

# LOCAL AND GLOBAL WELL-POSEDNESS FOR A CLASS OF NONLINEAR DISPERSIVE EQUATIONS

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ABSTRACT. We study the local and global well-posedness of the initial value problem for the class of nonlinear dispersive PDEs of the form

$$u_t - Mu_x + F(u)_x = 0, \quad t \in \mathbb{R},$$

where  $u = u(x, t)$ ,  $x \in \mathbb{R}$  or  $x \in \mathbb{T}$ . Here  $M$  is a linear operator, given in the Fourier space by the multiplication operator:  $\widehat{Mv}(\xi) = |\xi|^{2\beta} \widehat{v}(\xi)$ ,  $\beta \geq \frac{1}{2}$  and  $F$  is a nonlinear (sufficiently) smooth function. This equation is a generalization of the Korteweg-de Vries (KdV) equation ( $\beta = 1$ ), the Benjamin-Ono (BO) equation ( $\beta = \frac{1}{2}$ ) and the fifth-order KdV equation ( $\beta = 2$ ). The nonlinearity can be very general, but a certain growth condition must be imposed in order to obtain global results. Roughly speaking, we impose that  $(F'(r))_+$  grows at most like  $|r|^p$  as  $r \rightarrow \infty$ , for some  $p < 4\beta$ . Global existence of solutions is, therefore, intimately related to the balance between the strength of the nonlinearity and the dispersion relation. The semigroup methods developed by Goldstein-Oharu-Takahashi are being successfully applied here. Most of the results are presented in the periodic case (i.e.  $x \in \mathbb{T}$ ), but they are also valid in the real line case (when  $x \in \mathbb{R}$ ).

## 1. INTRODUCTION

In this paper<sup>1</sup> we focus on the local and global (in time) well-posedness for the following Cauchy problem written in general form:

$$\begin{aligned} \text{(NDE)} \quad & \frac{du}{dt} - Mu_x + F(u)_x = 0, \\ & u(0) = u_0, \end{aligned}$$

with the linear dispersive term  $-Mu_x$ ,  $M$  being the linear operator given by a multiplication operation in the Fourier space:  $\widehat{Mv}(\xi) = |\xi|^{2\beta} \widehat{v}(\xi)$ ,  $\beta \geq \frac{1}{2}$ . The nonlinearity is of the form  $F(u)_x = F'(u)u_x$ , with  $F : \mathbb{R} \rightarrow \mathbb{R}$  a (sufficiently) smooth function satisfying the following growth condition:

$$\text{(C)} \quad \limsup_{|r| \rightarrow \infty} \frac{F'(r)}{|r|^p} < \infty \text{ for some } p < 4\beta.$$

In many situations it is instructive to consider the particular nonlinearity  $F(u)_x = u^p u_x$ , for some power  $p \geq 1$ , (with the convention  $u^p = |u|^{p-1}u$  for non-integer powers  $p$ ). For integer powers  $p = k$ , one obtains the (KdV<sub>k</sub>) equations (see below), which exhibit many of the phenomena of interest to us. We address the question of global existence of solutions and well-posedness in the Sobolev spaces  $H^s(\mathbb{T})$ . Our

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results are also valid in the context of  $H^s(\mathbb{R})$ , but we shall not focus on this case since it is similar to and slightly easier than the  $H^s(\mathbb{T})$  case.

**1.1. Motivation.** We begin by mentioning some of the previous work done in the literature. Many authors have studied the initial-value problem for the generalized KdV equation:

$$\begin{aligned} \text{(GKdV)} \quad u_t + u_{xxx} + F(u)_x &= 0, \\ u(0) &= u_0, \end{aligned}$$

and, in particular,

$$\begin{aligned} \text{(KdV}_k\text{)} \quad u_t + u_{xxx} + u^k u_x &= 0, \\ u(0) &= u_0. \end{aligned}$$

Here  $x \in \mathbb{R}$  (real line case) or  $x \in \mathbb{T}$  (periodic case). The first result on global well-posedness for (GKdV) was proved by Bona and Smith [14] in 1978, for the classical KdV equation, i.e.  $F(u) = \frac{u^2}{2}$ . Later, Kato [46], in 1983, developed a theory for quasi-linear evolution equations, and applied it to equation (GKdV). This was the first place where it was realized that global existence of solutions depends in a precise way on the nonlinearity. In a series of papers, e.g. [1], [9], [13], Bona et al have studied the more general forms of this equation, with generalized dispersion relations. See also Dix [31], Ginibre-Velo [37], [38], Saut [65].

It is worth noting that the signs of the dispersion term and of the nonlinearity play an important role in the global existence of the solution  $u(t)$ . As it will be seen in Chapter 4, global existence in time is guaranteed by the uniform boundedness of the  $H^\beta(\mathbb{T})$  norm of the solutions. If condition (C) is assumed, this uniform bound can be automatically derived from the conserved quantities. The condition  $p < 4\beta$  imposed in (C) is essential to guarantee the boundedness of the norm mentioned above, hence the global in time existence of solutions. For  $\beta = 1$  and  $p \geq 4$ , Bona et al, [10], obtained numerical evidence of blow-up of  $u_x$  in finite time. In the critical case  $p = 4$ , Martel and Merle proved in a series of papers (e.g. [58], [59]) that finite time blow up indeed occurs for certain  $H^1$  data. An interesting analysis of self-similar blow-up solutions, but not in the usual energy spaces, appeared in [16]. In general, for  $p \geq 4\beta$ , one can still get uniform bounds for the  $H^\beta$ -norm of the solution if the initial data are small enough (in a sense to be made precise later).

A different but related question which has been extensively addressed in the literature is that of obtaining optimal results with respect to the smoothness of the solution. That is, one is interested in solving the Cauchy problem for "rough" data. This direction has been pursued in the literature through the leading work of Kenig, Ponce, Vega (see [49], [50], [51], [52], [53], [54]), Bourgain ([17], [18], [21]), Staffilani ([67]), Colliander et al ([28]). Most of the methods used are applied also to other dispersive equations, such as the nonlinear Schrödinger equation and the Zakharov system. A related question is the growth of the Sobolev norms of solutions of such systems. Results have been obtained in this direction in [67] and [22]. In the present work we do not concentrate on optimal values of  $s$ . Instead we plan to study the relationship between the dispersion effects and the strength of the nonlinearity in the usual functional setting of the Sobolev spaces.

**1.2. Review of semigroup theory.** In this section we place our results in the context of semigroup theory, by recalling two fundamental results, namely the Hille-Yosida theorem and the Crandall-Liggett theorem.

Let  $X$  be a Banach space. A continuous one parameter semigroup on  $X$  is a family of operators  $\mathcal{T} : \{T(t) : X \rightarrow X\}_{t \geq 0}$  satisfying the following properties:

$$\begin{aligned} T(t+s) &= T(t)T(s), \text{ for all } t, s \geq 0 \\ T(0) &= I \\ \lim_{t \rightarrow 0} T(t)f &= f, \text{ for all } f \in X. \end{aligned}$$

In case the operators  $T(t)$  are linear and bounded,  $\mathcal{T} = \{T(t)\}$  is usually called a  $C_0$ -semigroup and satisfies the growth condition  $\|T(t)\| \leq Me^{\omega t}$  for some constants  $M > 0$  and  $\omega \in \mathbb{R}$ . By change of norm tricks, one may consider only ( $C_0$ ) contraction semigroups:  $\|T(t)\| \leq 1$  for all  $t \geq 0$ . (See, for instance, [40].)

The linear case is well understood. Any  $C_0$ -semigroup can be represented as  $T(t)f = u(t)$ , where  $u(t)$  is the solution of the Cauchy problem

$$(1.1) \quad \begin{aligned} \frac{du}{dt} &= Au \\ u(0) &= f \in X. \end{aligned}$$

Moreover,  $A$  can be defined as

$$Af = \lim_{t \rightarrow 0} \frac{T(t)f - f}{t}, \quad \mathcal{D}(A) = \{f \in X \mid Af \in X\}.$$

Conversely,  $A$  generates a  $C_0$ -contraction semigroup iff  $\overline{\mathcal{D}(A)} = X$  and for all  $\lambda > 0$ ,

$$(1.2) \quad \text{Range}(I - \lambda A) = X,$$

$$(1.3) \quad \|(I - \lambda A)^{-1}\| \leq 1.$$

Then the semigroup  $T$  and the solution  $u$  of (1.1) are given by

$$(1.4) \quad u(t) = T(t)f = \lim_{n \rightarrow \infty} (I - \frac{t}{n}A)^{-n}f;$$

$u$  is a strong solution if  $f \in \mathcal{D}(A)$  and a mild solution in general (see [40]).

The Crandall-Liggett-Bénilan theory extends this to nonlinear operators. Let  $\omega \in \mathbb{R}$  and let  $A_\omega = A - \omega I$ . Suppose that, for all (small)  $\lambda > 0$ ,

$$(1.5) \quad \text{Range}(I - \lambda A_\omega) \supset \mathcal{D}(A_\omega),$$

$$(1.6) \quad \|(I - \lambda A_\omega)^{-1}\|_{Lip} \leq 1,$$

i.e. if  $\lambda > 0$  and  $u_i - \lambda A_\omega u_i = h_i, i = 1, 2$ , then  $\|u_1 - u_2\| \leq \|h_1 - h_2\|$ . Then

$$T(t)f = \lim_{n \rightarrow \infty} (I - \frac{t}{n}\bar{A})^{-n}f$$

(where  $\bar{A}$  is the closure of  $A$ ) defines a strongly continuous semigroup  $T$  on  $\overline{\mathcal{D}(A)}$  satisfying  $\|T(t)\|_{Lip} \leq e^{\omega t}$  for all  $t \geq 0$ . This is the Crandall-Liggett theorem (see [30]). Bénilan defined in [5] mild solutions and showed that  $u(t) = T(t)f$  is the unique (global) mild solution of (1.1) whenever  $f \in \overline{\mathcal{D}(A)}$ .

