Math 447/547 - FALL 2009 **STUDENT NAME:** ______________________

Dr. Radu C. Cascaval **STUDENT ID:** ______________________

EXAM I – Methods in Applied Math

Please read each problem carefully. *Please make sure you turn in all 5 pages!!*

.

Exercise 1 *[30pts]*

(a) Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0$$

with boundary conditions

$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(\pi, t) = 0, \quad t > 0$$

and initial condition

$$u(x,0) = \sin\frac{x}{2} - 4\sin\frac{3x}{2}, \quad 0 < x < \pi$$

(b) Using the 'energy method' with $E(t) = \int_0^\pi u(x,t)^2\,dx$, show that the solution found above is unique.

Exercise 2 *[30pts]*
Find the equilibrium temperature distribution between two spheres ($\rho_1 < \rho < \rho_2$) if both the inner and outer boundaries obey Newton's law of cooling $\frac{\partial u}{\partial n} + \alpha u = T_1$ and $\frac{\partial u}{\partial n} + \alpha u = T_2$ respectively. Here $\alpha > 0$ is constant.
[Hint: The spherically symmetric heat equation in shperical coordinates reads: $\frac{\partial u}{\partial t} = \frac{k}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial u}{\partial \rho}\right), k > 0$*].*

Exercise 3 *[30pts]*

(a) Using the separation of variables method, write down the *eigenvalue problem* that appear when solving the Laplace's equation

$$0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

in the planar region $R = \{(r,\theta)|\, a < r < b,\, 0 < \theta < \pi/2\}$ with boundary conditions:

$$u(r,0) = 0, \quad u(r,\frac{\pi}{2}) = f(r), \quad \frac{\partial u}{\partial r}(a,\theta) = 0, \quad \frac{\partial u}{\partial r}(b,\theta) = 0.$$

[**DO NOT solve the eigenvalue problem OR the Laplace's equation completely.**]

(b) Do the same thing for the boundary conditions

$$u(r,0) = 0, \quad u(r,\frac{\pi}{2}) = 0, \quad \frac{\partial u}{\partial r}(a,\theta) = 0, \quad \frac{\partial u}{\partial r}(b,\theta) = g(\theta).$$

Exercise 4 *[30pts]*
Solve the Laplace's equation in the semi-infinite strip ($0 < x < L$, $0 < y < \infty$) in the figure, subject to the boundary conditions

$$u(x,0) = f(x), \quad u(L,y) = 0, \quad \frac{\partial u}{\partial y}(0,y) = 0$$

[Hint: You may assume the solution $u(x,y)$ remains bounded as $y \to \infty$.]

Exercise 5 *[30pts]*

Find the solution of the Laplace's equation outside of the disk of radius 1 (hence $r > 1$) with Robin boundary conditions $\frac{\partial u}{\partial n} + u = f(\theta)$, when $r = 1$. Assume the solution $u(r, \theta)$ remains bounded as $r \to \infty$.