1.12 Rowing Dynamics

What determines the speed of a rowing boat? By pulling on their oars the crew push against the water far from the boat. This exerts a force, which *tries* to make the boat gain speed; we'll call it the tractive force and denote it by T. The water adjacent to the sides of the boat, however, exerts a force that *tries* to make the boat lose speed; we'll call this the drag force and denote it by D. If T exceeds D, then the boat really will gain speed, or *accelerate*; and the greater the difference between T and D, the greater this acceleration. On the other hand, if D exceeds T, then the boat will actually lose speed; and the greater the difference, the greater the speed loss.

Let's try to turn these ideas into mathematics. Let $u(t)$ be the speed of the boat at time t. Then du/dt is the rate of change of speed, which we have agreed to call the acceleration. What we have found is that

$$\frac{du}{dt} > 0 \quad \text{if } T > D, \qquad \frac{du}{dt} < 0 \quad \text{if } T < D, \qquad \left|\frac{du}{dt}\right| \text{ increases with } |T-D|.$$

The simplest hypothesis that explains all this is that du/dt is proportional to $T - D$. Let us therefore assume that

$$\frac{du}{dt} = \frac{1}{M}\{T - D\}, \tag{1.73}$$

where $1/M$ is a constant of proportionally. Notice that, for given tractive force and drag, large M is associated with small acceleration, and small M is associated with large acceleration. M is therefore a measure of the boat's reluctance to change speed, or *inertia*. We call M the *mass*.

Before proceeding, we pause to record a detail from physics. Heavy objects have greater reluctance to change speed than light ones (try pushing), hence greater mass. But heavier objects, by definition, have greater weight, denoted by W. Indeed the relationship between weight and mass is a simple one of proportionality, namely,

$$W = Mg. \tag{1.74}$$

The constant of proportionality, g, is known as the *acceleration due to gravity*. The justification for this terminology will emerge in Section 1.14.

Further to develop our model (1.73), we borrow again from the science of physics. In an infinitesimal time δt, the boat will move a distance $u\delta t$; and maintaining a tractive force T for this distance requires mechanical energy $Tu\delta t$. Where does this energy come from? From the muscular exertions of the crew. The rate at which energy is supplied by them is known as the *power*. But since humans are only about 25% efficient at converting the chemical energy of food into mechanical energy for useful work, the power actually supplied to the boat is only about a quarter of the rate at which oarsmen burn energy, the remaining three quarters being dissipated as heat. For this reason, some people prefer to call the rate of energy supply the *effective power*. Now, assume that there are eight in the boat's crew, and that they have entered a race. In training, they will have determined empirically the maximum effective power that an oarsman can sustain for the full length of the course; let's denote it by P. To make matters simple, let's assume that the oarsmen are identical, at least for the purpose of rowing a boat. Then, assuming that they would like to win the race, they will supply power $8P$ for the entire length of the course. Therefore, because power is rate of energy supply, the energy supplied in an infinitesimal time δt will be $8P\delta t$. This must equal $Tu\delta t$. Hence

$$\begin{array}{ccc} \text{Effective Power} & = & \text{Tractive Force} \times \text{Velocity} \\ 8P & = & T \times u. \end{array} \tag{1.75}$$

From fluid dynamics, we borrow the result that drag is proportional both to the square of the boat's velocity and to its wetted surface area, i.e., the area in contact with the water. Let's denote wetted surface area by S. Then

$$D = bSu^2, \tag{1.76}$$

where b is a constant of proportionality. From (1.73), (1.75), and (1.76):

$$\begin{array}{ccccc} \text{Mass} \times \text{Acceleration} & = & \text{Tractive Force} & - & \text{Drag Force} \\ M \times \dfrac{du}{dt} & = & \dfrac{8P}{u} & - & bSu^2. \end{array} \tag{1.77}$$

This is a first order ordinary differential equation for $u(t)$, which we propose as a model for the dynamics of rowing. The solution of this differential equation, subject to the initial condition $u(0) = 0$, is defined implicitly (Exercise 1.19) by

$$\ln\left\{\frac{u^2 + uV + V^2}{(u - V)^2}\right\} + \frac{\pi}{\sqrt{3}} - 2\sqrt{3}\tan^{-1}\left\{\frac{V + 2u}{V\sqrt{3}}\right\} = \frac{6bSVt}{M}, \tag{1.78}$$

where we have defined

$$V = 2\left(\frac{P}{bS}\right)^{1/3}. \tag{1.79}$$

The function $u(t)$ defined implicitly by (1.78) is plotted in Fig. 1.8. Notice that the initial acceleration is theoretically infinite according to (1.77), when in practice it must be finite; but this should not concern us any more than the fact that, in theory, the boat will actually reach speed V only as $t \to \infty$. A model is intended only to capture the essential features of reality, not to be an identical replica.

Yet how good is this model—does it fit the facts? Fig. 1.8 suggests that oarsmen supplying power at a constant rate would achieve maximum acceleration at the start of a race and subsequently reduce acceleration, until their speed levelled off at the constant value defined by (1.79). My intuition suggests that this is more or less what happens in a race; except perhaps that the oarsmen slow down slightly towards the end of it. But in

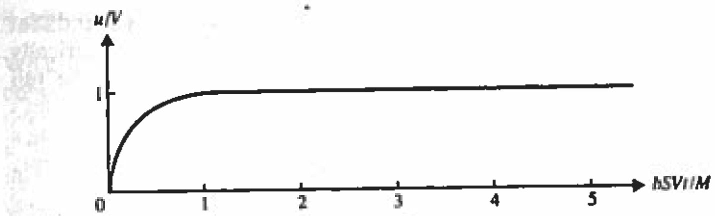

Fig. 1.8 Speed history of a rowing boat. Both dependent and independent variables are dimensionless; see Section 1.14 and Exercise 1.20.

science it's generally a bad idea to rely exclusively upon one's intuition. Therefore, in Chapter 4, we will test this model more objectively.

1.13 Traffic Dynamics

Suppose that there are N vehicles in a lane of traffic. We will label them $j = 1, \ldots, N$, with $j = 1$ corresponding to the leading vehicle. Let us assume that their masses and lengths are equal. This might be at least approximately true on a bridge or in a tunnel from which trucks have been barred. Let the front bumper of the jth vehicle have displacement $z_j(t)$ at time t, measured from the beginning of the road. Then the jth vehicle has velocity dz_j/dt and acceleration d^2z_j/dt^2; whereas its displacement and velocity *relative to the vehicle in front of it* are, respectively, $z_j(t) - z_{j-1}(t)$ and $dz_j/dt - dz_{j-1}/dt$.

Suppose that the road is quite congested, which bridges and tunnels usually are. Thus the average value of $|z_j(t) - z_{j-1}(t)|$ is not too large. The smaller it is, the more congested the road. Now, on a busy road, vehicles are prevented from crashing into one another, at least in theory, because drivers brake when they see that the gap in front of them is closing. The faster that gap closes, i.e., the greater the value of $dz_j/dt - dz_{j-1}/dt$, the harder they brake. Moreover, the greater the congestion, i.e., the smaller the value of $|z_j(t) - z_{j-1}(t)|$, the harder they brake. The simplest hypothesis incorporating these two observations is that

$$\text{Braking Force} = A\frac{z_j'(t) - z_{j-1}'(t)}{|z_j(t) - z_{j-1}(t)|}, \tag{1.80}$$

where $A(>0)$ is a constant and a prime denotes differentiation. Now, borrowing from physics, the braking force is $-M \times$ ACCELERATION, where M is the mass of a vehicle. You can see this by identifying the braking force with D in (1.73) above and setting $T = 0$. But drivers take a certain time, say τ, to respond to changing traffic conditions. Hence

$$-M\frac{d^2z_j(t+\tau)}{dt^2} = A\frac{z_j'(t) - z_{j-1}'(t)}{|z_j(t) - z_{j-1}(t)|}; \tag{1.81}$$

or, because $z_{j-1}(t) > z_j(t)$,

$$\frac{d^2z_j(t+\tau)}{dt^2} = \lambda\frac{d}{dt}\{\ln|z_j(t) - z_{j-1}(t)|\}, \tag{1.82}$$

where $\lambda = A/M$. Integrating, we obtain

$$\frac{dz_j(t+\tau)}{dt} = \lambda \ln|z_j(t) - z_{j-1}(t)| + \alpha_j, \tag{1.83}$$

where α_j is independent of t. Because this equation is valid for $2 \le j \le N$, we have a system of $N - 1$ nonlinear ordinary differential equations. This system is even more complex than (1.20) and (1.22) because of the delay, τ, in driver response to the stimulus of changing traffic conditions.

If we weren't able to solve (1.20) or (1.22) analytically, then what hope do we have of solving (1.83)? To be sure, we can always find solutions by numerical approximation. As we shall discover repeatedly throughout this text, however, there is a satisfaction in the simplicity but generality of analytic solutions which makes them always desirable and always preferable, when available, to numerical approximations. But the question still remains: how do we hope to obtain analytic solutions from equations as complicated as (1.83)?