For the semilinear problem

$$(1.7) \quad \begin{aligned} \frac{du}{dt} &= Au + B(u), \\ u(0) &= f, \end{aligned}$$

where  $A$  generates a (linear)  $C_0$ -contraction semigroup  $\mathcal{T}$  on  $X$  and  $B$  need not be linear, a mild solution is by definition a continuous function  $u : [0, \infty) \rightarrow X$  satisfying

$$u(t) = T(t)f + \int_0^t T(t-s)B(u(s))ds,$$

for all  $t \geq 0$ . Building upon earlier work of Oharu and Takahashi [61], [62], Goldstein, Oharu and Takahashi ([43], [42]) developed a semilinear Hille-Yosida theory and applied it to the generalized KdV equation (GKdV) corresponding to  $\beta = 1$ . For the usual KdV equation, if  $\mathcal{S} = \{S(t) : t \in \mathbb{R}\}$  is the group which governs the Cauchy problem, then on all the Sobolev spaces,  $\|S(t)\|_{Lip} = \infty$  for all  $t \neq 0$ . Thus, the Crandall-Liggett-Bénilan (C-L-B) theory does not apply.

Here is the precise statement of the Oharu-Takahashi theorem [61]. Consider  $C$  to be a closed convex subset in  $X$  and  $\varphi$  a convex, lower-semicontinuous functional on  $X$ , with  $C \subset \mathcal{D}(\varphi)$ .  $A$  generates a  $C_0$ -semigroup  $\mathcal{T}$  satisfying  $\|T(t)\| \leq e^{\omega_0 t}$  for some  $\omega_0 \in \mathbb{R}$  and  $B$  is a locally Lipschitz operator defined on  $C$  with respect to  $\varphi$ , in the following sense: For any  $r > 0$ , there exists  $\omega = \omega_r$  such that

$$(1.8) \quad |B(u) - B(v)| \leq \omega_r |u - v|, \text{ whenever } \varphi(u) \leq r, \varphi(v) \leq r.$$

Then the following theorem holds true.

**Theorem 1.1.** *Let  $a, b \geq 0$  be given. Then there is a locally Lipschitz semigroup  $\mathcal{S} = \{S(t) : C \rightarrow C | t \geq 0\}$ , with respect to  $\varphi$ , consisting of continuous mappings on  $C$ , and satisfying*

$$(1.9) \quad S(t)f = T(t)f + \int_0^t T(t-s)B(S(s)f)ds,$$

$$(1.10) \quad \varphi(S(t)f) \leq e^{at}(\varphi(f) + bt), \quad t \geq 0,$$

for all  $f \in C$ , if and only if, for all  $r > 0$ , there is  $\lambda_0(r) > 0$  such that for all  $f \in C_r$  and  $0 < \lambda < \lambda_0(r)$ , there is an  $f_\lambda \in \mathcal{D}(A) \cap C_r$  satisfying

$$(1.11) \quad f_\lambda - \lambda A f_\lambda - \lambda B(f_\lambda) = f$$

$$(1.12) \quad \varphi(f_\lambda) \leq \frac{\varphi(f) + b\lambda}{1 - a\lambda}.$$

In [43], the authors realized that in fact, the sufficiency in the theorem above remains valid even in the absence of the convexity of the functional  $\varphi$ . This led us to generalize the result for non-convex functionals and apply the abstract semilinear theory to the Cauchy problem (NDE). As it will become clear in the next chapter, in our application the functionals are not all convex. For the abstract result that we will use, see Theorem 3.1.

It is worth mentioning that the first treatment of dispersive equations of KdV type via semigroup methods was done by T. Kato in a series of papers [44], [45], [46]. Kato developed a semigroup theory for the local and global well-posedness of the Cauchy problem for a large class of quasi-linear equations and applied it to the generalized KdV. Our semilinear theory is different in nature, is based on a

different point of view, and gives well-posedness directly.

**1.3. Main results.** Our main goal here is to prove global well-posedness of the Cauchy Problem for (NDE) in the periodic case, using an explicit method for generating solutions and obtaining - at the same time - estimates for different norms of the solution. The generating method has been developed in a general setting by S. Oharu and T. Takahashi and applied to different systems (reaction-diffusion, Navier Stokes, phase-field equation). J. Goldstein has extended their method to cover equations such as (KdV) and (GKdV), and then applied this method successfully to the real line case, [41], [43]. One of our purposes here is to show that the semigroup approach fits also the periodic case. Our main result concerning (NDE) is stated below:

**Theorem 1.2.** *The Cauchy problem (NDE) is globally well-posed in  $H^s(\mathbb{T})$  with  $s = \max\{2\beta, \frac{3}{2} + \varepsilon\}$  for some  $\varepsilon > 0$ , provided that the growth condition (C) is satisfied and we are in any of the following cases:*

- $\beta = 1$  and  $s = 2$ ;
- $\beta = \frac{1}{2}$  and  $F'(u) = u$  and  $s \in (\frac{3}{2}, 2]$ ;
- $\beta > \frac{3}{2}$  and  $s = 2\beta$ .

For  $\frac{1}{2} < \beta \leq \frac{3}{2}$ ,  $\beta \neq 1$ , (NDE) is locally well-posed in  $H^s(\mathbb{T})$ , with  $s = 2\beta$  for  $\beta > \frac{3}{4}$  and  $s \in (\frac{3}{2}, 2\beta + 1]$  for  $\frac{1}{2} < \beta \leq \frac{3}{4}$ .

In particular, we obtain well-posedness results for (GKdV), when  $\beta = 1$ .

**Theorem 1.3.** *The Cauchy problem for the periodic (GKdV) equation is globally well-posed in  $H^2(\mathbb{T})$ .*

Theorems 1.2 and 1.3 also hold in the real-line case, i.e. on  $H^s(\mathbb{R})$ , with  $s = \max\{2\beta, \frac{3}{2} + \varepsilon\}$  and  $\varepsilon$  chosen as in Theorem 1.2. In both cases, the growth condition (C) on the nonlinearity is assumed.

The plan of this paper is as follows. In Section 2.1 we introduce the notation and some known facts we will use in the sequel. We then define the operators and the functionals which will allow us to treat the Cauchy problem (NDE) using an abstract theorem. Section 2.3 contains a motivation for our choice of the functionals  $\varphi_j$ , at least in the special cases when  $\beta = 1$  and  $\beta = \frac{1}{2}$ . Chapter 3 is devoted to the abstract semilinear Hille-Yosida theorem. The proof of the abstract theorem is contained in Sections 3.1 and 3.2. The continuous dependence of solutions on the initial data is formulated in Section 3.4. Finally, in Chapter 4 we prove that the operators defined earlier, in Section 2.2, satisfy all the assumptions that make the abstract semilinear theory work. We conclude with a discussion of our results and related open problems.

## 2. FORMULATION OF THE RESULTS

In this section we will state the main results, but we start by introducing the appropriate notations and analytic tools we will use throughout this work.

**2.1. Notations.** The standard notation for Sobolev spaces will be used.  $L^2(\mathbb{T})$  is the usual space of square integrable functions defined on  $[0, 1]$  with the  $L^2$  - norm denoted by  $|\cdot|$  and inner product  $(\cdot, \cdot)$ . For  $k = 1, 2, \dots$ ,  $H^k(\mathbb{T}) = \{u \in H^k_{loc}(\mathbb{R}) \mid u \text{ is 1-periodic}\}$  is the classical Sobolev space of periodic functions, which can be identified with  $\{u \in H^k[0, 1], \partial^j u(0) = \partial^j u(1) \text{ for } j = 0, 1, \dots, k-1\}$ . Here  $\partial = \partial_x$  is the first order differential operator with respect to the space variable  $x$ .

For  $u \in H^k(\mathbb{T})$  define the seminorm  $|\cdot|_{H^j}$ , respectively the norm  $|||\cdot|||_{H^k}$  by

$$(2.1) \quad |u|_{H^j} = |D^j u| = |\partial_x^j u|, 0 \leq j \leq k,$$

$$(2.2) \quad |||u|||_{H^k} = \left( \sum_{j=0}^k |u|_{H^j}^2 \right)^{\frac{1}{2}}.$$

The corresponding (semi) inner products are denoted by  $(u, v)_{H^j} = (\partial^j u, \partial^j v)$  and  $((u, v))_{H^k} = \sum_{j=0}^k (u, v)_{H^j}$ .

We now introduce the fractional order Sobolev spaces and fractional order derivatives. Recall the Fourier transform of  $u \in L^2(\mathbb{T})$ :

$$\widehat{u}(\xi) = \int_{\mathbb{T}} u(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{Z},$$

and the inverse Fourier transform

$$u(x) = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \widehat{u}(\xi) e^{i\xi x}, \quad x \in \mathbb{T}.$$

The fractional order derivative operator is defined in terms of  $D = (-\partial_x^2)^{\frac{1}{2}}$ , so that  $\widehat{D^s u}(\xi) = |\xi|^s \widehat{u}(\xi)$ .

For any  $s \geq 0$ , not necessarily an integer, the Sobolev space  $H^s(\mathbb{T})$  is defined as:

$$H^s(\mathbb{T}) = \{u \in L^2(\mathbb{T}) : \sum_{\xi \in \mathbb{Z}} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 < \infty\}.$$

An equivalent way to define these (fractional order) Sobolev spaces is  $H^s(\mathbb{T}) = \{u \in L^2(\mathbb{T}) : D^s u \in L^2(\mathbb{T})\}$ . An equivalent norm, which will be frequently used in the sequel, is given by

$$\|u\|_{H^s}^2 = \sum_{\xi \in \mathbb{Z}} (1 + |\xi|^{2s}) |\widehat{u}(\xi)|^2.$$

We will make use, throughout this work, of the spaces of periodic functions with ‘mean’ zero

$$\dot{H}^s(\mathbb{T}) = \{u \in H^s(\mathbb{T}) \mid \int_{\mathbb{T}} u = 0\}.$$

On these spaces, we will use the following (homogeneous) norm, equivalent to the  $H^s$ -norm introduced earlier:

$$|u|_{\dot{H}^s} = \left( \sum_{\xi \in \mathbb{Z}} |\xi|^{2s} |\widehat{u}(\xi)|^2 \right)^{\frac{1}{2}}.$$

Clearly,  $|u|_{\dot{H}^s} = |D^s u|$ .

