

# Elements of Optimal Control Theory

## Pontryagin's Principle

### STATEMENT OF THE CONTROL PROBLEM

Given a system of ODEs ( $x_1, \dots, x_n$  are state variables,  $\mathbf{u}$  is the control variable)

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n; \mathbf{u}) \\ \frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n; \mathbf{u}) \\ \dots \\ \frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n; \mathbf{u}) \end{cases}$$

with initial data  $x_j(0) = x_j^0$ ,  $j = 1, \dots, n$ , the objective is to find an optimal control  $\mathbf{u}^* = \mathbf{u}^*(t)$  and the corresponding path  $\mathbf{x}^* = \mathbf{x}^*(t)$ ,  $0 \leq t \leq T$  which **minimizes** the functional

$$J(\mathbf{u}) = c_1 x_1(T) + c_2 x_2(T) + \dots + c_n x_n(T) + \int_0^T F(x_1(t), \dots, x_n(t); \mathbf{u}(t)) dt$$

among all admissible controls  $u = u(t)$ . Pontryagin's Principle gives a **necessary condition** for  $\mathbf{u} = \mathbf{u}^*(t)$  and corresponding  $\mathbf{x} = \mathbf{x}^*(t)$  to be optimal, i.e.

$$I(\mathbf{u}^*) = \min_{\mathbf{u}} I(\mathbf{u})$$

### METHOD FOR FINDING OPTIMAL SOLUTION:

Construct the "Hamiltonian"

$$H(\psi_1, \dots, \psi_n, x_1, \dots, x_n, \mathbf{u}) = \psi_1 f_1 + \psi_2 f_2 + \dots + \psi_n f_n + F$$

where  $\psi_1, \dots, \psi_n$  are so-called adjoint variables satisfying the adjoint system

$$\frac{d\psi_j}{dt} = -\frac{\partial H}{\partial x_j}, \quad j = 1, \dots, n \quad \left( \text{Note: } \frac{dx_j}{dt} = \frac{\partial H}{\partial \psi_j}, \quad j = 1, \dots, n \right)$$

For each fixed time  $t$  ( $0 \leq t \leq T$ ), **choose  $\mathbf{u}^*$**  be the value of  $u$  that maximizes the Hamiltonian  $H = H(\mathbf{u})$  among all admissible  $\mathbf{u}$ 's ( $\psi$  and  $x$  are fixed):

$$H(\mathbf{u}^*) = \max_{\mathbf{u}} H(\mathbf{u})$$

#### Case 1. Fixed Terminal Time $T$ , Free Terminal State $\mathbf{x}(T)$

Set terminal conditions for the adjoint system:  $\psi_j(T) = c_j$ ,  $j = 1 \dots n$ . Solve the adjoint system backward in time.

#### Case 2. Time Optimal Problem. Free Terminal Time $T$ , Fixed Terminal State $\mathbf{x}(T)$ .

$c_j = 0$ ,  $j = 1, \dots, n$ ,  $F \equiv 1$ . No terminal conditions for the adjoint system. Solve the adjoint system with  $H(\psi(T), \mathbf{x}(T), \mathbf{u}(T)) = 0$ .

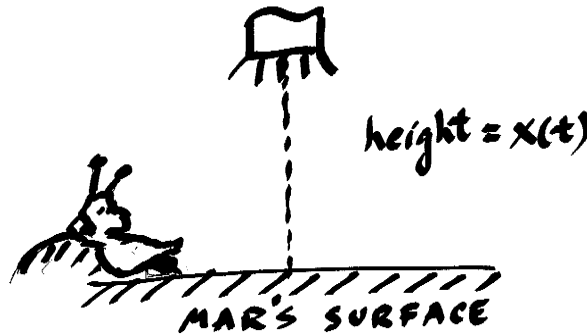
# Example 1. Landing Probe on Mars Problem

Consider the time-optimal problem of landing a probe on Mars. Assume the descent starts close to the surface of Mars and is affected only by gravitational field and by friction with 'air' (proportional with the velocity). The lander is equipped with a braking system, which applies a force  $u$ , in order to slow down the descent. The braking force cannot exceed a given (maximum) value  $U$ . Let  $x(t)$  be the distance from the landing site. Then Newton's Law gives  $m\ddot{x} = -mg - k\dot{x} + u$ . The objective is to find the optimal control  $u(t)$  such that the object lands in **shortest time**.