The answer lies in the observation that, in the real world, nothing can grow or decay forever: ultimately, growth or decay must cease. What happens then? An equilibrium or steady state is reached. In the present model, this steady state is a special solution of (1.83) which, as we shall see in Section 2.3, is readily found. But already we're talking about equilibrium, and this is the topic of Chapter 2.

1.14 Dimensionality, Scaling, and Units

The use of dynamic models is greatly facilitated by an understanding of three closely related concepts, namely, those of dimensionality, units, and scaling. We conclude our first chapter by discussing them.

Suppose that a friend of mine drove 20,000 miles last year and earned 20,000 dollars. Would I be right to say that he drove as much as he earned? No, of course not, because the equation

$$\$20,000 = 20,000 \text{ miles} \tag{1.84}$$

does not make sense. Distance is measured in miles, income is measured in dollars, and no amount of dollars can ever equal a mile. We formalize this

2.3 How Fast Do Cars Drive Through a Tunnel?

The more congested a tunnel is, the slower the cars move; the less congested, the faster.[3] Thus the speed of cars through a tunnel is a decreasing function of congestion. How does this function decrease? In this section, we'll attempt to answer that question by using an equilibrium model.

First we need a suitable measure of traffic congestion. One such measure is the traffic density, x, i.e., the number of cars per unit length of road. In general, the density x varies with time, t, and with distance, z, along the road; hence $x = x(z, t)$. We measure the density by taking aerial photographs at fixed times. Let one such time be $t = t_0$. Then, to measure the traffic density at a particular station $z = z_0$ on the road (at time t_0), we use the photograph to count the number of cars in a suitable interval of the form $z_0 - \epsilon/2 < z \le z + \epsilon/2$ and divide the answer by ϵ; this is our estimate of $x(z_0, t_0)$. The interval length ϵ must be chosen carefully. It cannot, of course, be less than a few car-lengths; neither can it be too long, for important spatial variations of density would be smoothed out. It is probably best to choose ϵ somewhere between 30 and 50 meters, and certainly much less than a mile (even though traffic density is traditionally quoted as number of cars per mile). Because z_0 is arbitrary, and because we may repeat this procedure for any value of z_0, a single photograph will enable us to plot $x(z, t_0)$ as a function of z. A series of such photographs taken at different times will enable us to estimate $x(z, t)$.

If speed v is a decreasing function of traffic congestion, and if density x is our chosen measure of congestion, then $v = v(x)$ where

$$v'(x) \le 0. \tag{2.22}$$

At one extreme, if traffic is very light, then cars are free to move at the speed limit, which we will denote by v_{max}. This may be the official speed limit if police are watching, or whatever speed drivers deem safe, if not; all that matters is that the limit exists and may be assumed the same for all drivers. Hence there exists some critical traffic density x_c such that

$$v(x) = v_{max} \qquad \text{if } 0 \le x \le x_c. \tag{2.23}$$

The condition $v(0) = v_{max}$ should be interpreted as meaning that a car that enters an empty road is free to drive at the speed limit. At the other extreme, if traffic is so heavy that cars cannot move, then traffic density has reached its maximum, which we will denote by x_{max}. Thus

$$v(x_{max}) = 0. \tag{2.24}$$

Now assume that $x_c < x < x_{max}$, so that there is some degree of congestion; and suppose that the cars have reached an equilibrium, in which

[3] You may wish to review Section 1.13 before proceeding.

all are cruising at the constant speed v, at the same distance d from the car in front. To make matters simple, we'll also assume that all cars have the same length, L. Then the traffic density is

$$x = \frac{1}{d+L} \tag{2.25}$$

(why?). Note that, although equilibrium density, x, and equilibrium velocity, $v = v(x)$, are independent of both z and t, their values may vary from tunnel to tunnel. In this section, we'll assume that our equilibrium is stable. We'll discuss whether that assumption is reasonable or not in Section 2.8.

Let's assume that the dynamics of approach to this equilibrium are governed by the model introduced in Section 1.13. Then, from (1.83),

$$\frac{dz_j(t+\tau)}{dt} = \lambda \ln |z_j(t) - z_{j-1}(t)| + \alpha_j, \tag{2.26}$$

where $z_j(t)$ is the displacement of the jth car in a platoon of N cars, and where τ is the driver response time. Now, we have just assumed that all cars are moving at the same speed v, with front bumpers a distance $d+L$ apart. Therefore, for all j and at all times, $dz_j(t+\tau)/dt = v$, $|z_j(t) - z_{j-1}(t)| = d+L$ and α_j is independent of j. Hence (2.26) yields $v = \lambda \ln(d+L) + \alpha$. From (2.25), and on using (2.24) to determine α, we deduce

$$v(x) = \lambda \ln\left(\frac{x_{max}}{x}\right), \qquad x_c < x \le x_{max}. \tag{2.27}$$

The value of λ is determined from the assumption that $v(x)$ is a continuous function throughout the interval $0 \le x \le x_{max}$ and hence, in particular, at $x = x_c$. Thus

$$v(x) = \begin{cases} v_{max}, & 0 \le x \le x_c \\ v_{max}\dfrac{\ln(x_{max}/x)}{\ln(x_{max}/x_c)}, & x_c < x \le x_{max}. \end{cases} \tag{2.28}$$

You can easily see that (2.22) is satisfied.

How good is this model—does it fit the facts? A possible objection to this model is that traffic density is not allowed to vary *across* the road, as it surely would if a road had several lanes. In a tunnel, however, cars are invariably prohibited from switching lanes, so that each lane behaves as though it were a separate tunnel and density varies only *along* the road. But does it vary along the road as predicted by (2.28)? Perhaps you would like to think about that—we'll return to this point in Chapter 4. Meanwhile, try your hand at Exercises 2.4, 2.5, and 2.10.

2.4 Salmon Equilibrium and Limit Cycles

The salmon population described in Section 1.8 will be in static equilibrium if the population (in hundreds of millions) persists forever at the

2.8 How Quickly Must Drivers React to Preserve an Equilibrium?

Consider a platoon of cars, each of length L, driving in equilibrium through a tunnel.[9] All cars are moving at the same speed, v. The gap between cars is a constant, d. The cars will remain in equilibrium only if they all continue to travel at exactly the same speed and hence remain a distance d apart. Now, suppose that the leading driver slows down for a moment, for whatever reason, before accelerating back to speed v. Thus she subjects the equilibrium of the platoon to a small perturbation. Following this perturbation, will the drivers regain their equilibrium? In other words, is the equilibrium stable? Intuition suggests that the drivers will regain equilibrium if they react sufficiently quickly to the first driver's braking, but if they react too slowly, then two or more of the cars will collide. But how quickly is quickly enough? In an attempt to answer that question, we will develop the model for a platoon of N cars that we introduced in Section 1.13. For the sake of definiteness, we will assume in this section that there are just ten cars in the platoon, but with obvious modifications, the analysis would apply just as well to any other number of cars.

Let x be the traffic density in this equilibrium; i.e., let $x = 1/(d+L)$. Then, on using (2.28) and assuming the tunnel to be at least so congested that $x > x_c$, we find that the speed of the platoon in the equilibrium is

$$v = v_{\max}\frac{\ln(x_{\max}/x)}{\ln(x_{\max}/x_c)}, \tag{2.94}$$

where $v_{\max}$ is the speed a car would have if the tunnel were empty. For reasons that will emerge in Section 3.5, and again for the sake of definiteness, we shall assume that the traffic density in equilibrium is

$$\frac{x_{\max}}{e} = 0.368x_{\max}, \tag{2.95}$$

where $e = \exp(1)$. Then, on using (2.94), we find that the speed of the platoon is

$$v = \frac{v_{\max}}{\ln(x_{\max}/x_c)}. \tag{2.96}$$

Suppose that the cars have remained in equilibrium throughout the interval $t < 0$ and that the first car of the platoon crosses station $z = 0$ at time $t = 0$. If this equilibrium persists then, for $t \geq 0$, the displacement of the jth car in the platoon is given by

$$\begin{aligned}\zeta_j(t) &= vt - (j-1)(d+L)\\ &= vt - \frac{(j-1)}{x}\\ &= vt - \frac{e(j-1)}{x_{\max}}, \qquad j = 1, 2, \ldots, 10.\end{aligned} \tag{2.97}$$

We will refer to $\zeta_j(t)$ as the *equilibrium displacement* of the jth car. This does not, of course, mean that the displacement is constant (it increases linearly with time); rather, the speed $d\zeta_j/dt = v$ of the car is constant, and $\zeta_j(t)$ is simply the displacement associated with that equilibrium speed.