Let  $V_0 = L^2(\mathbb{T})$  with the usual norm. For  $\beta \geq \frac{1}{2}$ , we introduce the following convenient notation:

$$(2.3) \quad V_1 = \dot{H}^\beta(\mathbb{T}), V_2 = \dot{H}^{2\beta}(\mathbb{T}), V_3 = \dot{H}^{2\beta+1}(\mathbb{T}).$$

We will choose to work with the following norms on the  $V_1, V_2, V_3$  spaces, respectively:

$$(2.4) \quad |u|_1 = |D^\beta u|,$$

$$(2.5) \quad |u|_2 = |D^{2\beta} u|,$$

$$(2.6) \quad |u|_3 = |D^{2\beta+1} u|.$$

The following inequalities are true, for any  $u \in \dot{H}^\beta(\mathbb{T})$ , with  $\beta > \frac{1}{2}$ :

$$(2.7) \quad |u|_\infty \leq c |D^\beta u|,$$

$$(2.8) \quad |u|_\infty^2 \leq C |u|^{2-\frac{1}{\beta}} |D^\beta u|^{\frac{1}{\beta}}.$$

The first inequality (2.7) is easy to prove. For  $u \in \dot{H}^\beta(\mathbb{T})$ , we have  $\widehat{u}(0) = 0$ . Thus,

$$\begin{aligned} |u|_\infty &\leq \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} |\widehat{u}(\xi)| = \frac{1}{2\pi} \sum_{\xi \neq 0} |\widehat{u}(\xi)| \\ &\leq \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} |\xi|^{\beta-\frac{1}{2}-\epsilon} |\widehat{u}(\xi)|, \quad \text{for some } 0 < \epsilon < \beta - \frac{1}{2}, \\ &\leq \frac{1}{2\pi} \left( \sum_{\xi \neq 0} |\xi|^{-1-2\epsilon} \right)^{\frac{1}{2}} \left( \sum_{\xi \neq 0} |\xi|^{2\beta} |\widehat{u}(\xi)|^2 \right)^{\frac{1}{2}} \\ &= c |u|_{H^\beta}. \end{aligned}$$

The second inequality (2.8) is a particular case of the more general Gagliardo-Nirenberg type inequality (see [2]).

**Proposition 2.1.** (*Gagliardo-Nirenberg*) For any  $u \in W^{\beta,r}(\mathbb{T})$ , and  $0 \leq \alpha \leq \beta$ ,  $\frac{\alpha}{\beta} \leq a \leq 1$ ,

$$(2.9) \quad |D^\alpha u|_{L^p} \leq C |u|_{L^q}^{1-a} |D^\beta u|_{L^r}^a,$$

where

$$\frac{1}{p} - \alpha = (1-a) \frac{1}{q} + a \left( \frac{1}{r} - \beta \right).$$

Notice the generality of the above result. Our inequality (2.8) is the particular case when  $\alpha = 0$ ,  $p = \infty$ ,  $q = r = 2$ ,  $a = \frac{1}{2\beta}$ .

The estimates (2.7) and (2.8) do not hold for  $\beta = \frac{1}{2}$ . This is because  $H^{\frac{1}{2}}$  is continuously imbedded in  $L^p$ , for any  $p < \infty$ , but not in  $L^\infty$ . Thus, for the Benjamin-Ono equation, one needs different estimates. See Section 2.3 for the case  $\beta = \frac{1}{2}$ .

**2.2. A, B, C and functionals  $\varphi_j$ .** The linear operator  $A = D^{2\beta}\partial_x$ , with domain  $\mathcal{D}(A) = H^{2\beta+1+s}(\mathbb{T})$ , is skew-adjoint on any of the spaces  $H^s(\mathbb{T})$ , and it generates a group of isometries  $\{T(t)|t \in \mathbb{R}\}$  on each  $H^s(\mathbb{T})$ . This is easy to see since  $\widehat{Au}(\xi) = i\xi|\xi|^{2\beta}\widehat{u}(\xi)$  for  $u \in H^s(\mathbb{T})$ . The same argument applies to the restriction of  $A$  to  $\dot{H}^{2\beta+1}(\mathbb{T}) = V_3$ , and  $\{T(t)|t \in \mathbb{R}\}$  is also a group of isometries on  $\dot{H}^s(\mathbb{T})$ . In particular,

$$|T(t)v|_k = |v|_k, \text{ for } v \in V_k, k = 0, 1, 2, 3.$$

Define a nonlinear operator  $B$  on  $\dot{H}^1(\mathbb{T})$  by

$$Bu = -\partial F(u) = -F'(u)\partial u.$$

Clearly,  $B : \dot{H}^1(\mathbb{T}) \rightarrow L^2(\mathbb{T})$  and, more generally,  $B : \dot{H}^k(\mathbb{T}) \rightarrow \dot{H}^{k-1}(\mathbb{T})$ . Note that the only thing needed so far is for  $F$  to be smooth enough.

Let us introduce the following functionals:

$$(2.10) \quad \varphi_0(u) = \frac{1}{2} \int_{\mathbb{T}} |u|^2, \quad \text{on } V_0$$

$$(2.11) \quad \varphi_1(u) = \frac{1}{2} \int_{\mathbb{T}} |D^\beta u|^2 - \int_{\mathbb{T}} G(u), \quad \text{on } V_1$$

$$(2.12) \quad \varphi_2(u) = \frac{1}{2} \int_{\mathbb{T}} |D^{2\beta} u|^2 - \frac{4\beta+1}{4\beta+2} \int_{\mathbb{T}} F(u) D^{2\beta} u + \frac{4\beta+1}{4\beta+2} \int_{\mathbb{T}} I(u), \quad \text{on } V_2$$

$$(2.13) \quad \varphi_3(u) = \frac{1}{2} \int_{\mathbb{T}} |D^{2\beta} \partial u + \partial F(u)|^2, \quad \text{on } V_3.$$

Here  $G(u) = \int_0^u F(r)dr$  and  $I(\cdot)$  satisfies  $I''(u) = (F'(u))^2$ ,  $I(0) = I'(0) = 0$ .

These functionals are lower semi-continuous on the corresponding spaces and, in particular, on  $H^{2\beta+1}$ . They are also bounded on bounded sets in  $V_j$ , more precisely:

$$B \text{ is a bounded set in } V_j, \text{ for } 0 \leq j \leq k \text{ and } k = 0, 1, 2, 3, \text{ implies} \\ \sup\{\varphi_j(f) : f \in B\} < \infty \text{ for all } 0 \leq j \leq k.$$

The reason for this choice of the functionals will be made clear in the next section. For now we just mention that in the integrable cases, Korteweg-de Vries ( $\beta = 1$ ) and Benjamin-Ono ( $\beta = \frac{1}{2}$ ), where  $F'(u) = u$ , these are just three of an infinite list of invariants.

Let  $g = g(r) > 0$  be a  $C^1$  increasing function and  $m = m(t, \alpha)$  the maximal solution of the initial value problem

$$(2.14) \quad \begin{aligned} m'(t) &= g(m(t)), \quad t \geq 0, \\ m(0) &= \alpha, \end{aligned}$$

where  $\alpha \in \mathbb{R}$ . When necessary, we will write  $m(t, \alpha) = m_g(t, \alpha)$ .

Our main result concerning the existence of solutions for the nonlinear dispersive equation (NDE) is stated here:

**Theorem 2.2.** *Let  $\beta \geq \frac{1}{2}$ . Under assumption (C) on the growth of the nonlinearity, the Cauchy problem (NDE) is well-posed in  $H^s(\mathbb{T})$ , with  $s = \max\{2\beta, \frac{3}{2} + \varepsilon\}$ , ( $\varepsilon > 0$  is as in Theorem 1.2), for arbitrary initial data  $u_0$ . Moreover, there exists a  $C^1$*



function  $g$  such that the following estimates hold when  $u_0 \in H^s(\mathbb{T})$ :

$$(2.15) \quad \varphi_0(u(t)) = \varphi_0(u_0),$$

$$(2.16) \quad \varphi_1(u(t)) = \varphi_1(u_0),$$

$$(2.17) \quad \varphi_2(u(t)) \leq m_g(t, \varphi_2(u_0)), \text{ for } t \geq 0,$$

where  $m_g$  is the solution of the initial value problem (2.14). If  $u_0 \in H^{2\beta+1}(\mathbb{T})$ , then there exists  $\omega_0$  depending only on a bound for  $|u|_{H^s}$  on  $[0, T]$  so that

$$\varphi_3(u(t)) \leq e^{\omega_0 t} \varphi_3(u_0), \text{ for } t \in [0, T],$$

for any  $T > 0$ .

From now on it is sufficient to assume  $F(0) = 0$  and to restrict all our calculations to initial data belonging to  $\dot{H}^s$ , i.e. with zero mean, instead of the whole  $H^s$ . This is possible (without loss of generality) due to the fact that the mean of the solution  $u(t)$  is constant in time, so  $F(u)$  in the equation can be replaced by  $\tilde{F}(u) = F(u + c) - F(c)$ ,  $c = \int u$ .

The choice of the function  $g$  mentioned in the theorem depends on  $\beta$  and on the nonlinearity  $F$ . Our well-posedness theorem is global in time, provided that the solution  $m$  of  $dm/dt = g(m)$  exists for all  $t \geq 0$ . We have shown this to be true whenever  $\beta > \frac{3}{2}$  and when  $\beta = 1$ . It also holds for  $\beta = \frac{1}{2}$  and  $F$  any quadratic polynomial. For any other case of a pair  $(\beta, F)$  for which one can show that  $m$  exists globally in time, then (NDE) is globally well-posed. Thus, local well-posedness assertions in our theorems are actually global in all such cases where  $m$  exists globally. Since  $(\beta, F)$  determine many possible choices of  $g$ , it seems likely that our result are of a global nature in cases where we do not assert this.

The proof of Theorem 2.2 will be given in Chapter 4 and will be a consequence of the abstract theorem presented in the next chapter. As it will become transparent later, if  $p \geq 4\beta$  in the assumption (C) on the growth of  $F$ , global existence of solutions will still be guaranteed, provided that the  $H^\beta$  norm of the solution is uniformly bounded. This can be accomplished, for example, by requiring the smallness of the initial data. See Proposition 4.8.