For convenience assume  $m = 1$ ,  $k = 1$ ,  $U = 2$  and  $g = 1$  and  $0 \leq u \leq 2$ . The system reads

$$\ddot{x} = -1 - \dot{x} + u$$

As initial condition take  $x(0) = 10$  and  $\dot{x}(0) = 0$ .



## Solution

This problem is a standard time optimal problem (when terminal time  $T$  for return to origin has to be minimized). It can be formulated as a Pontryagin Maximum Principle problem as follows.

Write the second order ODE for  $x = x(t)$  as a system of two equations for two state variables  $x_1 = x$  (position) and  $x_2 = \dot{x}$  (speed).

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -1 - x_2 + u \end{aligned} \tag{1}$$

and initial conditions  $x_1 = 10$  and  $x_2 = 0$ . The objective is to minimize  $T$  subject to the constraint  $0 \leq u \leq 2$  and terminal conditions  $x_1(T) = 0$ ,  $x_2(T) = 0$ .

We set the Hamiltonian (for time optimal problem)

$$H(\psi_1, \psi_2, x_1, x_2; u) = \psi_1 x_2 + \psi_2(-1 - x_2 + u) - 1$$

where  $\psi_1$  and  $\psi_2$  satisfy the adjoint system

$$\begin{aligned} \frac{d\psi_1}{dt} &= -\frac{\partial H}{\partial x_1} \equiv 0 \\ \frac{d\psi_2}{dt} &= -\frac{\partial H}{\partial x_2} = -\psi_1 + \psi_2 \end{aligned} \tag{2}$$

There are no terminal conditions for the adjoint system.

Pontryagin's principle asks to maximize  $H$  as a function of  $u \in [0, 2]$  at each fixed time  $t$ . Since  $H$  is linear in  $u$ , it follows that the maximum occurs at one of the endpoints  $u = 0$  or  $u = 2$ , hence the control

is bang-bang! More precisely,

$$\begin{aligned} u^*(t) &= 0 \text{ whenever } \psi_2(t) < 0 \\ u^*(t) &= 2 \text{ whenever } \psi_2(t) > 0 \end{aligned}$$

We now solve the adjoint system. Since  $\psi_1 \equiv c_1$  is constant, we obtain

$$\psi_2(t) = c_1 + c_2 e^t \tag{3}$$

It is clear that no matter what constants  $c_1$  and  $c_2$  we use,  $\psi_2$  can have no more than one sign change. In order to be able to land with zero speed at time  $T$ , it is clear that the initial stage one must choose  $u = 0$  and then switch (at a precise time  $t_s$ ) to  $u = 2$ .

The remaining of the problem is to find the switching time  $t_s$  and the landing time  $T$ . To this end, we solve the state system (1) for  $u(t) \equiv 0$ , for  $t \in [0, t_s]$  and  $u(t) \equiv 2$ , for  $t \in [t_s, T]$ :

In the time interval  $t \in [0, t_s]$ ,  $u \equiv 0$  yields:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= -1 - x_2 \end{aligned}$$

which has the solution  $x_1(t) = 11 - t - e^{-t}$  and  $x_2(t) = -1 + e^{-t}$

In the time interval  $t \in [t_s, T]$ ,  $u \equiv 2$  yields:

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= 1 - x_2 \end{aligned}$$

which has the solution  $x_1(t) = -1 + t - T + e^{T-t}$  and  $x_2(t) = 1 - e^{T-t}$

The matching conditions for  $x_1$  and  $x_2$  at  $t = t_s$  yield the system of equations for  $t_s$  and  $T$ . which can be solved (surprisingly) explicitly:

$$\begin{cases} 11 - t_s - e^{-t_s} = -1 + t_s - T + e^{T-t_s} \\ -1 + e^{-t_s} = 1 + e^{T-t_s} \end{cases}$$

Whether you use a computer or you do it by hand, the solution turns out to be  $t_s \approx 10.692$  and  $T \approx 11.386$ . (quite short braking period...)