Suppose that the equilibrium of the platoon is now perturbed, because the leading driver ($j = 1$) suddenly brakes. Let $z_j(t)$ denote the subsequent displacement of the jth car, where $2 \leq j \leq 10$. Then (Exercise 2.17), on using (2.26) and (2.27) we get

$$\frac{dz_j(t+\tau)}{dt} = v\ln|z_j(t) - z_{j-1}(t)| + \alpha_j, \qquad 2 \leq j \leq 10. \tag{2.98}$$

Here τ is the driver response time (assumed the same for all drivers) and α_j a constant. We will assume that for all j the constant α_j takes the value implied by the analysis of Section 2.3, namely, (verify) $\alpha = v\ln(x_{\max})$. Then (Exercise 2.17), because $z_{j-1}(t) > z_j(t)$, the subsequent displacements of cars 2 to 10 satisfy

$$\frac{dz_j(t+\tau)}{dt} = v\ln\left\{x_{\max}(z_{j-1}(t) - z_j(t))\right\}, \qquad 2 \leq j \leq 10. \tag{2.99}$$

Now suppose that at $t = 0$ the first driver brakes for exactly one second, reducing her car's speed to $v(1-\Delta)$ before accelerating back to the equilibrium speed v. Then a reasonable description of the car's velocity for $t \geq 0$ would be (Fig. 2.9)

$$\frac{dz_1}{dt} = v(1 - \Delta t e^{1-t}), \tag{2.100}$$

whence (verify)

$$z_1(t) = v\left[t + \Delta\left\{(t+1)e^{-t} - 1\right\}e\right]. \tag{2.101}$$

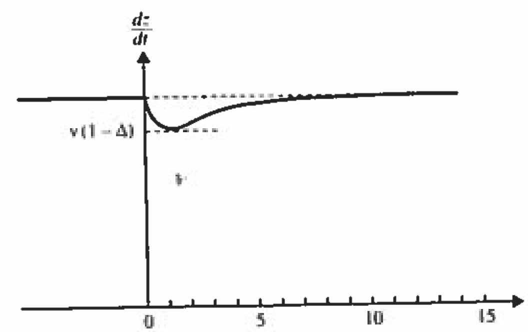

Fig. 2.9 Speed history of first car in a platoon.

[9] You may wish to review Section 2.3 before proceeding.

From (2.97), with $j = 1$, this displacement differs from the equilibrium displacement $\zeta_1(t)$ by an amount $z_j(t) - \zeta_j(t) = v\Delta\{(t+1)e^{-t} - 1\}\, e$, when $t \geq 0$. Let us define the *perturbation displacement* of the leading car at time t, $\phi_1(t)$, by $\phi_1(t) \equiv z_1(t) - \zeta_1(t)$. Then

$$\phi_1(t) = \begin{cases} v\Delta\{(t+1)e^{-t} - 1\}\, e, & \text{if } t \geq 0; \\ 0 & \text{if } t < 0. \end{cases} \tag{2.102}$$

Note that $\phi_1(t) < 0$ when $t > 0$ because as a result of braking, the leading driver is nearer to $z = 0$ at time t than she would have been otherwise.

Cars 2 to 10 will react to this perturbation by also braking. Let the perturbation displacement of the jth car be denoted by $\phi_j(t) \equiv z_j(t) - \zeta_j(t)$, $2 \leq j \leq 10$. Then, on using (2.97) we have

$$\phi_j(t) = \begin{cases} z_j(t) - vt + \dfrac{e(j-1)}{x_{\max}}, & \text{if } t \geq 0; \\ 0 & \text{if } t < 0. \end{cases} \tag{2.103}$$

On substituting $z_j(t) = \zeta_j(t) + \phi_j(t)$ into (2.99), we obtain (Exercise 2.17)

$$v + \frac{d\phi_j(t+\tau)}{dt} = v \ln\left[x_{\max}\left\{\frac{e}{x_{\max}} + \phi_{j-1}(t) - \phi_j(t)\right\}\right]. \tag{2.104}$$

Replacing t by $t + \tau$, we thus obtain (verify)

$$\frac{d\phi_j}{dt} = v \ln\left[1 + \frac{x_{\max}}{e}\left\{\phi_{j-1}(t-\tau) - \phi_j(t-\tau)\right\}\right]. \tag{2.105}$$

In this equation, ϕ_j is identically zero whenever its argument is negative. Thus $\phi_j(t - \tau) = 0$ whenever $\tau > t$.

The equilibrium described by (2.97) is stable if, following the perturbation depicted in Fig. 2.9, the cars all regain speed v without colliding. Now, two cars will collide if the relative displacement of their front bumpers falls below a car length, L; i.e., if $z_{j-1}(t) - z_j(t) < L$, at any time $t (\geq 0)$ and for any j such that $2 \leq j \leq 10$. From the definition of ϕ_j, the condition $z_{j-1}(t) - z_j(t) < L$ is equivalent to $\phi_{j-1}(t) - \phi_j(t) + \zeta_{j-1}(t) - \zeta_j(t) < L$ (verify). But from (2.97), $\zeta_{j-1}(t) - \zeta_j(t) = d + L$. Hence two cars will collide if, for any j satisfying $2 \leq j \leq 10$ and for any $t \geq 0$,

$$\phi_j(t) - \phi_{j-1}(t) > d. \tag{2.106}$$

For the sake of definiteness, let's choose a particular value for d. Now, as we will discuss in Section 3.3, an appropriate traffic density for New York's tunnels is about 65 cars per mile, and an appropriate equilibrium traffic speed is about 20 miles per hour. Working in feet and seconds, therefore, it seems reasonable to take $v = 20 \times 5280 \div 3600 = 88/3$ and $x = x_{\max}/e = 65/5280 = 13/1056$. Then $d + L = 1056/13$. Thus, if we choose $d = 65$, then the length of a car is a little more than sixteen feet, which does not seem unreasonable. Let's adopt these values (if you don't like them, then you can easily work through the analysis again using different ones). Then the collision criterion (2.106) becomes

$$-\phi_{j-1}(t) + \phi_j(t) > 65, \tag{2.107}$$

and on using our chosen values in (2.102)–(2.105), we find (verify)

$$\begin{aligned} \phi_j(t) &= 0, && \text{if } t < 0, 1 \leq j \leq 10; \\ \phi_1(t) &= \frac{88}{3}\Delta\{(t+1)e^{-t} - 1\}\, e, && \text{if } t \geq 0; \\ \frac{d\phi_j}{dt} &= \frac{88}{3}\ln\left[1 + \frac{13}{1056}\left\{\phi_{j-1}(t-\tau) - \phi_j(t-\tau)\right\}\right], && \text{if } t \geq 0, 2 \leq j \leq 10. \end{aligned} \tag{2.108}$$

These differential-delay equations are not especially easy to solve. We will therefore approximate them by recurrence relations. Now, the quantity $d\phi_j/dt$ is approximately equal to

$$\frac{\phi_j(t+\epsilon) - \phi_j(t)}{\epsilon} \tag{2.109}$$

if ϵ is small.[10] It will be convenient to suppose that $\epsilon = 1/M$, where M is a large enough integer, and measure time in units of one Mth of a second. Thus n units of new time correspond to $t = n/M$ seconds. Let $\Phi_j(n)$ denote the perturbation displacement of driver j after n units of new time; i.e., $\Phi_j(n) \equiv \phi_j(n/M)$. Then (2.109) becomes $M\{\Phi_j(n+1) - \Phi_j(n)\}$, and for large enough M we can approximate the last equation of (2.108) by (verify)

$$M\left\{\Phi_j(n+1) - \Phi_j(n)\right\} = \frac{88}{3}\ln\left[1 + \frac{13}{1056}\left\{\Phi_{j-1}(n-\sigma) - \Phi_j(n-\sigma)\right\}\right], \tag{2.110}$$

where σ is the reaction time measured in new units; i.e., in old units, the reaction time is $\tau = \sigma/M$ seconds.

How large should M be? Clearly, the larger the value of M we choose, the smaller the error in approximating $d\phi_j/dt$ by (2.109). On the other hand, you must perform $60M$ recursions of (2.110) for each minute of time during which you would like to know what happens to the platoon; and these recursions will be time-consuming if M is very large. I experimented with different values of M and decided that $M = 8$ gave an acceptable trade-off between accuracy and computing cost. Hence we use $M = 8$ in the remainder of this section. (You may wish to experiment with larger values, but you will find that they do not qualitatively alter our conclusions from the analysis.) With this value, and on replacing n by $n - 1$ in (2.110),

[10]See Supplementary Note 2.4.

