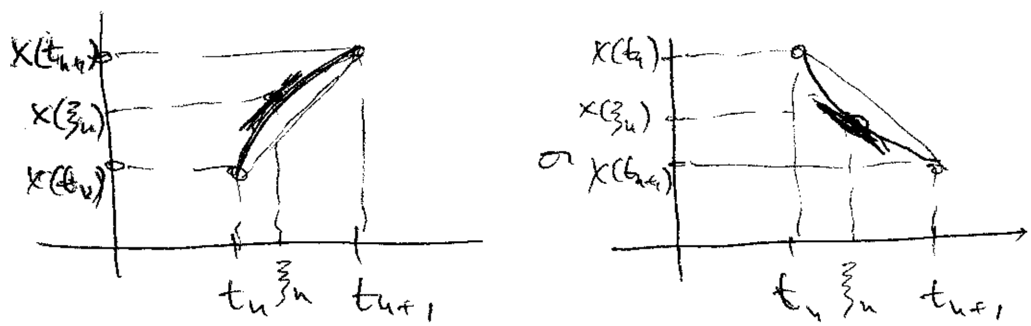
Consider $\{x(t_{n+1}) - x(t_n)\}$ a sequence that goes to 0.

By Mean Value Theorem, there exists $\xi_n \in [t_n, t_{n+1}]$ such that

$$x'(\xi_n) = \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \to 0$$

$\Rightarrow f(x(\xi_n)) \to 0$.

By the uniqueness of solutions, we also have $x(\xi_n)$ is between $x(t_n)$ and $x(t_{n+1})$



So $\lim\limits_{n \to \infty} x(\xi_n) = y$. Because f is continuous

$$f(y) = f\left(\lim_{n \to \infty} x(\xi_n)\right) = \lim_{n \to \infty} f(x(\xi_n)) = 0$$

$\Rightarrow y$ is an equilibrium point!

Note: There are several other solutions.

Exercise 2

Consider the planar nonlinear system

$$x' = -y$$
$$y' = x - y + y^3$$

(a) Find the linearization around the origin and determine the stability of its equilibrium.
(b) Show that $L(x,y) = x^2 + y^2$ is a Liapunov function for the nonlinear system.
(c) Determine the 'size' of the basin of attraction of the origin (e.g. find a ball of radius R centered at the origin, which is included in the basin of attraction of the origin, with R as large as you possibly can).

(a) $F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x-y+y^3 \end{bmatrix} \Rightarrow DF\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1+3y^2 \end{bmatrix}$

$\Rightarrow DF\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = A$. $\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -(1-\lambda) \end{vmatrix} =$

$= (-\lambda)(-1-\lambda)+1 = \lambda^2+\lambda+1 = 0$

$\operatorname{Re}\lambda = -\frac{1}{2} < 0$

$\Rightarrow$ the linearization has $(0,0)$ as a sink (spiraling in)

$\Rightarrow$ the nonlinear system has a nonlinear sink at $(0,0)$.

(b) $\frac{d}{dt}L(x(t),y(t)) = 2xx' + 2yy' = 2x(-y) + 2y(x-y+y^3)$

$= 2y(-y+y^3) = 2y^2(y^2-1) < 0$

provided $y^2-1<0 \Rightarrow |y|\le 1$ or $y\in(-1,1)$.

Since $\mathcal{O} = \{(x,y) \mid -1<y<1\}$ is an open set containing $(0,0) \Rightarrow L$ is a strict Liapunov function on $\mathcal{O}$.

$\Rightarrow (0,0)$ is asymptotically stable. To

(c) The basin of attraction contains the Ball $B_r = \{(x,y), x^2+y^2<$
which is bounded by the level curve $L(x,y)=1$.

Exercise 3

Given the 2nd order (scalar) nonlinear equation

$$x'' + x - \frac{1}{3}x^3 = 0.$$

(a) Write it as a system of 1st order equations and show the system is Hamiltonian, by explicitly computing a Hamiltonian function.

(b) Do the same thing for the more general equation

$$x'' + F(x) = 0,$$

where F is a smooth function on $\mathbb{R}$.