**2.3. Apriori estimates for the cases  $\beta = 1, \beta = \frac{1}{2}$ .** In this section we will find apriori estimates on the functionals introduced in the preceding section, for the special cases  $\beta = 1$  and  $\beta = \frac{1}{2}$ . In what follows we assume  $u(t)$  is a real, smooth, classical solution of the equation

$$(2.18) \quad \frac{\partial u}{\partial t} + \partial^3 u + \partial F(u) = 0.$$

Here  $\partial = \frac{\partial}{\partial x}$ . This is exactly (NDE) for  $\beta = 1$ , as  $D^2 = -\partial^2$ . In the sequel we show how the functionals introduced earlier give estimates on the  $H^s$  norms. Note also that we will work only with real valued solutions of (2.18). In general, if the initial data  $u_0$  is real valued, then  $x \mapsto u(x, t)$  is real valued on  $\mathbb{T}$  for all  $t \in \mathbb{R}$ . Thus, restriction to real solutions  $u(x, t)$  is equivalent to restricting to real initial data  $u_0(x)$ .

The first two functionals are conserved along solutions:

$$(2.19) \quad \varphi_0(u) = \frac{1}{2} \int_{\mathbb{T}} |u|^2,$$

$$(2.20) \quad \varphi_1(u) = \frac{1}{2} \int_{\mathbb{T}} |\partial u|^2 - \int_{\mathbb{T}} G(u),$$

with  $G'(r) = F(r)$ ,  $G(0) = 0$ . Indeed, using the fact that  $u_t = -(\partial^3 u + \partial F(u))$  and the fact that  $\partial$  is skew adjoint in  $L^2$ , one obtains

$$\begin{aligned} \frac{d}{dt} \varphi_0(u(t)) &= \int_{\mathbb{T}} u u_t = - \int_{\mathbb{T}} u (\partial^3 u + \partial F(u)) = \int_{\mathbb{T}} \partial u \partial^2 u + \int_{\mathbb{T}} F(u) \partial u \\ &= \int_{\mathbb{T}} \partial \left[ \frac{1}{2} (\partial u)^2 + G(u) \right] = 0, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} \varphi_1(u(t)) &= \int_{\mathbb{T}} \partial u \partial u_t - \int_{\mathbb{T}} F(u) u_t = - \int_{\mathbb{T}} (\partial^2 u + F(u)) u_t \\ &= \int_{\mathbb{T}} (\partial^2 u + F(u)) \partial (\partial^2 u + F(u)) = 0. \end{aligned}$$

The conservation of the first functional gives a uniform bound on the  $L^2$  norm of the solution. The second conserved functional gives a uniform bound on the  $H^1$ -norm only when the growth condition on the nonlinear function  $F$  is imposed. Thus, in the absence of this growth condition  $(\mathcal{C})$  for  $F$ , one needs to find apriori uniform bounds for solutions. This can be done if the initial condition is sufficiently small in  $H^1$  norm.

Let us consider now the third functional, which in general will not be conserved along smooth solutions. We will show how the value of the constant  $\gamma = \frac{5}{6}$  appears. Thus consider

$$\varphi_2(u) = \frac{1}{2} \int_{\mathbb{T}} (\partial^2 u)^2 + \gamma \int_{\mathbb{T}} \partial^2 u F(u) + \gamma \int_{\mathbb{T}} I(u)$$

where  $I''(u) = (F'(u))^2$ ,  $I(0) = I'(0) = 0$ . From the computation below it will be transparent that if  $F'(u) = u$  (corresponding to (KdV)) or  $F'(u) = u^2$ , (for (KdV<sub>2</sub>)), or even if  $F'(u)$  is a general quadratic function, then  $\varphi_2$  is indeed conserved, and belongs to the list of infinitely many conserved functionals for these

integrable equations.

$$\begin{aligned}
\frac{d}{dt}\varphi_2(u(t)) &= \int_{\mathbb{T}} \partial^2 u \partial^2 u_t + \gamma \int_{\mathbb{T}} \partial^2 u_t F(u) + \gamma \int_{\mathbb{T}} \partial^2 u F'(u) u_t + \gamma \int_{\mathbb{T}} I'(u) u_t \\
&= \int_{\mathbb{T}} \partial^3 u \partial(\partial^3 u + \partial F(u)) - \gamma \int_{\mathbb{T}} \partial^2 F(u) (\partial^3 u + \partial F(u)) \\
&\quad - \gamma \int_{\mathbb{T}} \partial^2 u F'(u) (\partial^3 u + \partial F(u)) - \gamma \int_{\mathbb{T}} I'(u) (\partial^3 u + \partial F(u)) \\
&= \int_{\mathbb{T}} \partial^3 u \partial^4 u + (1-\gamma) \int_{\mathbb{T}} \partial^3 u \partial^2 F(u) - \gamma \int_{\mathbb{T}} \partial^2 F(u) \partial F(u) \\
&\quad - \gamma \int_{\mathbb{T}} \partial^2 u F'(u) \partial^3 u - \gamma \int_{\mathbb{T}} (F'(u))^2 \partial u \partial^2 u - \gamma \int_{\mathbb{T}} I'(u) \partial^3 u \\
&\quad - \gamma \int_{\mathbb{T}} I'(u) F'(u) \partial u.
\end{aligned}$$

Note that we made use of the following facts:  $I'' = (F')^2$ ,  $\int_{\mathbb{T}} \partial^3 u \partial^4 u = 0$ ,  $\int_{\mathbb{T}} \partial^2 F(u) \partial F(u) = 0$  and  $\int_{\mathbb{T}} I'(u) F'(u) \partial u = \int_{\mathbb{T}} \partial J(u) = 0$ , where  $J(r) := \int_0^r I'(s) F'(s) ds$ .

Thus,

$$\begin{aligned}
\frac{d}{dt}\varphi_2(u(t)) &= (1-\gamma) \int_{\mathbb{T}} \partial^3 u \partial^2 F(u) - \gamma \int_{\mathbb{T}} \partial^2 u F'(u) \partial^3 u \\
&= (1-\gamma) \int_{\mathbb{T}} \partial^3 u \partial(F'(u) \partial u) - \gamma \int_{\mathbb{T}} \partial^2 u F'(u) \partial^3 u \\
&= (1-2\gamma) \int_{\mathbb{T}} F'(u) \partial^2 u \partial^3 u + (1-\gamma) \int_{\mathbb{T}} F''(u) (\partial u)^2 \partial^3 u \\
&= -\frac{1-2\gamma}{2} \int_{\mathbb{T}} F''(u) \partial u (\partial^2 u)^2 - 2(1-\gamma) \int_{\mathbb{T}} F''(u) \partial u (\partial^2 u)^2 \\
&\quad - (1-\gamma) \int_{\mathbb{T}} F'''(u) (\partial u)^3 \partial^2 u.
\end{aligned}$$

By the choice of  $\gamma = \frac{5}{6}$  we have the obvious equality  $-\frac{1-2\gamma}{2} - 2(1-\gamma) = 0$ , thus we conclude, after one integration by parts,

$$(2.21) \quad \frac{d}{dt}\varphi_2(u(t)) = -\frac{1}{6} \int_{\mathbb{T}} F'''(u) (\partial u)^3 \partial^2 u = \frac{1}{4!} \int_{\mathbb{T}} F^{(IV)}(u) (\partial u)^5.$$

Note that for the special equations (KdV) and (KdV<sub>2</sub>), we have  $F^{(IV)}(r) = 0$ . This also holds true for any  $F$  which is a polynomial of degree  $\leq 3$ . Thus, in all these cases, the functional  $\varphi_2$  is conserved.

For the general nonlinearity, using the Gagliardo-Nirenberg inequality (2.9) for  $w \in H^1$ ,  $2 \leq r \leq \infty$ , we have

$$|w|_{L^r} \leq C |w|_{L^2}^{\frac{1}{2} + \frac{1}{r}} |\partial w|_{L^2}^{\frac{1}{2} - \frac{1}{r}}$$

(see (2.9)), therefore we can estimate the last integral in (2.21) to obtain

$$\frac{d}{dt}\varphi_2(u(t)) \leq C |F^{(IV)}(u)|_{\infty} |\partial u|^{7/2} |\partial^2 u|^{3/2}.$$

It is clear from the definition of  $\varphi_2$  that if the  $H^1$ -norm of  $u(t)$  is uniformly bounded (apriori), then  $|\partial^2 u|^2 \leq c + \varphi_2(u(t))$ ,  $c$  depending on a bound for  $|u|_{H^1}$ .

This implies

$$\frac{d}{dt}\varphi_2(u(t)) \leq C(\varphi_2(u(t)))^{1-1/4}.$$

Note that the constant  $C$  (with or without indices) is allowed to change from line to line. It is easy to obtain now the estimate

$$\varphi_2(u(t)) \leq Ct^4 \text{ for } t \rightarrow \infty,$$

for some constant  $C$  depending only on  $|u|_{H^1}$ . Returning to the  $H^2$ -norm, we finally get

$$\|u(t)\|_2 \leq Ct^2 \text{ for } t \rightarrow \infty.$$

By interpolation inequalities for Sobolev norms, we obtain (with different constants)

$$\|u(t)\|_s \leq Ct^s \text{ for } t \rightarrow \infty$$

( $1 < s \leq 2$ ), as long as we have an *a priori* uniform bound on the  $H^1$ -norm of  $u(t)$ .