(2.108) is approximated (verify) by

$$\Phi_j(n) = 0 \qquad \text{if } n < 0,\, 1 \le j \le 10;$$

$$\Phi_1(n) = \frac{11}{3}\Delta\left\{(n+8)e^{-n/8} - 8\right\} e \qquad \text{if } n \ge 0;$$

$$\Phi_j(n) = \Phi_j(n-1) + \frac{11}{3}\ln\left[1 + \frac{13}{1056}\left\{\Phi_{j-1}(n-\sigma-1) - \Phi_j(n-\sigma-1)\right\}\right] \qquad \text{if } n \ge 0,\, 2 \le j \le 10. \tag{2.111}$$

These recurrence relations are easily solved by calculator or computer. The results are shown in Fig. 2.10 for $\Delta = 0.15$ and $\tau = 1.5$ seconds ($\sigma = 12$). In this particular case, the leading driver reduces speed from 20 m.p.h. to 17 m.p.h. and then accelerates back to 20 m.p.h. again, according to Fig. 2.9. The corresponding perturbation (2.102) is the uppermost graph in Fig. 2.10. The remaining graphs are (approximations to) the perturbation displacements $\{\phi_j(t): t \ge 0, 2 \le j \le 10\}$ of the following cars, plotted 0.35 inches apart, which corresponds to a real distance of 65 feet. Notice that equilibrium is restored within three-quarters of a minute, with all cars once again moving at 20 m.p.h., but (taking the limit of (2.102) as $t \to \infty$) in such a way that all cars are a fixed distance $v\Delta e \approx 12$ feet behind where they would have been if the first driver had not braked. Notice also that each of the following cars takes longer to adjust to the new equilibrium than the car in front of it. Its phase of adjustment is represented by an oscillation of decreasing amplitude about the new equilibrium, but the amplitude of this oscillation increases with j as the initial perturbation

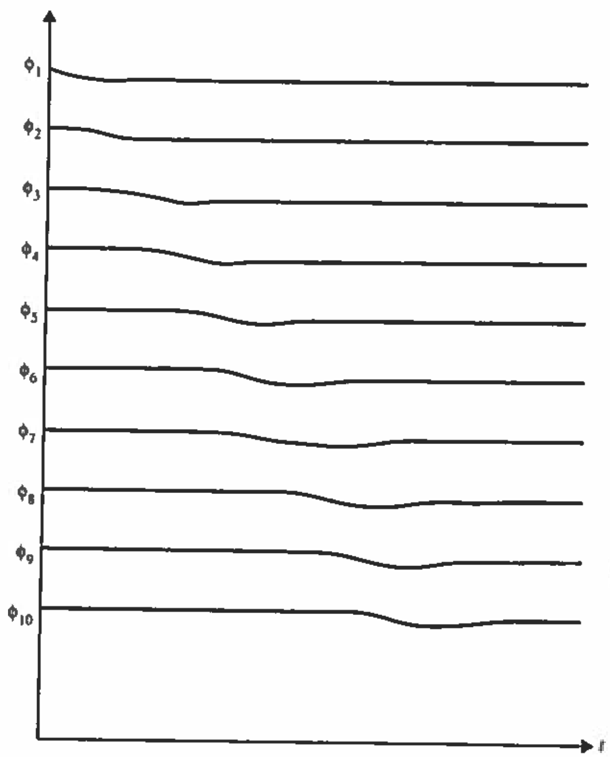

Fig. 2.10 Perturbation displacements of a platoon of ten cars in which drivers take 1.5 seconds to respond when the first car slows temporarily to 17 m.p.h., according to Fig. 2.9, from the equilibrium speed of 20 m.p.h. The horizontal scale is 1 inch to 14.3 seconds, the vertical scale 1 inch to 186 feet.

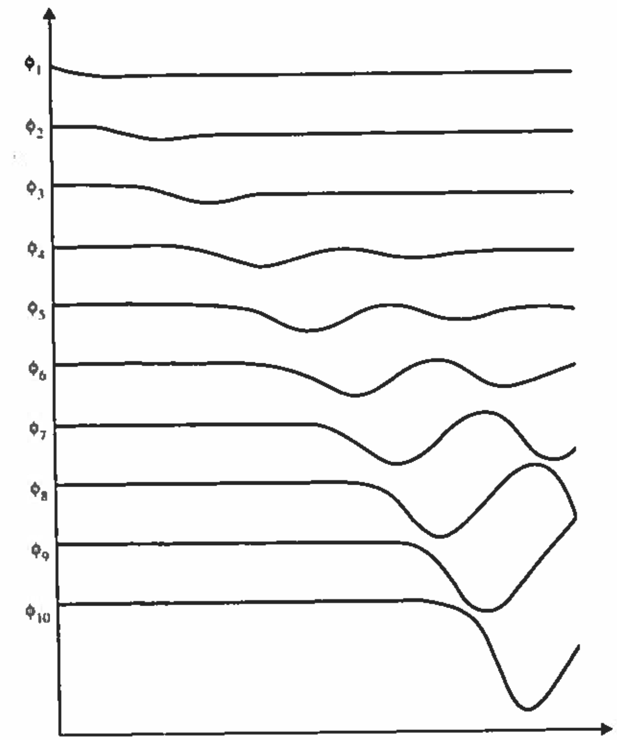

Fig. 2.11 Perturbation displacements of a platoon of ten cars in which drivers take 2.25 seconds to respond when the first car slows temporarily to 17 m.p.h., according to Fig. 2.9, from the equilibrium speed of 20 m.p.h. The horizontal scale is 1 inch to 14.3 seconds, the vertical scale 1 inch to 186 feet.

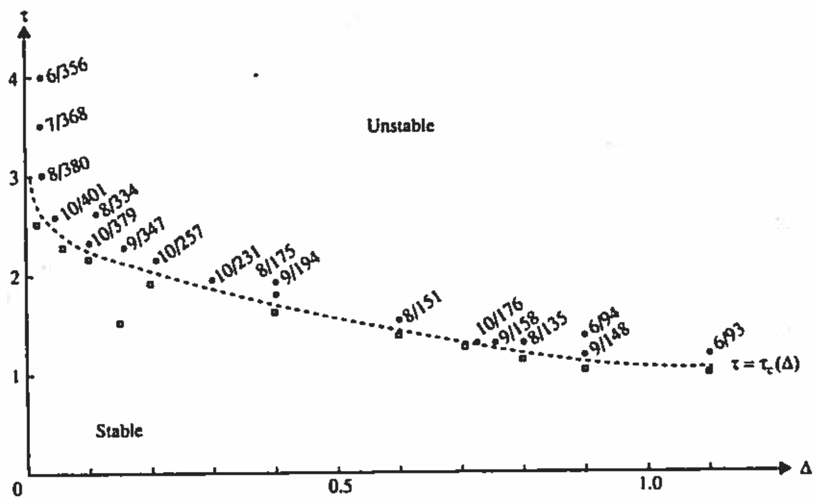

Fig. 2.12 Tentative stability boundary $\tau = \tau_c(\Delta)$ in the Δ-τ plane.

propagates backward through the platoon of cars. (From the point of view of pure mathematics, of course, the equilibrium is never restored. Rather, the cars will move toward the equilibrium forevermore, but will ultimately be an imperceptible distance away from it.)

For the same perturbation but the slower response time of $\tau = 2.25$ seconds ($\sigma = 18$), the graphs of $\phi_j(t)$ are plotted in Fig. 2.11. Notice that (2.107) is violated after 347 units of time (43.4 seconds), when the ninth car collides with the car in front of it. Thus, in the $\Delta - \tau$ plane, (0.15, 1.5) is a stable point for a platoon of ten cars; whereas (0.15, 2.25) is unstable.

The same analysis can be repeated for any other value of τ that is a multiple of an eighth of a second and for any other value of Δ (between 0 and 1). Thus we discover, for example, that (0.4, 1.625) is a stable point, but that (0.4, 1.75) is unstable, because the ninth car collides with the car in front of it after 24.25 seconds. A selection of such points is plotted in Fig. 2.12. A square denotes stability and a dot instability, with the label j/n recording that the jth car collided with the one in front after n units of time ($n/8$ seconds). By continuing in this fashion, we see that there is a boundary curve $\tau = \tau_c(\Delta)$ in the Δ-τ plane, below which the platoon is stable and above which it is unstable. On the unstable side of this boundary, for given Δ, a higher value of τ (slower response time) means a sooner collision between lower numbered cars; and, for given τ, a higher value of Δ (larger perturbation) means the very same thing. The boundary drawn (dashed) in Fig. 2.12 is a tentative one. To locate it precisely, you would have to solve (2.108) numerically and use a significant amount of computer time. You would find, however, a similar picture to the one in Fig. 2.12.

The analysis in this section is easily reproduced for a platoon that contains an arbitrary number of cars. Simply replace 10, wherever it appears, by N. Increasing N (higher number of cars in the platoon) will shift the $\tau_c(\Delta)$ curve downward and to the left; whereas decreasing N will shift it upward and to the right (as indicated by the labels j/n in Fig. 2.12).

But do you consider that Figs. 2.10–2.12 yield a reasonable picture? Perhaps you would like to mull this over, and then try Exercise 2.18.

Exercises

2.1 Verify (2.20).

2.2 Verify that ζ, as well as η, satisfies the differential equation (2.16).

2.3 By finding the roots of (2.19), classify the static equilibria encountered in
(i) Section 1.4,
(ii) Section 1.5,
(iii) Exercise 1.5.

***2.4** What would be the form of $v(x)$ in Section 2.3 if, in Section 1.13, we had assumed that braking drivers do not respond to the size of a gap but only to the rate at which it's closing?

***2.5** Suppose that we stand at a fixed point z_0 on a road and count the number of vehicles that pass that point in ϵ units of time, say during the interval $t_0 - \epsilon/2 < t \le t_0 + \epsilon/2$, then divide our answer by ϵ. We call the quantity thus estimated the *flux* of traffic (at station z_0, at time t_0) and denote it by $F(z_0, t_0)$. By repeating this experiment throughout the day, we obtain a function of time, $F(z_0, t)$, that records variations in the flux of traffic at that particular point on the road. The interval length ϵ must be neither so long that important variations in flux are not recorded (recall Fig. 1.6) nor so short that $F(z_0, t)$ would cease to be a smooth function of time (in the extreme case, it would take the values 0 or 1 in an apparently random sequence!). For a busy road, an hour is too long, a second too short, a minute about right; for a quiet road, we would not obtain a smooth function.