**Conclusion:** To achieve the optimal solution (landing in shortest amount of time), the probe is left to fall freely for the first 10.692 seconds and then the maximum brake is applied for the next 0.694 seconds.

## Example 2. Insects as Optimizers Problem

Many insects, such as wasps (including hornets and yellowjackets) live in colonies and have an annual life cycle. Their population consists of two castes: workers and reproductives (the latter comprised of queens and males). At the end of each summer all members of the colony die out except for the young queens who may start new colonies in early spring. From an evolutionary perspective, it is clear that a colony, in order to best perpetuate itself, should attempt to program its production of reproductives and workers so as to maximize the number of reproductives at the end of the season - in this way they maximize the number of colonies established the following year. In reality, of course, this programming is not deduced consciously, but is determined by complex genetic characteristics of the insects. We may hypothesize, however, that those colonies that adopt nearly optimal policies of production will have an advantage over their competitors who do not. Thus, it is expected that through continued natural selection, existing colonies should be nearly optimal. We shall formulate and solve a simple version of the insects' optimal control problem to test this hypothesis.

Let  $w(t)$  and  $q(t)$  denote, respectively, the worker and reproductive population levels in the colony. At any time  $t$ ,  $0 \leq t \leq T$ , in the season the colony can devote a fraction  $u(t)$  of its effort to enlarging the worker force and the remaining fraction  $1 - u(t)$  to producing reproductives. Accordingly, we assume that the two populations are governed by the equations

$$\begin{aligned}\dot{w} &= buw - \mu w \\ \dot{q} &= c(1 - u)w\end{aligned}$$

These equations assume that only workers gather resources. The positive constants  $b$  and  $c$  depend on the environment and represent the availability of resources and the efficiency with which these resources are converted into new workers and new reproductives. The per capita mortality rate of workers is  $\mu$ , and for simplicity the small mortality rate of the reproductives is neglected. For the colony to be productive during the season it is assumed that  $b > \mu$ . The problem of the colony is to maximize

$$J = q(T)$$

subject to the constraint  $0 \leq u(t) \leq 1$ , and starting from the initial conditions

$$w(0) = 1 \quad q(0) = 0$$

(The founding queen is counted as a worker since she, unlike subsequent reproductives, forages to feed the first brood.)

### Solution

We will apply the Pontryagin's maximum principle to this problem. Construct the Hamiltonian:

$$H(\psi_w, \psi_q, w, q, u) = \psi_w(bu - \mu)w + \psi_q c(1 - u)w,$$

where  $\psi_w$  and  $\psi_q$  are adjoint variables, satisfying the adjoint system

$$-\dot{\psi}_w = \frac{\partial H}{\partial w} = (bu - \mu)\psi_w + c(1 - u)\psi_q \tag{4}$$

$$-\dot{\psi}_q = \frac{\partial H}{\partial q} = 0 \tag{5}$$

and terminal conditions

$$\psi_w(T) = 0, \quad \psi_q(T) = 1$$

It is clear, from the second adjoint equation, that  $\psi_q(t) = 1$  for all  $t$ . Note also the adjoint equation for  $\psi_w$  cannot be solved directly, since it depends on the unknown function  $u = u(t)$ . The other necessary conditions must be used in conjunction with the adjoint equations to determine the adjoint variables.