For  $\varphi_3(u) = \frac{1}{2} \int_{\mathbb{T}} |\partial^3 u + \partial F(u)|^2$  we have ( $u$  is a real solution of (2.18)):

$$\begin{aligned} \frac{d}{dt}\varphi_3(u(t)) &= \int_{\mathbb{T}} (\partial^3 u + \partial F(u))(\partial^3 u_t + \partial F(u)_t) \\ &= - \int_{\mathbb{T}} u_t \partial^3 u_t + \int_{\mathbb{T}} (\partial^3 u + \partial F(u)) \partial F(u)_t \\ &= \int_{\mathbb{T}} (\partial^3 u + \partial F(u)) \partial(F'(u)u_t) \\ &= - \int_{\mathbb{T}} (\partial^4 u + \partial^2 F(u)) F'(u) u_t \\ &= \int_{\mathbb{T}} \partial(\partial^3 u + \partial F(u)) F'(u) (\partial^3 u + \partial F(u)) \\ &= \frac{1}{2} \int_{\mathbb{T}} \partial(\partial^3 u + \partial F(u))^2 F'(u) \\ &= -\frac{1}{2} \int_{\mathbb{T}} (\partial^3 u + \partial F(u))^2 F''(u) \partial u \\ &\leq \frac{1}{2} |F''(u) \partial u|_{\infty} \int_{\mathbb{T}} |\partial^3 u + \partial F(u)|^2 \\ &\leq \omega_0 \varphi_3(u), \end{aligned}$$

where  $\omega_0$  depends only on a bound on  $|u|_s$ , for  $s > \frac{3}{2}$ .

Thus we obtain

$$(2.22) \quad \varphi_3(u(t)) \leq e^{\omega_0 t} \varphi_3(u_0), \text{ for all } t \geq 0.$$

for which  $|u|_s$  has the assumed bound.

A few remarks deserve to be made about previously known results. In [20], [67], [68] the authors prove that the  $H^s$  norm of solutions to (GKdV) have polynomial growth assuming a uniform bound for the  $H^1$  norm. In the real line case the results are sharper than in the periodic case. In the first case, the growth is like  $t^{s-1}$ , in the latter case, the polynomial growth is only proven to be like  $t^{2s}$ , and the proof is only for (KdV $_k$ ),  $k = 1, 2, 3$ . So it seems that our estimate is a slight improvement in the periodic case. And although the result may not be optimal, the method to

obtain it is very elementary, compared to the methods used in the cited articles.

We now turn to the case  $\beta = \frac{1}{2}$  and obtain apriori estimates for the functionals introduced in Section 2.2. This part corresponds to the (generalized) Benjamin-Ono equation. We follow the ideas in Ginibre and Velo [38].

Thus, consider  $u$  to be a real valued, smooth solution of the initial value problem

$$(2.23) \quad \begin{aligned} \frac{\partial u}{\partial t} - D\partial u + \partial F(u) &= 0, \\ u(0) &= u_0. \end{aligned}$$

Here  $D = H\partial = |\partial|$ , where  $H$  is the well-known Hilbert transform. In Fourier space  $H$  is defined by the multiplication operator

$$\widehat{Hv}(\xi) = -i \frac{\xi}{|\xi|} \widehat{v}(\xi).$$

The Hilbert transform is an isometry on  $L^2$  and  $H^2 = -I$ . Moreover, the following Leibnitz formula holds (with  $v_1, v_2, v_3$  real valued):

$$(2.24) \quad \langle Hv_1, v_2 H v_3 \rangle + \langle H v_3, v_1 H v_2 \rangle + \langle H v_2, v_3 H v_1 \rangle = \langle v_1, v_2 v_3 \rangle.$$

This is equivalent to the following property of the Hilbert Transform:

$$(2.25) \quad H(v_1 H v_2 + v_2 H v_1) = H v_1 H v_2 - v_1 v_2$$

for any  $v_1, v_2 \in L^2$ , which can easily be verified in the Fourier space.

As in the case  $\beta = 1$ , the first two functionals, corresponding to  $\beta = \frac{1}{2}$ ,

$$\begin{aligned} \varphi_0(u) &= \frac{1}{2} \int_{\mathbb{T}} |u|^2, \\ \varphi_1(u) &= \frac{1}{2} \int_{\mathbb{T}} |D^{1/2} u|^2 - \int_{\mathbb{T}} G(u), \end{aligned}$$

are invariant under the flow generated by (2.23). This is obtained by scalar multiplication of (2.23) with  $u$  and, respectively, with  $Du - F(u)$ . Thus, we shall concentrate here on estimating the third functional, which (for  $\beta = \frac{1}{2}$ ) has the form

$$\varphi_2(u) = \frac{1}{2} \int_{\mathbb{T}} |Du|^2 - \frac{3}{4} \int_{\mathbb{T}} F(u) Du + \frac{3}{4} \int_{\mathbb{T}} I(u).$$

Here  $I(u)$  is defined in the same way as before:  $I'' = (F')^2$ ,  $I(0) = I'(0) = 0$ . In the sequel, denote  $\gamma = \frac{3}{4}$ . In a similar manner to the case  $\beta = 1$ , we obtain

$$\begin{aligned} \frac{d}{dt} \varphi_2(u) &= \int_{\mathbb{T}} Du Du_t - \gamma \int_{\mathbb{T}} F'(u) Du u_t - \gamma \int_{\mathbb{T}} F(u) Du_t + \gamma \int_{\mathbb{T}} I'(u) u_t \\ &= \int_{\mathbb{T}} D^2 u (D\partial u - \partial F(u)) - \gamma \int_{\mathbb{T}} F'(u) Du (D\partial u - \partial F(u)) \\ &\quad - \gamma \int_{\mathbb{T}} DF(u) (D\partial u - \partial F(u)) + \gamma \int_{\mathbb{T}} I'(u) \partial (Du - F(u)) \\ &= - \int_{\mathbb{T}} D^2 u \partial F(u) - \gamma \int_{\mathbb{T}} F'(u) Du D\partial u + \gamma \int_{\mathbb{T}} (F'(u))^2 Du \partial u \\ &\quad - \gamma \int_{\mathbb{T}} DF(u) D\partial u - \gamma \int_{\mathbb{T}} I''(u) \partial u (Du - F(u)) \\ &= (1 - \gamma) \int_{\mathbb{T}} DF(u) D\partial u - \gamma \int_{\mathbb{T}} F'(u) Du D\partial u. \end{aligned}$$

We used the fact that  $u$  is real-valued solution,  $D$  is self-adjoint and  $\partial$  is skew-adjoint in  $L^2$ . It then follows that

$$\begin{aligned} \frac{d}{dt}\varphi_2(u) &= (1-\gamma)(DF(u), D\partial u) - \gamma(F'(u)Du, D\partial u) \\ &= -(1-\gamma)(D\partial F(u), Du) + (1-\gamma)(F'(u)Du, D\partial u) - (F'(u)D\partial u, Du) \\ &= -(1-\gamma)[(D\partial F(u), Du) - (F'(u)Du, D\partial u)] + \frac{1}{2}(F''(u)\partial u Du, Du) \\ &= -(1-\gamma)[(D\partial F(u), Du) - (F'(u)D\partial u, Du) - 2(F''(u)\partial u Du, Du)], \end{aligned}$$

because  $2(1-\gamma) = \frac{1}{2}$ . Now, using the fact that  $D = H\partial$ , we evaluate the quantity  $(D\partial F(u), Du) - (F'(u)D\partial u, Du) - 2(F''(u)\partial u Du, Du) =$

$$\begin{aligned} &= -(H\partial F(u), H\partial^2 u) - (F'(u)H\partial^2 u, H\partial u) - 2(F''(u)\partial u H\partial u, H\partial u) \\ &= -(F'(u)\partial u, \partial^2 u) - \frac{1}{2}(F'(u), \partial(H\partial u)^2) - 2(F''(u)\partial u H\partial u, H\partial u) \\ &= \frac{1}{2}(F''(u)\partial u \partial u, \partial u) - \frac{3}{2}(F''(u)\partial u H\partial u, H\partial u) \\ &= \frac{1}{2}[2(H(F''(u)\partial u)\partial u, H\partial u) + (F''(u)\partial u H\partial u, H\partial u)] - \frac{3}{2}(F''(u)\partial u H\partial u, H\partial u) \\ &= (H(F''(u)\partial u)\partial u, H\partial u) - (F''(u)H\partial u, \partial u H\partial u) \\ &= ([H, F''(u)]\partial u, \partial u H\partial u), \end{aligned}$$

where  $[H, F''(u)]$  is the commutator of the operators  $H$  and the operator given by multiplication with  $F''(u)$ . We have also used the Leibnitz rule (2.24), applied to  $v_1 = F''(u)\partial u, v_2 = v_3 = \partial u$ .

Thus, we obtain

$$\frac{d}{dt}\varphi_2(u) = ([H, F''(u)]\partial u, \partial u H\partial u).$$

Clearly, if  $F''(u) = \text{constant}$ , that is if  $F'(u)$  is linear in  $u$  (the classical Benjamin-Ono equation), then  $[H, F''(u)] = 0$ , therefore  $\frac{d}{dt}\varphi_2(u) = 0$ , i.e.  $\varphi_2(u)$  is constant along solutions.

For general nonlinearities, the estimates are much weaker. We need the following commutator estimate (see [38]):

$$|[H, F''(u)]\partial u|_{L^\infty} \leq c|\partial u|_{L^2}^2,$$

where  $c$  depends only on a bound for  $|F'''(u)|_{L^\infty}$ . Making the extra assumption that  $F'''(\cdot) \in L^\infty$ , we conclude that

$$\frac{d}{dt}\varphi_2(u) \leq c|\partial u|_{L^2}^4,$$

which then implies

$$(2.26) \quad \frac{d}{dt}\varphi_2(u) \leq c_1(\varphi_2(u) + c_2)^2$$

with  $c_1, c_2$  depending only on a bound for  $|u|_{H^{1/2}}$ .

The inequality (2.26) is a result of the following estimates (see also Lemma 4.4), given in terms of  $|\partial u|_{L^2}^2 = |Du|_{L^2}^2$ , for each expression appearing in  $\varphi_2(u)$ .

$$\begin{aligned} |(F(u), Du)| &\leq |F(u)|_{L^2} |Du|_{L^2} \\ &\leq \left( C_1 |u|_{L^2} + C_2 |u|_{L^{p+1}}^{p+1} \right) |Du|_{L^2} \\ &\leq C(|D^{1/2}u|) |Du|_{L^2} \end{aligned}$$

and

$$\begin{aligned} I(u) &\leq C_1 + C_2 |u|_{L^{2p+2}}^{2p+2} \\ &\leq C(|D^{1/2}u|). \end{aligned}$$

Thus, from (2.26), we get an estimate of the form

$$\varphi_2(u(t)) \leq \frac{K_0}{T_0 - t} \text{ for } 0 \leq t < T_0$$

where  $K_0, T_0$  depend only on a bound for  $\varphi_2(u_0)$  (or, equivalently, on a bound for  $|u_0|_{H^1}$ ).