Once traffic flux F and traffic density x have been defined, we may define $v \equiv F/x$ to be the velocity of traffic at station z at time t. Thus, in general:

$$\begin{array}{ccccc} \text{Number of cars passing per unit of time} & = & \text{Number of cars per unit length of road} & \times & \text{Length of road covered per unit of time} \\ F(z,t) & = & x(z,t) & \times & v(z,t). \end{array}$$

But it is sometimes reasonable to assume that F is some function $f(x)$ of the traffic density, so that $F(z,t) = f(x(z,t))$. Then, in equilibrium, we have

$$\begin{array}{ccccc} \text{Number of cars passing per unit of time} & = & \text{Number of cars per unit length of road} & \times & \text{Length of road covered per unit of time} \\ f(x) & = & x & \times & v(x), \end{array}$$

where v is the velocity of traffic. The graph of f as a function of x is known as the fundamental diagram of traffic flow. Its shape will be similar to that in Fig. E2.1. In particular, $f(0) = 0 = f(x_{max})$ and f has a global maximum at $x = x^*$. We say traffic is *light* if $x < x^*$ and *heavy* if $x > x^*$, and $f(x^*)$ is the *capacity* of the road.

Find the capacity of the road when $v(x)$ is given
(i) by the model of Exercise 2.4;
(ii) by the model of Section 2.3.

2.6 Verify (2.38).

2.7 If r lies to the right of the cusp in Fig. 2.3 then the arguments of the absolute value function in (2.40) are negative. For such r, $\Delta(r) = (1-ry(r))\cdot(2r-ry(r)-1)$. Hence $\Delta(r_2) = 1$ is equivalent to $f(r_2) = 0$, where $f(r) \equiv r^2y(r)\cdot(y(r)-2) + 2(r-1)$. Find r_2 by using the Newton-Raphson iterative technique to solve $f(r) = 0$.
Hint: You will need a computer. First differentiate $g(w) = 2 - w$ implicitly with respect to r and arrange to obtain

$$y'(r) = \frac{y(r)\cdot(y(r)-2)\cdot(1-y(r))}{2-2ry(r)+ry(r)^2}.$$

This will enable you to express $f'(r)$ in terms of r, $y(r)$, and $y'(r)$. Estimate r_2 from the graph. This will be your first value in the iterative sequence. For each subsequent value, you will need a separate subroutine to supply the value of $y(r)$. The Newton–Raphson technique will be required within this subroutine as well.

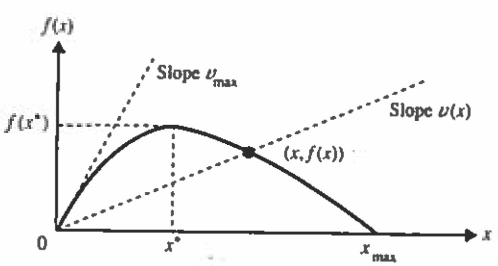

Fig. E2.1

2.8 By solving (2.36) numerically, show that there is a period-4 limit cycle $\{v_1, v_2, v_3, v_4\}$ when $r = 2.56$. To four significant figures, the values of $\{v_k\}$ are, in sequence, 0.2101, 1.587, 0.3531, and 1.850. Show similarly that the steady state when $r = 2.67$ is the 8-period limit cycle {1.924, 0.1631, 1.524, 0.3763, 1.990, 0.1417, 1.401, 0.4798} and that the steady state when $r = 2.69$ is the period-16 limit cycle {2.000, 0.1357, 1.388, 0.4890, 1.933, 0.1570, 1.516, 0.3781, 2.014, 0.1315, 1.360, 0.5160, 1.897, 0.1699, 1.585, 0.3288}.

2.9 For $r = 2.56$, the period-2 equilibrium cycle $\{y(r), z(r)\} = \{0.26357, 1.7364\}$ is unstable, because $r > r_2$. By solving (2.36) numerically with either y or z as initial value, show that the equilibrium cycle diverges toward the period-4 limit cycle in Exercise 2.8. (To keep the population in period-2 equilibrium, the exact value of y or z would be needed. Even if $y(= 0.263573742169)$ or $z(= 1.73642625783)$ were correct to 12 significant figures, the rounding error would be sufficient to shift the period-2 cycle toward the period-4 one, though the divergence would be extremely slow.)

***2.10** The wall of a building is in thermal equilibrium if it transfers heat at a constant rate (just as a rowing boat is in mechanical equilibrium if it moves at a constant speed). Let T_1 be the temperature of air inside a building and T_2 the temperature of the outside air. Assume that $T_2 < T_1$ (it's winter). Let F denote the outward heat flux per unit area, i.e., the amount of heat energy per unit of time that flows across unit area of the wall. Let d be the thickness of the wall.
(i) Which kind of wall will lose heat at the greater rate, a thick one or a thin one? Accordingly, how do you think F varies with d?
(ii) When will the rate of heat loss be greater, when $T_2 << T_1$ (much colder outside than inside) or when T_2 is just a little lower than T_1? Accordingly, how do you think F varies with $T_1 - T_2$?
(iii) Can you suggest an expression for F, in terms of T_1, T_2, d, and some arbitrary constant, which embodies (i) and (ii)? Make the simplest hypothesis consistent with your observations.

***2.11** Instead of installing double-glazing, some folks create a "storm window" by using a sheet of transparent plastic to trap a layer of stagnant air between the air in a room and the pane of glass in the existing window. Adapt the model of Section 2.5 to calculate the heat loss prevented by this contraption. Show, in particular, that it is always less than the loss prevented by double-glazing. (You do not need to know the thermal conductivity of the plastic. Why?)

***2.12** The outward heat flux across area A of wall in Section 2.5 is $AF(x)$ only because we have assumed that heat flux and temperature do not vary within any x = constant plane. If heat flux and temperature also varied with respect to y and z, however, where x, y, and z are orthogonal Cartesian coordinates, then the outward heat flux across area A would be

$$\int_A F(x,y,z)dA = \int_A F(x,y,z)dydz = -\int_A k\frac{\partial T(x,y,z)}{\partial x}dydz,$$

on using (2.45). Similarly, the outward heat flux across area A of a sphere of radius x would be

$$\int_A F(x,\theta,\phi)dA = \int_A F(x,\theta,\phi)x^2\sin(\theta)d\theta d\phi$$
$$= -\int_A k\frac{\partial T(x,\theta,\phi)}{\partial x}x^2\sin(\theta)d\theta d\phi,$$

where x, θ, and ϕ are spherical polar coordinates. Use this result to solve the following problem.

A spherically symmetric planet is in equilibrium with radius a. It consists of a radioactive core $0 \le x \le c$ surrounded by a solid mantle $c \le x \le a$, where x is the distance from the center of the planet. The core generates a steady heat flux $4\pi c^2 F$, which is conducted through the mantle to the surface of the planet. If the thermal conductivity of the mantle is k_M, and if the mantle heat flux per unit area is given by (2.45), show that the temperature drop across the mantle is

$$\frac{c(1 - c/a)F}{k_M}.$$

2.13 Using the methods of Section 1.14, verify that kinematic viscosity has dimensions $[\nu] = L^2/T$.

2.14 Establish (2.71). *Hint*: Use the transformation to elliptical cylinder coordinates,

$$x = ra\cos(\theta), \qquad y = br\sin(\theta),$$

to change the region of integration from that given in (2.71) to $0 \le \theta < 2\pi$, $0 \le r \le 1$.

***2.15** Water is flowing in equilibrium with circular symmetry through a cylindrical pipe with radius a. The pressure drops from p_2 to p_1 along length L of the pipe. Consider a cylinder of water with radius r and length L, whose axis of symmetry coincides with that of the pipe.
(i) What is the (outward) flux of momentum *per unit area* across this cylinder?
(ii) What is therefore the momentum flux?
(iii) What is the net pressure force on this cylinder of water?
(iv) Obtain a differential equation for the velocity $w(r)$ and solve it, subject to an appropriate boundary condition.
(v) Hence obtain Poiseuille's formula (2.72).

***2.16** (i) In the model of Exercise 1.17, if x is measured in tonnes and time in years, what are the dimensions of the constant a?
(ii) If the fish are harvested, as in Exercise 1.14, at rate qux, where u is a constant, show that there are two equilibria (other than $x = 0$) provided $u < aK/4q$. Use a graphical method.
(iii) What happens if $u > aK/4q$? We will ignore the possibility that $u = aK/4q$, precisely, on the grounds that the parameters are subject to so much uncertainty. In deterministic models of natural phenomena, only strict inequalities between parameters are realistic.
(iv) Assume that $u < aK/4q$. Let x_L denote the lower equilibrium and x_U the upper one. Use the graph from (ii) to show that $0 < x_L < K/2 < x_U < K$. Using the equilibrium shift technique, show that x_U decreases with u and q but increases with a. Verify your result graphically.
(v) Why may the shift technique not be applied to the equilibrium x_L? Show that it would lead to absurd results.