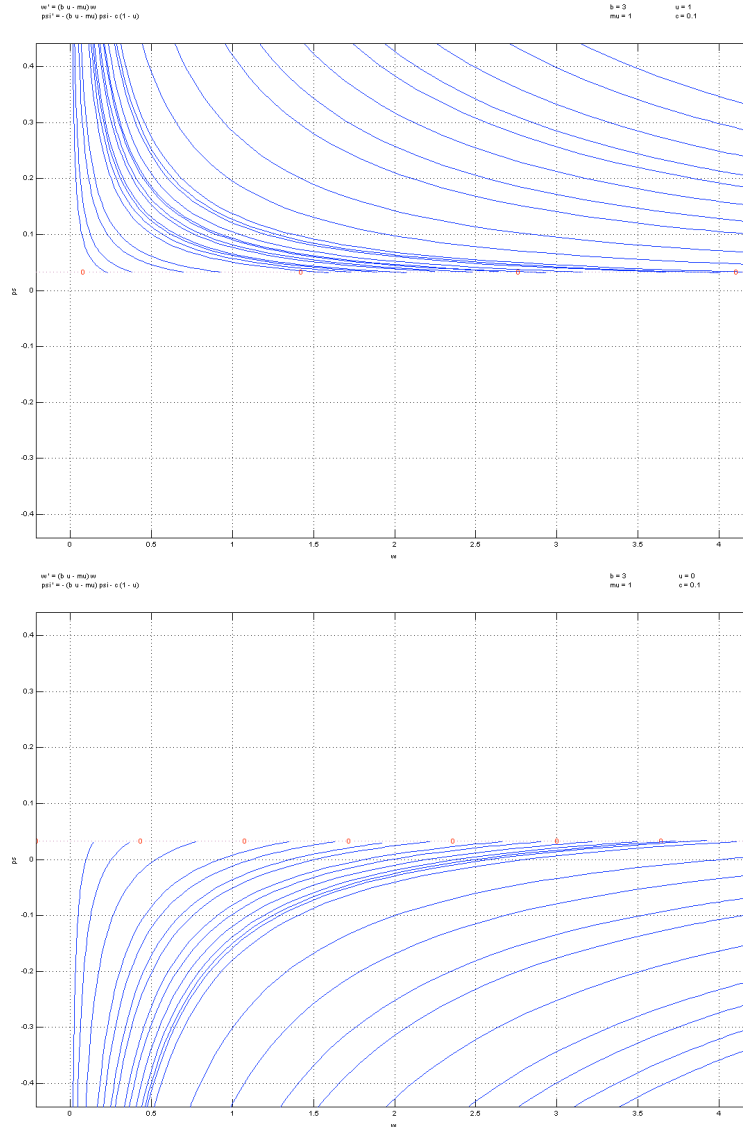
Since this Hamiltonian is linear as a function of  $u$  (for a fixed time  $t$ ),

$$H = H(u) = [(\psi_w b - c)w]u + (c - \psi_w \mu)w$$

and since  $w > 0$ , it follows that it is maximized with respect to  $0 \leq u \leq 1$  by either  $u^* = 0$  or  $u^* = 1$ , depending on whether  $\psi_w b - c$  is negative or positive, respectively. Using `pplane7.m`, we plot the solution curves in the two regions where the **'switch' function**

$$S(w, \psi_w) = \psi_w b - c$$

is positive (then  $u = 1$ ) and negative (then  $u = 0$ ) respectively. (See figures.)



It is now possible to solve the adjoint equations and determine the optimal  $u^*(t)$  by moving backward in time from the terminal point  $T$ . In view of the known conditions on  $\psi_w$  at  $t = T$ , we find  $\psi_w(T)b - c = -c < 0$ , and hence the condition for maximization of the Hamiltonian yields  $u^*(T) = 0$ . Therefore near the terminal time  $T$  the first adjoint equation becomes

$$-\dot{\psi}_w(t) = -\mu\psi_w(t) + c, \quad \psi_w(T) = 0$$

which has the solution

$$\psi_w(t) = \frac{c}{\mu} \left(1 - e^{\mu(t-T)}\right), \quad t \in [t_s, T]$$

Viewed *backward* in time it follows that  $\psi_w(t)$  increases from its terminal value of 0. When it reaches a point  $t_s < T$  where  $\psi_w(t) = \frac{c}{b}$ , the value of  $u^*$  switches from 0 to 1.

[The point  $t_s$  is found to be  $t_s = T + \frac{1}{\mu} \ln \left(1 - \frac{c}{b}\right)$ .]

Past that point  $t = t_s$ , the first adjoint equation becomes

$$-\dot{\psi}_w(t) = (b - \mu)\psi_w, \quad \text{for } t < t_s.$$

which, in view of the assumption that  $b > \mu$ , implies that, moving backward in time,  $\psi_w(t)$  continues to increase. Thus, there is no additional switch in  $u^*$ .

**Exercise:** Show that the optimal number  $q^*(T) = \frac{c}{b} e^{(b-\mu)t_s} = \frac{c}{b} e^{(b-\mu)\left[T + \frac{1}{\mu} \ln \left(1 - \frac{c}{b}\right)\right]}$ . (exercise!).