Thus global existence of solutions for the Benjamin-Ono equation (2.23) is guaranteed only in the case  $F(u)$  is a quadratic function of  $u$ . For general nonlinearity, the life-span of the solution depends on the size of the initial data. A related analysis (but with completely different tools) has been carried out in [53], where it has been shown that, at least when  $F$  is a polynomial, global existence of solutions holds for small initial data.

### 3. SEMILINEAR HILLE-YOSIDA THEORY

The abstract semilinear Hille-Yosida theory has been developed in [61], [62], and adapted in [43] for some situations not covered by the standard Crandall-Liggett theory. Here is the setting of the abstract theorem.

Let  $X$  be a real Hilbert space with norm  $\|\cdot\|$  and inner product  $(\cdot, \cdot)$ . In the applications we may consider a complex Hilbert space  $(H, \langle \cdot, \cdot \rangle)$  to be a real one  $(H, (\cdot, \cdot))$  by taking  $(x, y) = \operatorname{Re} \langle x, y \rangle$ .

$C$  is a subset of  $X$ ,  $u_0 \in C$ .  $\varphi = (\varphi_1, \dots, \varphi_N)$  are  $N$  lower semicontinuous functionals on  $X$ , with  $D(\varphi_k) = C$ , for  $k = 1, \dots, N$ . For  $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ , define  $C_\alpha = \{u \in C \mid \varphi_k(u) \leq \alpha_k \text{ for } k = 1, \dots, N\}$ .

Of concern is the abstract Cauchy problem in  $X$

$$(3.1) \quad \begin{aligned} \frac{du}{dt} &= Au + B(u), \\ u(0) &= u_0. \end{aligned}$$

Throughout the rest of this chapter we make the following assumptions.

(A1)  $A : D(A) \subset X \rightarrow X$  is the linear generator of a  $C_0$ -semigroup  $\{T(t)\}_{t \geq 0}$  with

$$\|T(t)\| \leq e^{\omega_0 t} \text{ for some } \omega_0 \in \mathbb{R}.$$

(A2) In addition, assume that  $\overline{D(A)} \cap \overline{C_\alpha} \subset C$ , for all  $\alpha$ .

$B : C \rightarrow X$  is a nonlinear operator satisfying:

(B1)  $B$  is weakly locally sequentially continuous, i.e.

$$(3.2) \quad \{u_n\} \subset C_\alpha \text{ for some } \alpha > 0 \text{ and } u_n \rightarrow u \text{ implies } Bu_n \rightharpoonup Bu.$$

(Here  $\rightarrow$  [resp.  $\rightharpoonup$ ] means norm [resp. weak] convergence in  $X$ .)

(B2)  $B$  is locally quasi-dissipative in the sense that:

For all  $\alpha > 0$ , there exists  $\omega_\alpha$  such that

$$(3.3) \quad (Bu - Bv, u - v) \leq \omega_\alpha |u - v|^2, \text{ for all } u, v \in C_\alpha$$

Then the following theorem holds.

**Theorem 3.1.** *Let  $a_k, b_k \geq 0$  be fixed numbers,  $k \in 1, \dots, N$ . Assume the following condition holds true:*

*For  $v \in C, \varepsilon > 0$ , there exists  $0 < \lambda_0 = \lambda_0(v, \varepsilon)$  such that  
for all  $0 < \lambda < \lambda_0$ , there exists  $u = u_\lambda \in D(A) \cap C$  satisfying*

$$(3.4) \quad |u - \lambda Au - \lambda B(u) - v| < \lambda \varepsilon,$$

$$(3.5) \quad \varphi_k(u) \leq \frac{\varphi_k(v) + (b_k + \varepsilon)\lambda}{1 - a_k \lambda}, \quad k = 1, \dots, N.$$

*Then there exists a nonlinear semigroup  $\{S(t)\}_{t \geq 0}$  of continuous mappings from  $C$  to  $C$  with the following properties:*

$$(3.6) \quad S(t + s) = S(t)S(s), S(0) = I,$$

$$(3.7) \quad S(t)u_0 = T(t)u_0 + \int_0^t T(t-s)B(S(s)u_0)ds,$$

$$(3.8) \quad \varphi_k(S(t)u_0) \leq e^{a_k t}(\varphi_k(u_0) + b_k t), \quad t \geq 0, \quad k = 0 \dots N - 1,$$

*and the mapping  $u_0 \mapsto u(\cdot) = S(\cdot)u_0$  is continuous from  $C$  into  $C(\mathbb{R}_+, X)$  and locally Lipschitzian for  $u_0 \in D(A)$ , i.e.*

$$|u(t) - v(t)| \leq e^{\Omega t} |u_0 - v_0|, \text{ for some } \Omega = \Omega(T, \alpha)$$

*whenever  $u_0, v_0 \in D(A) \cap C_\alpha, t \in [0, T]$ .*

Note that the growth condition (3.8) has as special cases linear growth (when  $a_k = 0$ ) as well as exponential growth (when  $b_k = 0$ ). The proof is given in a slightly more general setting in the next section. The theorem actually holds if  $X$  is a general Banach space. In this case, (3.3) should be replaced by:

$$B - \omega_\alpha I \text{ is dissipative on } C_\alpha.$$

Condition (3.3) is correctly stated if  $(\cdot, \cdot)$  is interpreted as a semi-inner product (in the sense of Lumer) on  $X$ , i.e., for  $x, y \in X$  :

$$(3.9) \quad (x, y) = \operatorname{Re} \phi(x), \text{ where } \phi \in X^* \text{ is such that } \phi(y) = \|y\|_X^2 = \|\phi\|_{X^*}^2.$$

This fact is basically contained (but not proved) in [41], [43]. For a formulation of our result in general Banach spaces with  $N$  functionals, see Remark 3.9. In the next section, we give the proof of our theorem, generalizing the estimates on the functional  $\varphi_j$ .

**3.1. Abstract theorem. Existence of solutions.** Of concern is the abstract Cauchy problem in  $X$

$$(3.10) \quad \begin{aligned} \frac{du}{dt} &= Au + B(u), \\ u(0) &= x_0 \in X. \end{aligned}$$

We assume all the hypothesis on  $A, B, C$  and  $\varphi$  from the preceding section, i.e. (A1),(A2), (B1) and (B2), hold throughout this chapter.



Let  $g = g(r) > 0$  be a  $C^1$  increasing function and  $m = m(t, \alpha)$  the maximal solution of the initial value problem

$$(3.11) \quad \begin{aligned} m'(t) &= g(m(t)), \quad t \geq 0, \\ m(0) &= \alpha, \end{aligned}$$

where  $\alpha \in \mathbb{R}$ . For the sake of simplicity, we will assume that the maximal solution is defined for all  $t \geq 0$ . Otherwise, when we say "for all  $t \geq 0$ " we mean "for all  $t \in [0, T_{max})$ ". When necessary, we will write  $m(t, \alpha) = m_g(t, \alpha)$ .

Here is the main abstract theorem, formulated for a single functional  $\varphi$  ( $N = 1$ ). The general case is very similar and will be explained later.

**Theorem 3.2.** *Assume that, for each  $x_0 \in C$ , the following hypothesis holds:*

$$(\mathcal{H}) \quad \left\{ \begin{array}{l} \text{For every } \varepsilon > 0, \text{ there exists } \delta = \delta(x_0, \varepsilon) \text{ such that} \\ \text{for all } h \in (0, \delta), \text{ there exists } x_h \in D(A) \cap C \text{ satisfying} \\ |x_h - x_0 - h(Ax_h + B(x_h))| \leq \varepsilon h, \\ \frac{1}{h}[\varphi(x_h) - \varphi(x_0)] \leq g(\varphi(x_h)) + \varepsilon. \end{array} \right.$$

Then there exists a nonlinear semigroup  $\{S(t) : C \rightarrow C\}_{t \geq 0}$  of continuous operators on  $C$  with the following properties:

$$(3.12) \quad S(t)x_0 = T(t)x_0 + \int_0^t T(t-s)BS(s)x_0 ds,$$

$$(3.13) \quad \varphi(S(t)x_0) \leq m(t, \varphi(x_0)),$$

for all  $x_0 \in C$  and all  $t \geq 0$ .

Most of Theorem 3.1 is obtained from Theorem 3.2 when we use  $N$  functions  $g_k(r) = a_k r + b_k$ , for  $k = 1, \dots, N$ . Extra arguments are needed for the conclusion involving local Lipschitz conditions.

We will organize the proof of the theorem in several lemmas. The ideas come from the work of Kobayashi, [55], and consist of constructing, for each  $x_0 \in C$ , the map  $t \rightarrow S(t)x_0$  as the limit of a sequence of approximations  $\{u_n(t)\}_n$ , where  $u_n(t)$  are step functions constructed by means of an implicit difference scheme. In the first subsection (Lemma 3.3), we show that such approximations exist, then, in subsection 3.3 (Lemma 3.6), that the approximations converge, and finally that the limit satisfies (3.12) and (3.13). Ideas similar to [55] are contained in Evans [33].

**3.2. Approximate solutions.** Let  $x_0 \in C_{r_0}$  (for some fixed  $r_0$ ) and  $T > 0$  be fixed. Choose also  $\eta > 1$  (close to 1) and  $R = R(r_0, T)$  such that

$$R > m_{\eta g}(r_0, T) = m_g(\eta r_0, \eta T).$$

Denote  $\omega = \omega_R$  the dissipativity constant for the operator  $A + B_R$  on  $C_R$ . Throughout this section,  $B_R$  is the restriction of  $B$  to  $C_R$ . Let  $l$  be a lower bound for  $\varphi$  on  $C_R$ . Let  $K = \max\{\omega, \sup_{l \leq m \leq R} g'(m)\}$  and denote  $\hat{\varepsilon} = \frac{\eta-1}{\eta K}$ .