2.17 (i) Verify (2.98) and (2.99).
(ii) Verify (2.104).

***2.18** (i) According to (1.81), following drivers in Fig. 2.11 will stop braking and start to accelerate 2.25 seconds after they have observed the preceding car begin to move away. Study the graphs in Fig. 2.11 and convince yourself that this is so. What changes geometrically when a driver stops braking and starts to accelerate?
(ii) The restoration of equilibrium in Fig. 2.10 is exceptionally smooth. I would have anticipated larger ripples. What do you think? Is there anything about (1.81) that produces a stabilizing effect that would not be observed in practice? If so, how could (1.81) be improved? *Hint*: Consider how drivers accelerate as the gap between them and the preceding car increases.

***2.19** (You will need a computer for this one.) In 1959, according to Section 3.3, cars were sent through New York's Lincoln tunnel in platoons of 22. Assume that they drove at 20 m.p.h., in equilibrium. Would such a platoon remain in equilibrium if the driver response time were 1.75 seconds and the leading driver slowed temporarily to 16 m.p.h. according to Fig. 2.9? Now that you have written a computer program to solve (2.111), you can reproduce Figs. 2.10–2.12 for yourself, as well as experiment with other values of d and other forms of $\phi_1(t)$.

***2.20** Suppose that the species of fish in Exercise 2.16 is being "overfished," i.e., that $u > aK/4q$. Show that, although x tends to zero as $t \to \infty$, x never actually reaches zero. Thus, according to the productivity law

$$F(x) = ax^2\left(1 - \frac{x}{K}\right),$$

the population would be certain to recover from any threat of extinction if the harvest were ever stopped. In practice, however, we might expect that the fish would not recover from an extremely low population level, even if that level were greater than zero (for otherwise, why would species ever become extinct?). In other words, our model does not allow for the real possibility of extinction. Can you suggest an improved form of $F(x)$ that would correct this deficiency without destroying any of the other properties which, according to Exercise 1.17, it is desirable for F to have?

Supplementary Notes

2.1 A more general system of differential equations than (2.12) would have the form

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \dots, x_n), \qquad 1 \le i \le n.$$

An extension of the analysis performed above would lead to the linear system of differential equations $d\eta/dt = G\eta$, where η is an n-dimensional vector of relative displacements and the $n \times n$ matrix **G** is defined by

$$g_{ij} = \frac{\partial f_i}{\partial x_j}(x_1^*, \dots, x_n^*), \qquad 1 \le i, j \le n.$$

The matrix **G** is called the Jacobian matrix of the system, and the static equilibrium $\mathbf{x}^*$ is stable if and only if all the eigenvalues of **G** have negative real parts. The special case of this result obtained in the text is sufficiently general for the purposes of this book.

2.2 For suppose that there are two solutions, say w_1 and w_2, and let $u = w_1 - w_2$. Denote the ellipse (2.54) by ∂A and the region it bounds by A. Then both w_1 and w_2 satisfy (2.69) and (2.55), and so we have $u = 0$ on ∂A and $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. Put $m = u\partial u/\partial x$ and $n = u\partial u/\partial y$. Then, on using Green's theorem

$$\oint_{\partial A} (m\,dy - n\,dx) = \iint_A \left(\frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}\right) dxdy$$

(which has essentially been proved in reaching (2.66)), we have

$$\iint_A \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} dxdy = \iint_A \left(\frac{\partial m}{\partial x} + \frac{\partial n}{\partial y}\right) dxdy$$

$$= \oint_{\partial A} u\left(\frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx\right) = 0.$$

Therefore $\partial u/\partial x = 0 = \partial u/\partial y$, implying that u is a constant. Because $u = 0$ on ∂A, the value of the constant must be zero. Hence $w_1 = w_2$; i.e., the solution is unique.

2.3 Principally, the Jacobian matrix **G**, defined by

$$g_{ij} = \frac{\partial f_i}{\partial x_j}(x_1^*, \dots, x_n^*), \qquad 1 \le i, j \le n,$$

must be nonsingular. By even mentioning **G**, of course, we presume that **f** is at least once differentiable with respect to **x**.

2.4 In view of (1.55) and (2.44), you may be wondering why $d\phi_j/dt$ is approximated by (2.109), and not by

$$\frac{M}{2}\left\{\Phi_j(n+1) - \Phi_j(n-1)\right\}. \tag{a}$$

The answer is that using (a) instead of (2.109) would mean approximating a *first order* ordinary differential equation by a *second order* recurrence relation, instead of by a first order one, and would lead to *numerical instability*; i.e., it would lead to spurious oscillations that predict collisions even for very low values of Δ and τ, which are bound to be stable.

To appreciate this phenomenon of numerical instability, consider the simple first order ordinary differential equation

$$\frac{dx}{dt} = -ax, \tag{b}$$

where a (> 0) is a constant. If we approximate the left-hand side of (b) by $x(n+1) - x(n)$, then the result is

$$x(n+1) - (1-a)x(n) = 0. \tag{c}$$

This first order recurrence relation has solution

$$x(n) = x(0) \cdot (1-a)^n. \tag{d}$$

If we approximate the left-hand side of (b) by $\{x(n+1) - x(n-1)\}/2$, then the result is

$$x(n+1) + 2ax(n) - x(n-1) = 0. \tag{e}$$

This second order recurrence has solution

$$x(n) = \frac{(x(1) - \mu x(0))\lambda^n + (\lambda x(0) - x(1))\mu^n}{\lambda - \mu}, \tag{f}$$

where

$$\lambda = -a + \sqrt{a^2+1}, \qquad \mu = -a - \sqrt{a^2+1}. \tag{g}$$

Now the solution of (b) for integer values of time, $t = n$, is

$$x(0) = x(0)e^{-an} = x(0)(e^{-a})^n, \tag{h}$$

so that (d) correctly predicts the qualitative behavior of the solution, namely, that it is decaying exponentially; though it overestimates the decay, because $1-a < e^{-a}$. Because $|\mu| > 1$, however, (f) introduces an oscillation of increasing amplitude that is completely spurious. For $a = 0.5$, values predicted by (h), (d), and (f) are recorded in the following table.

3.5 How Dense Should Traffic Be In a Tunnel?

Tunnels are notorious bottlenecks where motorists converge from several directions to compete for the means to cross a river.[4] To reduce these traffic delays, the traffic flux, i.e., number of vehicles using the tunnel per hour, should be maximized. This means solving an optimal control problem for which the utility is traffic flux.

We have already seen in Chapter 2 that traffic flux depends upon traffic density, i.e., number of cars per mile. Accordingly, let the control function, $u(t)$, be the traffic density at time t. To make matters simple, however, we'll assume that u is independent of t. The traffic flow is then in equilibrium, and we'll assume that its velocity is given by (2.28). Thus, from Exercise 2.5, the traffic flux is

$$J(u) = \begin{cases} uv_{max}, & \text{if } 0 \le u \le x_c; \\ Vu\ln(x_{max}/u), & \text{if } x_c < u \le x_{max}; \end{cases} \tag{3.45}$$

where we have defined

$$V \equiv \frac{v_{max}}{\ln(x_{max}/x_c)} \tag{3.46}$$

and x_c is the critical traffic density below which cars may travel at the speed limit, v_{max}.

We also saw in Exercise 2.5 that the maximum of (3.45) occurs at $u = x_{max}/e = 0.37x_{max}$ if $x_{max}/e \ge x_c$ and at $u = x_c$ if $x_{max}/e < x_c$. Which of these two inequalities is likely to hold? Let d_{min} denote the distance between cars at maximum density and d_c the distance between cars at the critical density, x_c. Then

$$\frac{x_{max}}{x_c} = \frac{d_c + L}{d_{min} + L}, \tag{3.47}$$

[4]You may wish to review Section 2.3 before proceeding.

where L is the length of a car. Cars move if they are more than a fraction of a carlength apart. Let this fraction be ϵ (to be determined). Then

$$d_{min} = \epsilon L, \tag{3.48}$$

where $0 < \epsilon < 1$. On the other hand, many state laws declare that for each 10 m.p.h. of speed you should be at least a carlength behind the car in front of you. Let's suppose that this applies to the speed limit. Then $d_c > v_{max}L/10$. Hence, on using (3.47) and (3.48), the maximum of $J(u)$ depends upon whether

$$\frac{\frac{v_{max}}{10} + 1}{\epsilon + 1} > e = 2.718. \tag{3.49}$$

Because $\epsilon < 1$, (3.49) is sure to be satisfied if v_{max} is 45 m.p.h. or greater. In practice, $v_{max} \ge 45$ is very likely (even if the official speed limit is lower); and ϵ is probably quite close to zero, making (3.49) valid even for much lower values of v_{max} than 45. Let us therefore assume that $x_{max}/e > x_c$. Then the optimal density is $u = u^*$, where

$$u^* = \frac{x_{max}}{e}; \tag{3.50}$$

while the maximum traffic flux, or tunnel capacity, is

$$J(u^*) = Vu^*. \tag{3.51}$$

Thus V, as defined by (3.46), is the optimal speed of traffic through the tunnel.