**Conclusion:** In terms of the original colony problem, it is seen that the optimal solution is for the colony to produce only workers for the first part of the season ( $u^*(t) = 1, t \in [0, t_s]$ ), and then beyond a critical time to produce only reproductives ( $u^*(t) = 0, t \in [t_s, T]$ ). Social insects colonies do in fact closely follow this policy, and experimental evidence indicates that they adopt a switch time that is nearly optimal for their natural environment.

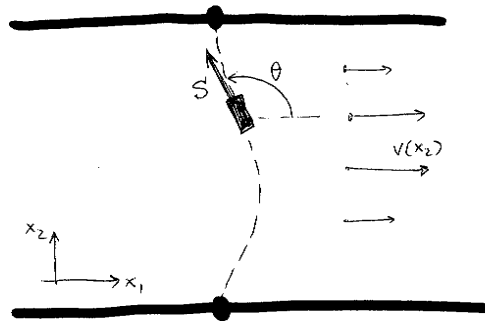
## Example 3: Ferry Problem

Consider the problem of navigating a ferry across a river in the presence of a strong current, so as to get to a specified point on the other side in minimum time. We assume that the magnitude of the boat's speed with respect to the water is a constant  $S$ . The downstream current at any point depends only on this distance from the bank. The equations of the boat's motion are

$$\dot{x}_1 = S \cos \theta + v(x_2) \tag{6}$$

$$\dot{x}_2 = S \sin \theta, \tag{7}$$

where  $x_1$  is the downstream position along the river,  $x_2$  is the distance from the origin bank,  $v(x_2)$  is the downstream current, and  $\theta$  is the heading angle of the boat. The heading angle is the control, which may vary along the path.



## Solution

This is a time optimal problem with terminal constraints, since both initial and final values of  $x_1$  and  $x_2$  are specified. The objective function is (negative) total time, so we choose  $f^0 = -1$ .

The adjoint equations are found to be

$$-\dot{\psi}_1 = 0 \quad (8)$$

$$-\dot{\psi}_2 = v'(x_2)\psi_1 \quad (9)$$

There are no terminal conditions on the adjoint variables. The Hamiltonian is

$$H = \psi_1 S \cos \theta + \psi_1 v(x_2) + \psi_2 S \sin \theta - 1 \quad (10)$$

The condition for maximization yields

$$0 = \frac{dH}{d\theta} = -\psi_1 S \sin \theta + \psi_2 S \cos \theta$$

and hence,

$$\tan \theta = \frac{\psi_2}{\psi_1}$$

We then rewrite (5) (along an optimal path)

$$H = \psi_1 \left( \frac{S}{\cos \theta} + v(x_2) \right) - 1. \quad (11)$$

Next, differentiating (5) with respect to  $t$ , we observe

$$\begin{aligned} \frac{d}{dt}H &= \dot{\psi}_1 S \cos \theta - \psi_1 S \sin \theta \dot{\theta} + \dot{\psi}_1 v(x_2) + \psi_1 v'(x_2) \dot{x}_2 + \dot{\psi}_2 S \sin \theta + \psi_2 S \cos \theta \dot{\theta} \\ &= \dot{\psi}_1 (S \cos \theta + v(x_2)) + (-\psi_1 S \sin \theta + \psi_2 S \cos \theta) \dot{\theta} + \psi_1 v'(x_2) \dot{x}_2 + \dot{\psi}_2 S \sin \theta \\ &= 0 \end{aligned}$$

(where we used (2), (3), and (4)). Thus, along an optimal path,  $H$  is constant in time.

$$H \equiv \text{const.} \quad (12)$$

Using (6) and (7) and the fact that  $\psi_1$  is constant ( $\dot{\psi}_1 = 0$ ), we obtain:

$$\frac{S}{\cos \theta} + v(x_2) = A$$

for some constant  $A$ . Once  $A$  is specified this determines  $\theta$  as a function of  $x_2$ .

$$\cos \theta = \frac{S}{A - v(x_2)} \quad (13)$$

Back in the state space, (1) and (2) become:

$$\dot{x}_1 = \frac{S^2}{A - v(x_2)} + v(x_2) \quad (14)$$

$$\dot{x}_2 = S \sqrt{1 - \frac{S^2}{(A - v(x_2))^2}}, \quad (15)$$

The constant  $A$  is chosen to obtain the proper landing point.

