Fall 2017 **STUDENT NAME:** ____________________________

. **Student ID:** ________________________________

FINAL EXAM – Math 4/5650 - Numerical Analysis
Take Home Part - Due Thursday, Dec 14, 12:00pm (noon) - ENG 271

No team work of any kind! Answer each problem on a separate sheet of paper, with your name clearly visible.

1. Let $f(x) = e^{-x}$.

(a) Find the n-th Taylor polynomial $P_n(x)$ about $x_0 = 0$.

(b) Find the smallest value of n such that $|P_n(x) - f(x)| < 10^{-5}$, for all $x \in [0, 1]$.

(c) From part (b), use $P_n(0.5)$ to approximate $f(0.5)$ and compare it to the actual value.

(d) Compute $\lim_{x\to 0} \dfrac{f(x) - (1 - x)}{x^2}$.

(e) Determine the rate of convergence for part (d).

2. The equation $x^3 - 2x - 1 = 0$, has a root in $[1, 2]$.

(a) Which of the functions below have a fixed point in the given interval? Show details.

$$g_1(x) = \frac{2x+1}{x^2}, \quad g_2(x) = \sqrt{2 + \frac{1}{x}}, \quad \text{and} \quad g_3(x) = x^3 - x - 1.$$

(b) Starting with $x_0 = 1.5$, compute x_4 using the appropriate fixed point function(s) found in part (a).

(c) Estimate the number of iterations necessary to obtain approximations accurate to within 10^{-6}, using part (b).

(d) Use Newton's Method to find the root to within 10^{-6}. State the number of iterations.

3. Let $f(x) = \sqrt{(x+1)}$ and consider the values of $f(x)$ at the points $x = 0, 0.25, 0.5, 0.75$.

(a) Find the natural cubic polynomial spline function.

(b) Find the clamped cubic polynomial spline function.

(c) Use both splines to approximate $f(0.3)$.

(d) Use both splines to approximate $f'(0.3)$.

4. Let $f(x)$ be a smooth function with $f(0) = 8$, $f(1) = 5$, $f(2) = 3$, $f(3) = 2$, and $f(4) = 3$.

(a) Use a 3-point centered difference scheme to approximate $f''(2)$.

(b) Use Richardson extrapolation to improve this result. Hint: first, use $h = 2$, and so, $h/2 = 1$.

(c) Use (composite) Trapezoidal Rule and Simpson's Rule to approximate $\int_0^4 f(x)\,dx$

5. Consider the following differential equation with exact solution $y(t) = t^2(e^t - e)$

$$y' = \frac{3y}{t+1} + t + 1, \quad 0 \le t \le 2, \quad y(0) = 0.$$

(a) Approximate the solution using Taylor's method of order two and $h = 0.1$. Compare with the exact solution.

(b) Approximate the solution using Runge-Kutta method of order 4 and $h = 0.1$. Compare with the exact solution.

(c) Plot all three solutions on the same graph.

(d) Use linear interpolation and the values from part (b) to find an approximate value of $y(1.75)$ and compare to the exact value.

6. Let

$$\begin{array}{rcrcrcl} 0.5x_1 & + & 1.1x_2 & + & 3.1x_3 & = & 6.0 \\ 2.0x_1 & + & 4.5x_2 & + & 0.4x_3 & = & 0.02 \\ 5.0x_1 & + & 1.0x_2 & + & 3.5x_3 & = & 1.0 \end{array}$$

(a) Solve the given system using naive Gauss elimination.

(b) Rearrange the given linear system such that the convergence of the Jacobi and Gauss-Seidel methods is guaranteed. Explain.

(c) Using part (b) and the initial approximation $\mathbf{x}^{(0)} = [-1, 1, 2]^T$, use the Jacobi method to find $x^{(2)}$. Compute an error bound for the approximation.

(d) Using part (b) and the initial approximation $\mathbf{x}^{(0)} = [-1, 1, 2]^T$, use the Gauss-Seidel method to find an approximate solution. [Hint: Use the jacobi.m and gaussseidel.m codes posted on the course website http://academics.uccs.edu/rcascava/Math4650