(c) Compare the phase portrait of the system in (a) with the phase portrait of the nonlinear pendulum

$$x'' + \sin x = 0$$

Set $x' = y \Rightarrow y' = x'' = -x + \frac{1}{3}x^3$

(a) $\begin{cases} x' = y & = f(x,y) \\ y' = -x + \frac{1}{3}x^3 = g(x,y) \end{cases}$ $\quad \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0 + 0 = 0$

f, g cont on $\mathbb{R}^2$ - simply connected

$\Rightarrow$ System is Hamiltonian. To find H:

$f(x,y) = \frac{\partial H}{\partial y} \Rightarrow y = \frac{\partial H}{\partial y} \Rightarrow H = \frac{1}{2}y^2 + c(x)$

$g(x,y) = -\frac{\partial H}{\partial x} \Rightarrow -x + \frac{1}{3}x^3 = -c'(x) \Rightarrow c'(x) = x - \frac{1}{3}x^3 \Rightarrow$

$\Rightarrow c(x) = \frac{1}{2}x^2 - \frac{1}{12}x^4$

$\Rightarrow \boxed{H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{12}x^4}$ on $\mathbb{R}^2$

(b) Similar computations lead to

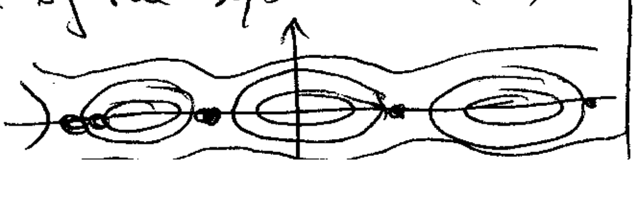
$H(x,y) = \frac{1}{2}y^2 + \int F(x)\,dx$ on $\mathbb{R}^2$

to be the Hamiltonian of the system $\begin{cases} x' = y \\ y' = -F(x) \end{cases}$

(c) The phase portraits are conjugate near $(0,0)$, but not past the equilibria of the system in (a)

Can you find a conjugacy map???

(a) 



Exercise 4

Consider the dynamical system on $\mathbb{R}^2 \setminus \{(0,0)\}$

$$x' = -\frac{y}{x^2+y^2} \quad [=: f(x,y)]$$
$$y' = \frac{x}{x^2+y^2} \quad [=: g(x,y)]$$

(a) Verify that $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$.
(b) Prove that the system is NOT a gradient system.
(c) Show that the system is Hamiltonian, by explicitly computing a Hamiltonian function.

(a) direct verification

(b) The system on $\mathbb{R}^2 \setminus \{(0,0)\}$ cannot be gradient. The simplest way is the show that there is a nonconstant periodic solution

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases}, \quad t \in [0, 2\pi]$$

which cannot happen for gradient systems! (See lecture notes).

(c) Notice that $H(x,y) = -\frac{1}{2}\ln(x^2+y^2)$

satisfies $x' = \frac{\partial H}{\partial y} = -\frac{y}{x^2+y^2}$

$y' = -\frac{\partial H}{\partial x} = \frac{x}{x^2+y^2}$

So the system is Hamiltonian on $\mathbb{R}^2 \setminus \{(0,0)\}$.

Exercise 5

Consider the differential system ($X = (x, y)^T \in \mathbb{R}^2$)

$$X' = AX + \lambda |X|^2 X$$

where $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, and α, β are real numbers, with $\alpha < 0$. Here $|X|^2 = x^2 + y^2$.

(a) Show that $L(x, y) = x^2 + y^2$ is a Liapunov function and determine the basin of attraction of the origin.

(b) Determine whether any bifurcation occurs with respect to the paramenter λ or not.

(c) Do the same conclusions hold if $\alpha = 0$? Explain.

$$\begin{cases} x' = \alpha x + \beta y + \lambda(x^2+y^2)x \\ y' = -\beta x + \alpha y + \lambda(x^2+y^2)y \end{cases}$$

$$\frac{d}{dt}L(x(t),y(t)) = 2xx' + 2yy' = \dots = 2\alpha(x^2+y^2) + 2\lambda(x^2+y^2)^2 = \left(2\alpha + 2\lambda(x^2+y^2)\right)(x^2+y^2)$$