A special case is when  $v(x_2) = v$  is independent of  $x_2$ , hence so is  $\theta$ . Then the optimal paths are then straight lines.

## Example 4. Harvesting for Profit Problem

Consider the problem of determining the optimal plan for harvesting a crop of fish in a large lake. Let  $x = x(t)$  denote the number of fish in the lake, and let  $u = u(t)$  denote the intensity of harvesting activity. Assume that in the absence of harvesting  $x = x(t)$  grows according to the logistic law  $x' = rx(1 - x/K)$ , where  $r > 0$  is the intrinsic growth rate and  $K$  is the maximum sustainable fish population. The rate at which fish are caught when there is harvesting at intensity  $u$  is  $h = qux$  for some constant  $q > 0$  (catchability). (Thus the rate increases as the intensity increases or as there are more fish to be harvested). There is a fixed unit price  $p$  for the harvested crop and a fixed cost  $c$  per unit of harvesting intensity. The problem is to maximize the total profit over the period  $0 \leq t \leq T$ , subject to the constraint  $0 \leq u \leq E$ , where  $E$  is the maximum harvesting intensity allowed (by the "WeCARE" environmental agency). The system is

$$\begin{aligned}x'(t) &= rx(t)(1 - x(t)/K) - h(t) \\x(0) &= x_0\end{aligned}$$

and the objective is to maximize

$$J = \int_0^T [ph(t) - cu(t)] dt$$

[The objective function represents the integral of the profit rate, that is the total profit.]

1. Write the necessary conditions for an optimal solution, using the maximum principle.
2. Describe the optimal control  $u^* = u^*(t)$  and resulting population dynamics  $x^* = x^*(t)$ .
3. Find the maximum value of  $J$  for this optimal plan.

[You may assume (at some point) the following numerical values: the intrinsic growth rate  $r = 0.05$ , maximum sustainable population  $K = 250,000$ ,  $q = 10^{-5}$ , maximum level of intensity  $E = 5000$  (in boat-days). The selling price per fish  $p = 1$  (in dollars), unit cost  $c = 1$  (in dollars), initial population  $x_0 = 150,000$  (fish) and length of time (in years)  $T = 1$ . ]

### Solution

This is an optimal control problem with fixed terminal time, hence the Pontryagin maximum principle can be used. We first set up the problem:

The objective function is

$$J = \int_0^T [pqx(t)u(t) - cu(t)] dt.$$

The state variable is  $x = x(t)$  and satisfies the differential equation

$$x' = rx(1 - x/K) - qxu$$

with initial condition  $x(0) = x_0$ .

The Hamiltonian

$$\begin{aligned}H(\psi, x, u) &= pqxu - cu + \psi(rx(1 - x/K) - qxu) \\ &= [(qx(p - \psi) - c]u + \psi rx(1 - x/K)\end{aligned}$$

is linear in  $u$ , hence to maximize it over the interval  $u \in [0, E]$  we need to use boundary values, i.e.  $u = 0$  or  $u = E$ , depending on the sign of the coefficient of  $u$ . More precisely



$$\begin{aligned} u^*(t) &= 0 & \text{whenever} & & qx(t)(p - \psi(t)) - c < 0 \\ u^*(t) &= E & \text{whenever} & & qx(t)(p - \psi(t)) - c > 0 \end{aligned}$$

The adjoint variable  $\psi$  satisfies the differential equation

$$\psi' = -\frac{\partial H}{\partial x} = -(pqu + (r(1 - 2x/K) - qu)\psi).$$

and the terminal condition

$$\psi(T) = 0.$$

Based on the computed values of  $u$  (either 0 or  $E$ ) we get two systems ( $x$  and  $\psi$ ) to consider.

For  $u \equiv 0$  we get

$$\begin{cases} x' &= rx(1 - x/K) \\ \psi' &= -r(1 - 2x/K)\psi \end{cases}$$

For  $u \equiv E$  we get

$$\begin{cases} x' &= rx(1 - x/K) - qx E \\ \psi' &= -(pqE + (r(1 - 2x/K) - qE)\psi) \end{cases}$$

We can employ the use of Matlab code `pplane7.m` to help decide if any switching is needed. Here is a phase portrait for each  $(x, \psi)$  system, and the zero level set of the **'switch' function**

$$S(x, \psi) = qx(p - \psi) - c,$$

using the numerical values given in the problem:  $q = 10^{-5}$ ,  $p = 1$ ,  $c = 1$ .