**Lemma 3.3.** *Assume the hypothesis  $(\mathcal{H})$  holds. Then, for  $x_0 \in C_{r_0}$  and  $0 < \varepsilon < \widehat{\varepsilon}$ , the family*

$$\Gamma_{x_0, \varepsilon} := \left\{ (x_k, h_k)_{k=1}^n \mid (x_k)_{k=1}^n \subset D(A) \cap C_R, 0 < h_k < \varepsilon, \sum_{k=1}^{n-1} h_k < T, \sum_{k=1}^n h_k \geq T, \right.$$

$$(3.14) \quad |x_k - x_{k-1} - h_k(Ax_k + B(x_k))| \leq \varepsilon h_k,$$

$$(3.15) \quad \frac{1}{h_k} [\varphi(x_k) - \varphi(x_{k-1})] \leq g(\varphi(x_k)) + \varepsilon, \text{ for } k = 1, \dots, n \}$$

is nonempty.

*Proof.* The existence of a sequence  $(x_k, h_k)$  satisfying the requirements above follows inductively, using the hypothesis  $(\mathcal{H})$ , which can be conveniently written as a "tangential condition":

For all  $\varepsilon > 0$ ,  $x \in C_r$  such that there exists  $\delta > 0$ , such that

$$(3.16) \quad h \in (0, \delta) \text{ implies } \text{dist}(x, \text{Range}(I - h(A + B_R))) < \varepsilon h,$$

or,

$$(3.17) \quad \lim_{h \rightarrow 0} \frac{1}{h} \text{dist}(x, \text{Range}(I - h(A + B_R))) = 0, \text{ for all } x \in C_r,$$

where  $r < R$  must satisfy  $m_{\eta g_\varepsilon}(r, T) < R$  (see Proposition 3.4 below).

Let  $\varepsilon < \widehat{\varepsilon}$  be fixed and denote  $\bar{\delta} = \bar{\delta}(x, \varepsilon)$  the supremum of  $\delta$ ,  $\delta < \varepsilon$ , satisfying (3.16).

The induction argument is as follows. Assuming we have generated  $x_i \in C_R$ ,  $i = 1, \dots, k-1$ ,  $\sum_{i=1}^{k-1} h_i < T$ , we can find  $\bar{\delta} = \bar{\delta}(x_{k-1}, \varepsilon)$ ,  $h_k \in (0, \bar{\delta})$  and  $x_k = x_{h_k}$  as in  $(\mathcal{H})$ . We choose  $h_k$ , in each step, such that

$$(3.18) \quad \frac{1}{2} \bar{\delta}(x_{k-1}, \varepsilon) < h_k < \varepsilon.$$

Our first claim is that  $x_k \in C_R$  for as long as  $\sum_{i=1}^{k-1} h_i < T$ . Indeed, we can prove that

$$(3.19) \quad \varphi(x_i) \leq m_{\eta g_\varepsilon}(\varphi(x_{i-1}), h_i), \text{ for all } i = 1, \dots, k;$$

here  $g_\varepsilon := g + \varepsilon$ . This follows from the following

**Proposition 3.4.**

$$(3.20) \quad (I - hg)^{-1}z \leq m_{\eta g}(z, h), \text{ for any } z < R, h < \widehat{\varepsilon}.$$

To prove the above inequality, it is enough to rewrite it as

$$z \leq m_{\eta g}(z, h) - hg(m_{\eta g}(z, h)) =: \Phi(h).$$

Recall that  $g$  is increasing function. Then  $\Phi'(h) = \eta g(m) - g(m) - hg'(m)\eta g(m) \geq (\eta - 1 - hK\eta)g(m) \geq 0$ , for  $h < \widehat{\varepsilon}$ , which implies  $\Phi(h) \geq \Phi(0) = z$ , if  $0 < h < \widehat{\varepsilon}$ . This proves the claim of Proposition 3.4.

Using (3.20) with  $g_\varepsilon = g + \varepsilon$  instead of  $g$  and the fact that  $\varphi(x_i) \leq (I - h_i g_\varepsilon)^{-1} \varphi(x_{i-1})$  (by construction, see (3.15)) we get

$$\begin{aligned} \varphi(x_i) &\leq (I - h_i g_\varepsilon)^{-1} \varphi(x_{i-1}) \\ &\leq m_{\eta g_\varepsilon}(\varphi(x_{i-1}), h_i) \end{aligned}$$

and therefore (3.19) holds. Iterating (3.19), we obtain

$$\begin{aligned}
\varphi(x_k) &\leq m_{\eta g_\varepsilon}(\varphi(x_{k-1}), h_k) \\
&\leq m_{\eta g_\varepsilon}(m_{\eta g_\varepsilon}(\varphi(x_{k-2}), h_{k-1}), h_k) = m_{\eta g_\varepsilon}(\varphi(x_{k-2}), h_{k-1} + h_k) \\
&\quad \dots \\
&\leq m_{\eta g_\varepsilon}(\varphi(x_0), \sum_{i=1}^k h_i) \\
&\leq m_{\eta g_\varepsilon}(r_0, T) \leq m_{\eta g_\varepsilon}(r_0, T) < R.
\end{aligned}$$

As mentioned above, this guarantees the induction can continue until  $\sum_{k=1}^n h_k \geq T$  for some  $n$ . We claim that this is indeed reached in a finite number of steps, which is the conclusion of Lemma 3.3. We argue by contradiction. Suppose that we can indefinitely generate  $(x_i, h_i)$ , with  $\sum_{i=1}^k h_i < T$ , for every integer  $k \geq 1$ . Then  $s_0 := \sum_{k=1}^{\infty} h_k \leq T < \infty$ . Without loss of generality, we can assume  $x_0 \in \mathcal{D}(A)$ . We will reach a contradiction by showing that the sequence  $(x_k) \subset C_R$  has a limit point  $x^* \in C_R$  which fails to satisfy the hypothesis  $(\mathcal{H})$ .

For convenience, we introduce the following notation:

$$(3.21) \quad t_n := \sum_{k=1}^n h_k, \quad \gamma_n := \prod_{k=1}^n (1 - \omega h_k) \text{ for all } n \geq 1,$$

where  $\omega = \omega_R$  is the dissipativity constant for the operator  $\mathcal{B} := A + B_R$  on  $C_R$ . We make the conventions  $t_0 := 0$  and  $\gamma_0 := 1$ .

It is worth noting that our previous computations, applied in the particular case  $g(r) = \omega r$ , lead us to the estimates:

$$(3.22) \quad 1 - \omega h_k \leq e^{\eta \omega h_k},$$

$$(3.23) \quad \prod_{i=1}^k (1 - \omega h_i) \leq e^{\eta \omega t_k},$$

because of our choice of  $h_k < \hat{\varepsilon}$ , for all  $k \geq 1$ .

The next step is to prove by induction that for all  $i \geq j \geq 0$ ,

$$(3.24) \quad |x_i - x_j| \leq e^{\eta \omega (t_i + t_j)} [(t_i - t_j) |\mathcal{B}x_0| + \varepsilon (t_i + t_j)].$$

But before doing this, we need the following proposition (cf. Kobayashi [55]).

**Proposition 3.5.** *Let  $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subset X \rightarrow X$  be such that  $A - \omega I$  is dissipative.*

(i) *For all  $\lambda, \mu > 0$ ,  $x_\lambda, x_\mu \in \mathcal{D}(\mathcal{B})$ ,*

$$(3.25) \quad (\lambda + \mu - \omega \lambda \mu) |x_\lambda - x_\mu| \leq \lambda |x_\mu - x_\lambda - \mu \mathcal{B}x_\mu| + \mu |x_\lambda - x_\mu - \lambda \mathcal{B}x_\lambda|.$$

(ii) *For all  $\lambda > 0$ ,  $x, u \in \mathcal{D}(\mathcal{B})$ ,*

$$(3.26) \quad (1 - \lambda \omega) |x - u| \leq |x - u - \lambda \mathcal{B}x| + \lambda |\mathcal{B}u|.$$

*Proof.* We have

$$\begin{aligned}
(\lambda + \mu) |x_\lambda - x_\mu|^2 &= \lambda |x_\lambda - x_\mu|^2 + \mu |x_\lambda - x_\mu|^2 \\
&= \lambda \langle x_\mu - x_\lambda - \mu \mathcal{B}x_\mu, x_\mu - x_\lambda \rangle \\
&\quad + \mu \langle x_\lambda - x_\mu - \lambda \mathcal{B}x_\lambda, x_\lambda - x_\mu \rangle \\
&\quad + \lambda \mu \langle \mathcal{B}x_\lambda - \mathcal{B}x_\mu, x_\lambda - x_\mu \rangle \\
&\leq (\lambda |x_\mu - x_\lambda - \mu \mathcal{B}x_\mu| + \mu |x_\lambda - x_\mu - \lambda \mathcal{B}x_\lambda| + \lambda \mu \omega) \\
&\quad \cdot |x_\lambda - x_\mu|,
\end{aligned}$$

which implies (i).

To prove (ii), let  $\lambda = \mu$  and  $x_\lambda = x, x_\mu = u$  in (i):

$$\begin{aligned}
(2 - \omega\lambda) |x - u| &\leq |u - x - \lambda \mathcal{B}u| + |x - u - \lambda \mathcal{B}x| \\
&\leq |x - u| + \lambda |\mathcal{B}u| + |x - u - \lambda \mathcal{B}x|
\end{aligned}$$

from which it follows

$$(1 - \lambda\omega) |x - u| \leq \lambda |\mathcal{B}u| + |x - u - \lambda \mathcal{B}x|,$$

concluding the proof of the proposition.  $\square$

Returning to the proof of Lemma 3.3, our goal is to prove the estimate (3.24) on  $|x_i - x_j|$ . To reach this goal, we make use of the inequalities (3.22), (3.23) and prove instead the following estimate, for all  $i \geq j \geq 0$ :

$$(3.27) \quad a_{i,j} := \gamma_i \gamma_j |x_i - x_j| \leq (t_i - t_j) |\mathcal{B}x_0| + \varepsilon(t_i + t_j)$$

We show the above inequality by double induction in  $(i, j)$ . First, let  $j = 0$ . Then

$$\begin{aligned}
(1 - \omega h_i) |x_i - x_0| &\leq |x_i - x_0 - h_i \mathcal{B}x_i| + h_i |\mathcal{B}x_0| \\
&\leq |x_{i-1} - x_0| + |x_i - x_{i-1} - h_i \mathcal{B}x_i| + h_i |\mathcal{B}x_0| \\
&\leq |x_{i-1} - x_0| + h_i |\mathcal{B}x_0| + \varepsilon h_i.
\end{aligned}$$

$$\begin{aligned}
a_{i,0} &= \gamma_i |x_i - x_0| \leq \gamma_{i-1} |x_{i-1} - x_0| + \gamma_{i-1} (h_i |\mathcal{B}x_0| + \varepsilon h_i) \\
&\leq a_{i-1,0} + h_i (|\mathcal{B}x_0| + \varepsilon).
\end{aligned}$$

Iterating this last inequality, we get

$$(3.28) \quad a_{i,0} \leq \left( \sum_{l=1}^i h_l \right) (|\mathcal{B}x_0| + \varepsilon) = t_i |\mathcal{B}x_0| + \varepsilon t_i.$$

This proves our inequality (3.27) for  $j = 0$ .