How would this optimal solution be implemented? In practice, one would control the optimal "throughput" of traffic itself, by placing a stop light at the start of the tunnel that limited influx to $Vu^*/60$ cars per minute. According to data presented by Prigogine and Herman (1971, p. 9), the optimal density for New York's tunnels appears to be about 65 cars per mile, and the optimal speed is about 20 m.p.h. (though it varies slightly from tunnel to tunnel, the newer ones being more efficient.) Thus according to (3.51), the capacity of a New York tunnel is about 1300 cars per hour. This optimal traffic flux can be achieved by using a stop light to allow only about 22 cars (per lane) per minute to enter the tunnel. Gazis (1972, p. 422) reports that, starting in 1959, this very strategy was used to increase the throughput of the Lincoln tunnel. According to Gazis, the strategy worked very well for part of the time, but it was difficult to reset the (optimal) equilibrium after perturbations due to any sudden increase in congestion.

What does this mean? Is this a good model because it sometimes mimics reality well, or a bad model because it sometimes fails to mimic reality at all? You should mull this question over. We'll return to it in Chapter 4.

4.6 The Speed of Cars in a Tunnel

In Section 2.3, we used the conceptual approach to derive a model for the speed, $v(x)$ miles per hour, of cars through a tunnel when the traffic density is x cars per mile.[6] The model we obtained was

$$v(x) = \begin{cases} v_{max}, & \text{if } 0 \leq x \leq x_c; \\ v_{max}\dfrac{\ln(x_{max}/x)}{\ln(x_{max}/x_c)}, & \text{if } x_c < x \leq x_{max}. \end{cases} \tag{4.26}$$

Here x_c is the highest density at which drivers are free to move at the speed limit v_{max}, and x_{max} is the density that will just bring traffic to a halt.

We have already explained, in Chapter 2, how density is measured on an actual road. Data points obtained in this way from New York's Holland tunnel are depicted in Fig. 4.3. Think of the data point (x, v) as recording the average value of all speeds observed when the density was measured and found to be x. The dashed curve in the diagram is (4.26), with v_{max} about 55 m.p.h., x_c about 10 cars per mile, and x_{max} about 175 cars per mile. Observe that this curve fits the data quite well in the range $x_c < x \leq x_{max}$. (We do not have data for $0 \leq x \leq x_c$. The effect of this is considered toward the end of the section.)

We have raised the point, in Section 4.3, that a model may yield several successive generations of predictions, and that any of these may be tested.

[6] You may wish to review Sections 2.3 and 3.5 before proceeding.

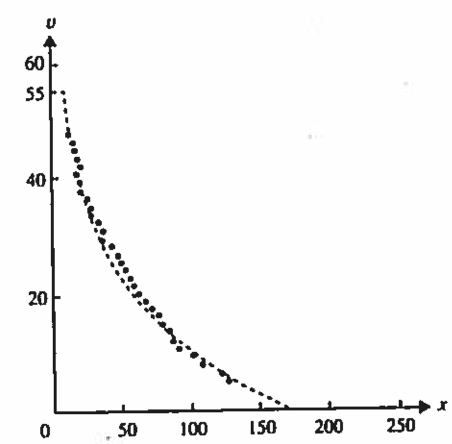

Fig. 4.3 Speed, v m.p.h., versus traffic density, x cars per mile, for New York's Holland tunnel. The 32 data points were obtained from a sample of over 20,000 cars. The dashed curve is (4.26). The sketch is adapted by permission of the publisher from Prigogine and Herman, *Kinetic Theory of Vehicular Traffic*, p. 6. (Copyright 1971 by Elsevier Science Publishing Co., Inc.)

In Section 4.3, for example, a test was applied to a second generation prediction, the growth in value of a forest. Viewed in this light, (4.26) is a first generation prediction, which, by virtue of Fig. 4.3, has passed a first generation test. But the same model has already yielded, in Section 3.5, the second generation prediction that the optimal speed for cars through a tunnel is $v_{max}\ln(x_{max}/x_c)$, the optimal flux being $x_{max}/e = 0.368x_{max}$ times that speed. We already know from Chapter 3 that these predictions are compatible with data from the Lincoln tunnel. Thus the model has also passed a second generation test. Moreover, the fact that the model, as mentioned in Chapter 3, fails to mimic reality in the immediate aftermath of perturbations, due to any sudden congestion, is of no concern to us, because we cannot expect an equilibrium model to describe nonequilibrium conditions! We therefore conclude that the model provides an acceptable description of equilibrium traffic flow, the purpose for which it was designed.

For $x_c < x \leq x_{max}$, the dashed curve in Fig. 4.3 is the curve of the form (4.26) which best fits the data, in the sense of Section 8.4. It is found that $x_{max} \approx 175$ and

$$\frac{v_{max}}{\ln(x_{max}/x_c)} \approx 19, \tag{4.27}$$

if x_{max} and the parameter on the left-hand side of (4.27) are the two parameters that are varied, to give the dashed curve its best fit property. In view of (3.46) and (3.51), it is also found that the optimal speed for cars through New York's Holland tunnel is about 19 m.p.h.; see Prigogine and Herman (1971, p. 9).

Because no data are available for $0 \le x \le x_c$, however, the appropriate values of v_{max} and x_c cannot be determined accurately. All we know is that they are constrained, so that (4.27) is satisfied and the horizontal section of the dashed curve lies above the leftmost data point; but within these constraints they are arbitrary. Thus the horizontal section of curve in Fig. 4.3 is purely speculative and should, perhaps, be lower. As we have already seen in Chapter 3, however, this is a matter of minor importance when using the model to determine optimal flux.

4.7 The Stability of Cars in a Tunnel

Sometimes, even without a conscious effort to compare your model with data, it will dawn on you that one of your predictions is somewhat dubious.[7] Improvements that emerge as the result of such doubts are the subject of this and the following section.

In Section 2.8 we developed a model for instability of traffic in a tunnel. We found that for any given amplitude of the initial disturbance Δ there is a critical value, $\tau_c(\Delta)$, which driver response time must not exceed if a platoon of cars is to stay in equilibrium. We found, moreover, that $\partial\tau_c/\partial\Delta < 0$ (see Fig. 2.12). This picture is surely in qualitative agreement with everyone's highway experience. But it seems that the return to equilibrium in Fig. 2.10 is almost too smooth to be true. If this is right, then something is wrong with the assumptions that led to Fig. 2.10; and hence with equation (1.81), i.e.,

$$\frac{d^2 z_j(t+\tau)}{dt^2} = -\lambda \frac{z_j'(t) - z_{j-1}'(t)}{|z_j(t) - z_{j-1}(t)|}. \tag{4.28}$$

Recall that the car with index j is following the car with index $j-1$. Thus the equation says two things. First, if two cars are moving toward one another then the closer they are, the harder the second one brakes. With this I agree. Second, if two cars are moving away from one another then the further away they are, the more gradually the second one accelerates. I suspect that this isn't true (though the follower will certainly accelerate if the leader is moving away).

A possible improvement of (4.28) would be to combine it with the model of Exercise 2.4 in the following way:

$$\frac{d^2 z_j(t+\tau)}{dt^2} = \begin{cases} -\lambda \dfrac{z_j'(t) - z_{j-1}'(t)}{|z_j(t) - z_{j-1}(t)|}, & \text{if } z_j'(t) > z_{j-1}'(t); \quad (4.29a) \\ -\mu\left\{z_j'(t) - z_{j-1}'(t)\right\}, & \text{if } z_j'(t) < z_{j-1}'(t); \quad (4.29b) \end{cases}$$

[7] You may wish to review Section 2.8 before proceeding.

where λ and μ are constants. Thus, if a gap between cars is closing, the following driver responds both to the size of the gap and the rate at which it is closing; but if a gap between cars is widening, then the following driver responds only to the rate at which it is widening. Integrating, and choosing the constants of integration so that the equilibrium solution of (4.29) will satisfy $v(x_{max}) = 0$, we find (verify) that

$$\frac{dz_j(t+\tau)}{dt} = \begin{cases} \lambda \ln\left(x_{max}(z_{j-1}(t) - z_j(t))\right), & \text{if } z_j'(t) > z_{j-1}'(t); \\ \mu\left(z_{j-1}(t) - z_j(t) - \dfrac{1}{x_{max}}\right), & \text{if } z_j'(t) < z_{j-1}'(t). \end{cases} \tag{4.30}$$

Because (4.29a) and (4.29b) must yield the same equilibrium solution, we have

$$\lambda \ln\left(\frac{x_{max}}{x}\right) = v(x) = \mu\left(\frac{1}{x} - \frac{1}{x_{max}}\right). \tag{4.31}$$