We plot separately the solutions curves only in the appropriate regions of the phase plane, that is where  $S(x, \psi) < 0$  for  $u \equiv 0$  and  $S(x, \psi) > 0$  for  $u \equiv E$ . The direction of the solution curves is determined by the two systems above, as follows: increasing  $x$  (left to right) for first graph ( $u = 0$ ) and decreasing  $x$  (right to left) for second plot ( $u = E$ ).

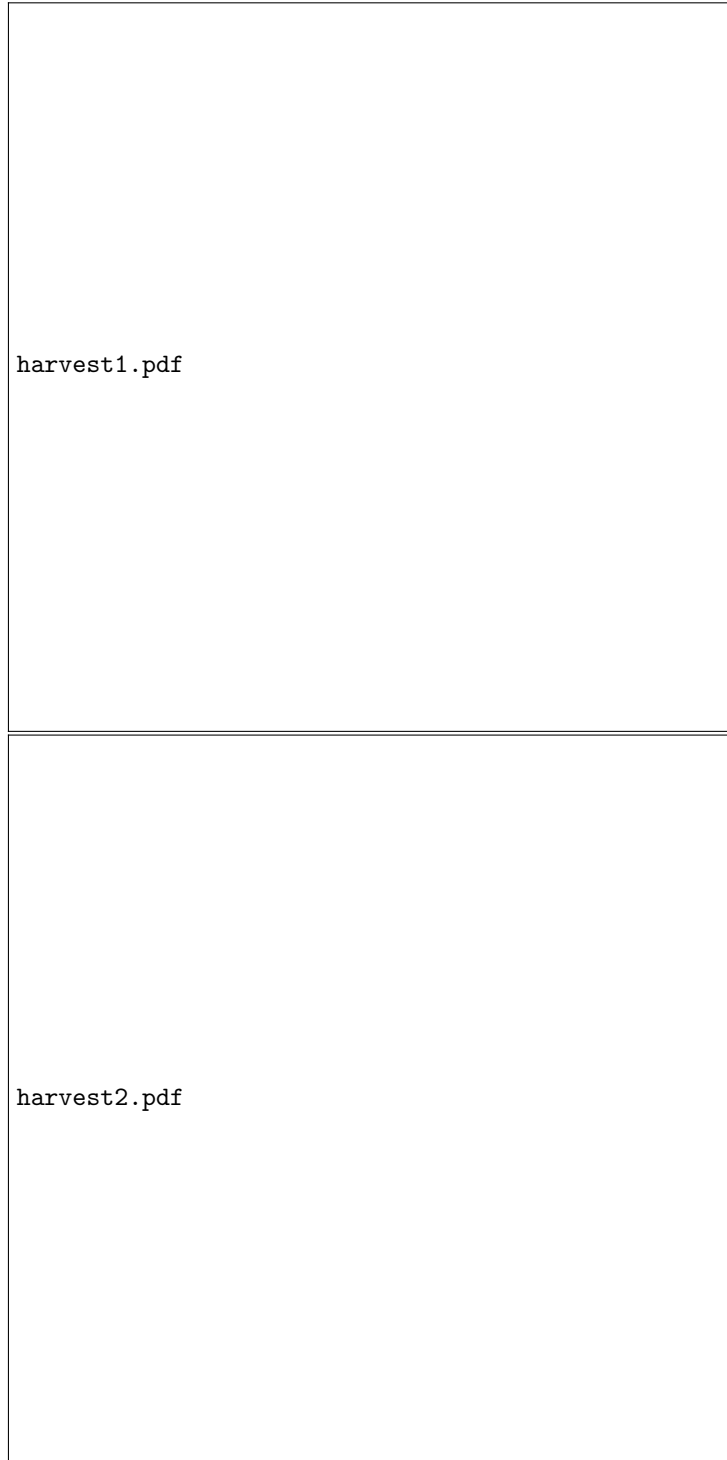


Figure 1. Solution curves  $(x(t), \psi(t))$ : in the region  $S(x, \psi) < 0$  (TOP,  $u = 0$ )  
and in the region  $S(x, \psi) > 0$  (BOTTOM,  $u = E$ )

