SAMPLE EXAM I – Math 4/5650 - Numerical Analysis - Fall 2017

- **Problem 1**
  Consider applying Newton's method to find the zeros of the function

  $$f(x) = x^2 - 2x + 1.$$

  (a) Write the iteration obtained from Newton's method.

  (b) Determine what initial guesses x_0 will lead to convergence.

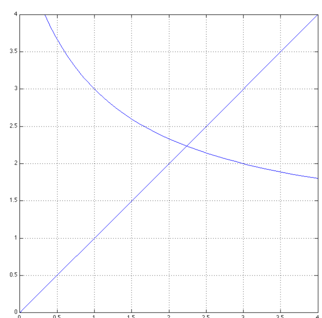
  (c) Show that the convergence is linear and not quadratic. Why is that?

  (d) Can you apply bisection method in this example? What about secant method? Which has a better order of convergence? Explain.

- **Problem 2**
  (a) Show that $x^* = \sqrt{5}$ is a fixed point for the function

  $$g(x) = \frac{x+5}{x+1}$$

  and that the method of fixed point iterations $x_n = g(x_{n-1})$ can be applied to approximate $x^* = \sqrt{5}$. What is the interval I where x_0 can be chosen so that convergence is guaranteed? What is the order of convergence in this case? Show this analytically and geometrically (graph of g is included for convenience).

  (b) Use the theoretical bound $|x_n - x^*| \leq \frac{k^n}{1-k}|x_1 - x_0|$ (where $k < 1$ is an upper bound for $|g'(x)|, x \in I$) to obtain a theoretical bound on the number of iterations needed to approximate the fixed point to within 10^{-4}? You may pick $x_0 = 3$.

- **Problem 3**
  Using divide differences, show that the polynomial interpolating the following data has degree three:

x	-2	-1	0	1	2	3
$f(x)$	1	4	11	16	13	-4

- **Problem 4**

  Consider the interpolation of the function $f(x) = x^2$ in $[0, 1]$ at two points x_1, x_2 by a polynomial $p_1(x)$ of degree 1.

  (a) Find a theoretical upper bound for the error magnitude

  $$\|f - p_1\|_\infty := \max_{0 \le x \le 1} |f(x) - p_1(x)|$$

  [Hint: Error in Lagrange interpolation is $f(x) - p_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n} (x - x_i)$.]

  (b) Show that the function $f - p_1$ is concave up (i.e., its second derivative is positive) and that $|f - p_1|$ has 3 maxima in the interval $[0, 1]$. Then compute $\|f - p_1\|_\infty$ and verify that the actual error magnitude does not exceed the upper bound found in (a).

  (c) Determine the points x_1 and x_2 in $[0, 1]$ which yield the smallest $\|f - p_1\|_\infty$ found in (b).

- **Problem 5**

  Given the function

  $$f(x) = \frac{1}{1 + x}$$

  (a) Find the third degree Taylor polynomial expanded about $x_0 = 0$ and use it to approximate $f(0.1)$. Find a bound for the error in this approximation.
  [Hint: The remainder in Taylor approximation is $f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x - x_0)^{n+1}$].

  (b) Use Lagrange interpolation to find a third degree polynomial interpolating $f(x)$ at the points $x_0 = 0$, $x_1 = 0.5$ $x_2 = 1$ and $x_3 = 2$ and use it to approximate the value of $f(0.1)$. What is the error estimate now?

  (c) Write down (without computing the coefficients explicitly) the clamped cubic spline interpolation $S(x)$ using the three points $x_0 = 0$, $x_1 = 0.5$ and $x_2 = 1$. What is the error in approximating the value $f(0.1)$?
  [Hint: The error in clamped cubic spline approximation is $|f(x) - S(x)| \le \frac{5M}{384} \max_j |x_{j+1} - xj|^4]$, where $M = \max_x |f^{(4)}(x)|$