Assuming, as in Section 2.8, that $x = x_{max}/e$ (the optimal density for maximum traffic flow) and denoting $v(x_{max}/e)$ simply by v, we determine the constants λ and μ to be

$$\lambda = v, \qquad \mu = \frac{v x_{max}}{e-1}, \tag{4.32}$$

where $e = \exp(1) = 2.718$. On writing $z_j(t)$ as the sum of the equilibrium displacement $\zeta_j(t)$, given by (2.97), and a perturbation displacement $\phi_j(t)$, substituting into (4.30), and using (4.32), we find (Exercise 4.4) that the differential-delay equations for the perturbation displacements are

$$\frac{d\phi_j(t+\tau)}{dt} = \begin{cases} v \ln\left\{1 + \dfrac{x_{max}}{e}(\phi_{j-1}(t) - \phi_j(t))\right\}, & \text{if } \phi_j'(t) > \phi_{j-1}'(t); \quad (4.33) \\ \dfrac{x_{max} v}{e-1}\left\{\phi_{j-1}(t) - \phi_j(t)\right\}, & \text{if } \phi_j'(t) < \phi_{j-1}'(t). \quad (4.34) \end{cases}$$

In view of Exercise 4.4(ii), the expression for $d\phi_j(t+\tau)/dt$ given by (4.33) exceeds that given by (4.34) if

$$-0.632 < \frac{x_{max}(\phi_{j-1} - \phi_j)}{e} < 0. \tag{4.35}$$

Equations (4.33)–(4.34) were discretized and solved for a platoon of 10 cars, as described in Section 2.8 for equations (2.105). I used the same perturbation displacement for the leading car as in Fig. 2.9 and the same parameter values $d = 65$, $v = 88/3$, and $x_{max}/e = 13/1056$ as were used in the earlier model. Thus (4.35) became

$$-51.3 < \phi_{j-1} - \phi_j < 0. \tag{4.36}$$

If you completed Exercise 2.19, then a simple modification of your computer program will enable you to do this for yourself. The results for $\Delta = 0.15$ and $\tau = 0.15$ are shown in Fig. 4.4 as the solid graphs. The

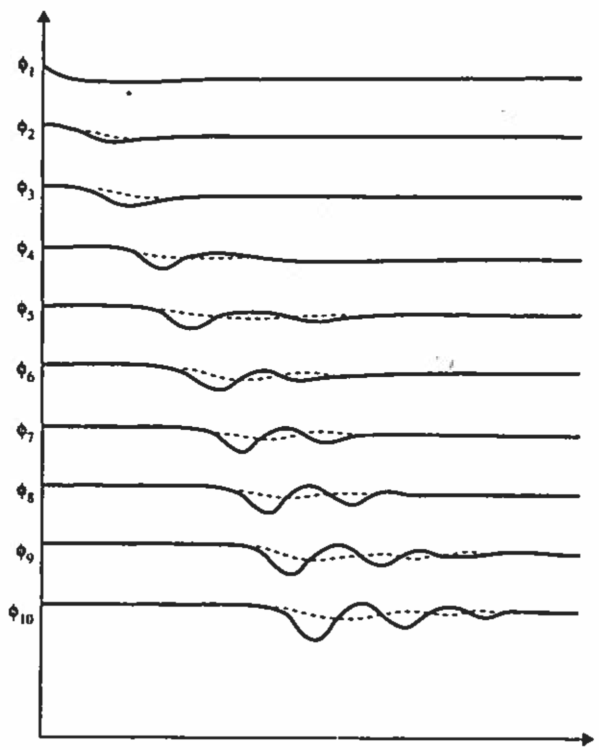

Fig. 4.4 Restoration of equilibrium for a platoon of 10 cars in a tunnel. The dashed curves are identical to those of Fig. 2.10. The solid curves are predicted by the model of (4.29). Scales are as in Fig. 2.10.

dashed ones, for comparison, are those of Fig. 2.10. The strong upward curvature of the solid curves during the accelerative phase is a symptom of (4.34), i.e., of acceleration being independent of spacing between cars. Notice that the solid curves always dip below the dashed ones when they first depart from them. The reason is that when driver j first reacts to driver $j-1$, $\phi_{j-1} < 0$ and $\phi_j = 0$, in such a way that (4.36) is satisfied. Then (4.33) yields a less negative slope than (4.34).

The results for $\Delta = 0.15$ and $\tau = 2.25$ are shown in Fig. 4.5 as the solid graphs. The dashed ones, for comparison, are those of Fig. 2.11. The eighth car collides with the seventh one after only 26.25 seconds (210 time units), much sooner than a collision occurs in Fig. 2.11. The effect introduced by (4.29b) is clearly a destabilizing one.

A selection of points is labeled in Fig. 4.6, according to whether the points are stable or unstable. As in Section 2.8, a square denotes stability,

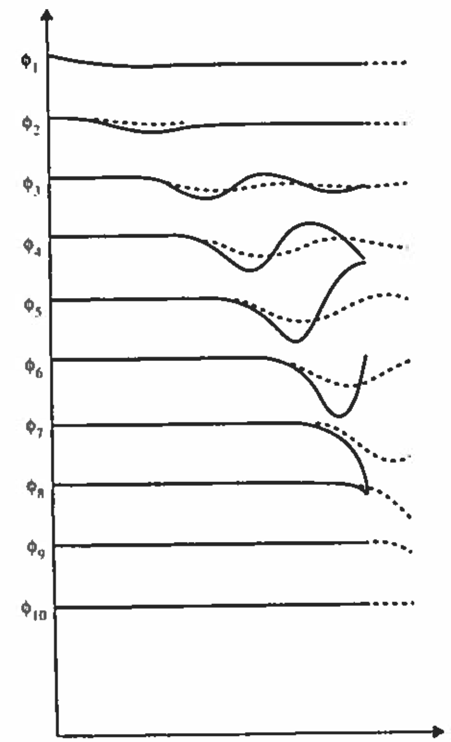

Fig. 4.5 Instability of a platoon of 10 cars when $\Delta = 0.15$ and $\tau = 2.25$, for first 30 seconds following perturbation. The dashed curves are identical to those of Fig. 2.11. No collision has yet occurred. The solid curves are the predictions of (4.29). The eighth car collides with the seventh after 26.25 seconds. Scales are as in Fig. 2.11.

a dot instability, and j/n indicates that the jth car collided with the car in front of it after $n/8$ seconds. A tentative stability boundary $\tau = \tau_c(\Delta)$ is drawn as a solid curve. The dashed curve, for comparison, is that of Fig. 2.12. Once again, the effect of (4.29b) is seen to be a destabilizing one.

But is this a genuine improvement? How do drivers actually drive—according to (4.28), according to (4.29), or according to neither of these? In the absence of hard data, the best we can do is to compare the predictions of (4.28), namely, Figs. 2.10–2.12, with those of (4.29), namely, Figs. 4.4–4.6, and use our personal experience to guess which picture is a fairer one of reality. Perhaps you would like to pursue this matter and obtain some data (though this will be far from easy). It is even possible that one model describes drivers in, say, Boston, and the other describes drivers in New

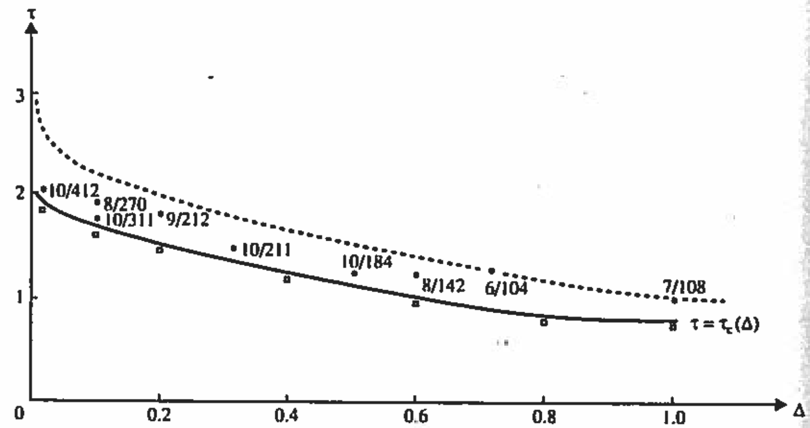

Fig. 4.6 Tentative stability boundary (solid curve) in the Δ-τ plane for the model of (4.29). The dashed curve is identical to that of Fig. 2.12.

York. But we now at least have a rational basis for accepting one model and rejecting the other (or rejecting both), even though, without the data, we cannot actually make that choice. Therefore, as good scientists, we can conclude only that the matter remains open to question.

On a real highway, of course, both τ and L would be functions of j, but such complexity could not usefully be embodied in a deterministic model. We would expect, however, that such variations make a real highway even more unstable than our models predict, and that a collision is likely if any driver (except the first) has a slower reaction time than $\tau_c(\Delta)$. Individual values of τ are utterly beyond the control of traffic police, and so they must endeavor instead to keep Δ as close as possible to zero, i.e., to ensure that the stability region in Fig. 2.12 or Fig. 4.6 is as high as possible. It's no wonder that police are so much in evidence in New York's tunnels, urging traffic to maintain equilibrium!

4.8 The Forest Rotation Time

In Exercise 3.14, we considered the effect of reducing a forest rotation period by a unit of time, from $t = s + 1$ to $t = s$.[8] We found that the forester's loss from unrealized tree growth would be approximately $V'(s)$; whereas her gain from alternative investment would be approximately $\delta V(s)$. Thus, according to Section 3.4, it is too soon to harvest if

[8]Before proceeding, you may wish to review your solutions to Exercise 3.9 and 3.14.