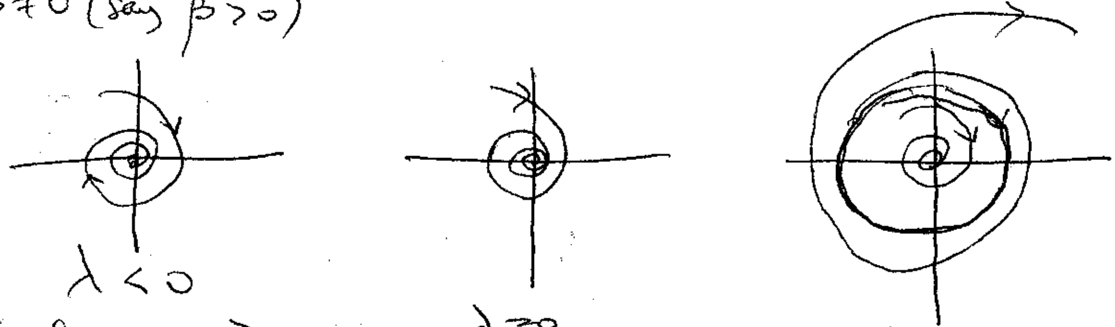
Since $\alpha < 0$, we get

$\lambda \le 0 \Rightarrow \dot{L} < 0$ for all (x,y) $\Rightarrow$ basin of attraction is all $\mathbb{R}^2$

$\lambda > 0 \Rightarrow \dot{L} < 0$ for $(x^2+y^2) < -\frac{\alpha}{\lambda}$ $\Rightarrow$ basin of attraction includes $\left\{(x,y) \mid x^2+y^2 < -\frac{\alpha}{\lambda}\right\}$

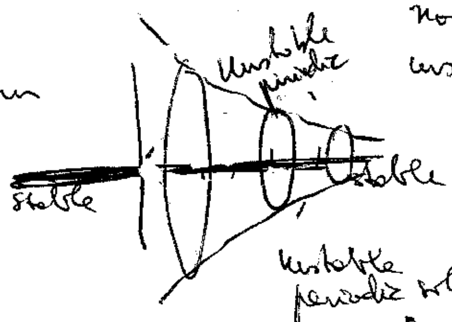
b) In polar coordinates $\begin{cases} r' = \alpha r + \lambda r^3 \\ \theta' = -\beta \end{cases}$ $\left(\begin{matrix} rr' = xx' + yy' \\ r^2\theta' = xy' - x'y \end{matrix}\right)$

For $\beta \neq 0$ (say $\beta > 0$)

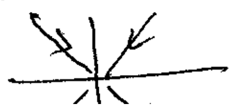


$\lambda < 0$ (nonlinear sink) $\lambda = 0$ $\lambda > 0$ nonlinear sink unstable periodic orbit

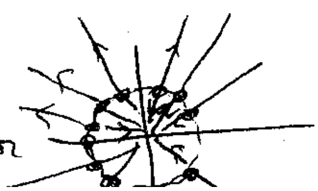
$\Rightarrow$ Bifurcation diagram



For $\beta = 0 \Rightarrow \theta' = 0 \to \theta =$ constant $\Rightarrow$

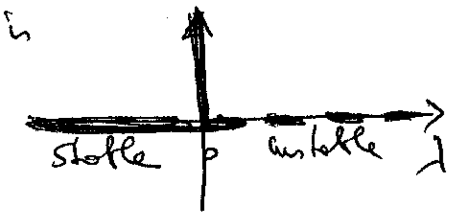


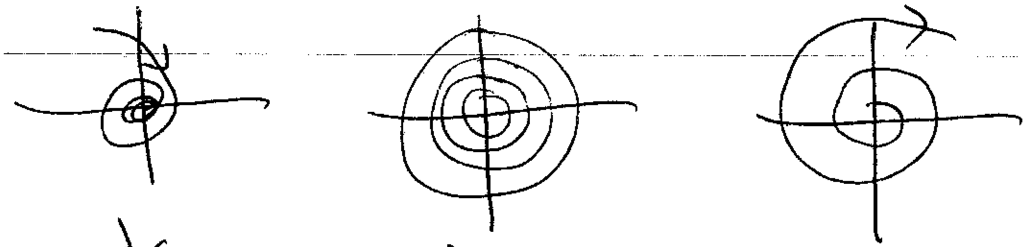
or



(c) The case $\alpha = 0$: $L(x,y)$ is Liapunov function only if $\lambda \leq 0$

Bifurcation is





$\lambda < 0$ $\lambda = 0$ $\lambda > 0$