It is deduced from these plots that in order to achieve  $\psi(T) = 0$  when we start with  $x(0) = 150000$ , we must have  $u^*(T) = E$  (that is finish the year with maximum level of effort). Hence,  $u^*(t) = E$  for

$t \in [t_s, T]$  for some  $t_s < T$ . By solving backward in time the system (with  $u = E$ ), we can compute the value of the switching time  $t_s$ .

It turns out that, for the given values of the parameters  $x_0, p, c$ , the switching time  $t_s = 0$  (see below), in other words the optimal control is  $u^*(t) = E$  for all times  $t \in [0, T]$ , meaning that **NO SWITCHING** is required in the effort of fishing (being always at its maximum).

As a matter of fact, we can solve both systems explicitly. Notice that the numerical values are such that  $qE = r = 0.05$ , hence the system ( $u = E$ ) simplify to

$$\begin{cases} x' &= -r/Kx^2 \\ \psi' &= -pr + 2r/Kx\psi. \end{cases}$$

Hence, when  $u \equiv E$ , we can explicitly solve for  $x = x^*$  first (separation of variables)

$$x^*(t) = \frac{1}{\frac{1}{x_0} + \frac{r}{K}t} \quad \text{for } t \in [0, T]$$

The equation for  $\psi$  can also be solved explicitly (it's linear equation in  $\psi$  hence method of integrating factor applies)

$$\psi^*(t) = p(\tilde{K} + rt) + \text{const}(\tilde{K} + rt)^2, \quad \text{where } \tilde{K} = \frac{K}{x_0} (= 5/3)$$

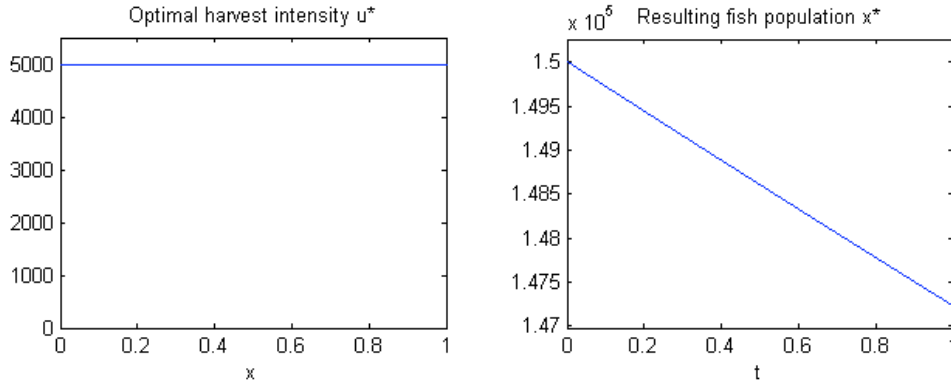
It is clear that one can choose the constant above such that  $\psi(T) = 0$ . More precisely, choose  $\text{const} = -p(\tilde{K} + rT)^{-1}$ .

One can verify that, in this case,  $S(x^*(t), \psi^*(t)) > 0$  for all  $t \in [0, T]$ , hence  $t_s = 0$ .

All of the above imply that the optimal control and optimal path are

$$u^*(t) = E, \quad x^*(t) = \frac{1}{\frac{1}{x_0} + \frac{r}{K}t}, \quad t \in [0, T]$$

The graphs are included below.



(c) The maximum value for the profit, given by the optimal strategy found above, is then

$$\begin{aligned} \max_u J(x, u) &= J(x^*, u^*) = \int_0^T (pqEx^*(t) - cE) dt \\ &= \int_0^1 \frac{0.05}{1/150,000 + 0.05/250000t} - 5000 dt \approx 2389 \quad (\text{ using Matlab!}) \end{aligned}$$

**Conclusion:** To achieve maximum profit, the company needs to fish at its maximum capacity for the entire duration of the year:  $u^*(t) = E = 5000$  boat-days,  $t \in [0, T]$ .