Obviously,  $a_{i,i} = 0$ , for all  $i \geq 0$ .

**The inductive step:** Let us prove the inequality (3.27) for the pair  $(i, j)$ , where  $i > j > 0$ , assuming it holds for  $(i-1, j)$  and  $(i, j-1)$ .

$$\begin{aligned}
(h_i + h_j - \omega h_i h_j) |x_i - x_j| &\leq h_j |x_i - x_j - h_i \mathcal{B}x_i| + h_i |x_j - x_i - h_j \mathcal{B}x_j| \\
&\leq h_j |x_j - x_{i-1}| + h_i |x_i - x_{j-1}| + 2\varepsilon h_i h_j
\end{aligned}$$

Multiply both sides by  $\gamma_i \gamma_j$ :

$$(h_i + h_j - \omega h_i h_j) a_{i,j} \leq h_j (1 - \omega h_i) a_{i-1,j} + h_i (1 - \omega h_j) a_{i,j-1} + 2\varepsilon h_i h_j \gamma_i \gamma_j.$$

From the induction hypothesis, (3.27) holds for  $a_{i-1,j}$  and  $a_{i,j-1}$ . This implies

$$\begin{aligned} (h_i + h_j - \omega h_i h_j) a_{i,j} &\leq h_j(1 - \omega h_i) [(t_{i-1} - t_j) |Ax_0| + \varepsilon(t_{i-1} + t_j)] + \\ &\quad h_i(1 - \omega h_j) [(t_i - t_{j-1}) |Bx_0| + \varepsilon(t_i + t_{j-1})] + \\ &\quad \varepsilon h_i h_j (1 - \omega h_i) + \varepsilon h_i h_j (1 - \omega h_j) \\ &\leq (h_i + h_j - \omega h_i h_j) [(t_i - t_j) |Bx_0| + \varepsilon(t_i + t_j)] \end{aligned}$$

which completes the inductive argument and the proof of the inequality (3.27).

We now use the estimate (3.27), which holds for  $\{(x_k, h_k)\}_{k \geq 1}$ , to obtain a similar estimate for  $\{(x_k, h_k)\}_{k \geq p}$ , for some  $p$ , that is,

$$(3.29) \quad |x_i - x_j| \leq e^{\eta\omega(t_i+t_j-2t_p)} [(t_i - t_j) |Bx_0| + \varepsilon(t_i + t_j - 2t_p)]$$

for every  $i \geq j \geq p$ . As we take  $i, j \rightarrow \infty$ ,  $t_i, t_j \rightarrow s_0$  in (3.29) and we conclude

$$\limsup_{i,j \rightarrow \infty} |x_i - x_j| \leq 2\varepsilon(s_0 - t_p) e^{2\eta\omega(s_0 - t_p)}$$

for all  $p$ . Now take  $p \rightarrow \infty$ ,  $t_p \rightarrow s_0$ , which leads us to the conclusion

$$\lim_{i,j \rightarrow \infty} |x_i - x_j| = 0$$

i.e., the sequence  $\{x_k\}_{k \geq 1}$  is Cauchy, therefore convergent to some  $x^* \in C_R$ .

By the tangential condition, there exists  $\delta^* = \delta(x^*, \frac{\varepsilon}{2})$  such that for all  $0 < \lambda < \delta^*$ , there exists  $\xi_\lambda \in D(A) \cap C$  such that

$$(3.30) \quad |\xi_\lambda - x^* - \lambda(A\xi_\lambda + B(\xi_\lambda))| < \lambda \frac{\varepsilon}{2}.$$

Because  $h_k \rightarrow 0$  (as  $\sum_{k=1}^{\infty} h_k \leq T < \infty$ ), and  $\frac{1}{2}\bar{\delta}(x_k, \varepsilon) \leq h_k$  (by our choice of  $h_k$ , see (3.18)), it follows that  $\bar{\delta}(x_k, \varepsilon) \rightarrow 0$  as  $k \rightarrow \infty$ . Thus there exists  $k^*$  such that  $\bar{\delta}(x_k, \varepsilon) < \frac{\delta^*}{2}$ , for all  $k > k^*$ . This guarantees the existence of an infinite sequence  $(\lambda_k)_k \in [\frac{\delta^*}{2}, \delta^*]$  and  $(\xi_{\lambda_k})_k \in D(A) \cap C_R$  satisfying, for all  $\lambda < \delta^*$ ,

$$|\xi_\lambda - x_k - \lambda_k(A\xi_\lambda + B(\xi_\lambda))| \geq \varepsilon \lambda_k, \text{ for all } k \geq k^*.$$

If  $\lambda^* \in [\frac{\delta^*}{2}, \delta^*]$  is an accumulation point for  $(\lambda_k)_k$ , then we have

$$|\xi_{\lambda^*} - x_k - \lambda_k(A\xi_{\lambda^*} + B(\xi_{\lambda^*}))| \geq \varepsilon \lambda_k, \text{ for all } k \geq k^*,$$

and, in the limit  $k \rightarrow \infty$ ,

$$|\xi_{\lambda^*} - x^* - \lambda^*(A\xi_{\lambda^*} + B(\xi_{\lambda^*}))| \geq \varepsilon \lambda^*,$$

which contradicts (3.30).

The contradiction we reached shows that our assumption made earlier, that we can generate an infinite sequence  $(x_k, h_k)_k$ , is false. Therefore the set  $\Gamma_{x_0, \varepsilon}$  is indeed nonempty. This concludes the proof of Lemma 3.3.  $\square$

Lemma 3.3 allows us to construct an approximate solution to the initial value problem

$$(3.31) \quad \begin{aligned} u'(t) &= Au(t) + B(u(t)), t \geq 0, \\ u(0) &= x_0 \in C_{r_0}. \end{aligned}$$

More precisely, for a finite sequence  $(x_k, h_k)_{k=1}^n \in \Gamma_{x_0, \varepsilon}$ , we define the step function  $u_\varepsilon : [0, T] \rightarrow C_R \cap \mathcal{D}(A)$ , such that

$$(3.32) \quad u_\varepsilon(t) = x_k \text{ if } t \in [t_{k-1}, t_k]$$

for  $k = 1, \dots, n$ . We will prove that for  $\varepsilon = \varepsilon_n \rightarrow 0$ , the sequence  $\{u_\varepsilon(t)\}$  converges uniformly to a continuous function  $u(t)$ , which turns out to be the desired  $S(t)x_0$ , satisfying

$$u(t) = T(t)x_0 + \int_0^t T(t-s)B(u(s))ds$$

for all  $t \in [0, T]$ , i.e.  $u$  is the mild solution for the initial value problem (3.31).

**3.3. Convergence of the approximate solutions.** The convergence of the approximate solutions constructed in the previous section follows from the following estimate.

**Lemma 3.6.** *Let  $0 < \tilde{\varepsilon} < \varepsilon$  and  $\{(x_k, h_k)\}_k \in \Gamma_{x_0, \varepsilon}$  and  $\{(\tilde{x}_j, \tilde{h}_j)\}_j \in \Gamma_{x_0, \tilde{\varepsilon}}$  be finite sequences constructed as in the proof of Lemma 3.3. Then, for all  $u \in D(\mathcal{B})$ ,*

$$(3.33) \quad |x_k - \tilde{x}_j| \leq e^{\eta\omega(t_i + \tilde{t}_j)} \{|x_0 - u| + [(t_k - \tilde{t}_j)^2 + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j]^{\frac{1}{2}} |\mathcal{B}u| + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j\}$$

for all  $k, j$ .

*Proof.* Denote

$$(3.34) \quad a_{k,j} := \gamma_k \tilde{\gamma}_j |x_k - \tilde{x}_j|$$

(see (3.21)). We show that for  $u \in D(\mathcal{B})$ ,  $k, j \geq 0$ ,

$$(3.35) \quad a_{k,j} \leq |x_0 - u| + [(t_k - \tilde{t}_j)^2 + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j]^{\frac{1}{2}} |\mathcal{B}u| + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j.$$

This is done by double induction in  $(k, j)$ .

First, for  $j = 0, k \geq 1$ ,

$$\begin{aligned} a_{k,0} &= \gamma_k |x_k - x_0| \leq \gamma_k |x_0 - u| + \gamma_k |x_k - u| \\ &\leq |x_0 - u| + t_k |\mathcal{B}u| + \varepsilon t_k \text{ (see (3.28)).} \end{aligned}$$

Similarly, for  $k = 0, j \geq 1$ ,

$$(3.36) \quad a_{0,j} \leq |x_0 - u| + \tilde{t}_j |\mathcal{B}u| + \tilde{\varepsilon} \tilde{t}_j.$$

**The inductive step:** Assume (3.35) holds for  $a_{k-1,j}$  and  $a_{k,j-1}$ ; we shall prove it for  $a_{k,j}$  :

$$\begin{aligned} (h_k + \tilde{h}_j - \omega h_k \tilde{h}_j) a_{k,j} &\leq \tilde{h}_j (1 - \omega h_k) a_{k-1,j} + h_k (1 - \omega \tilde{h}_j) a_{k,j-1} + \\ &\quad (\varepsilon + \tilde{\varepsilon}) h_k \tilde{h}_j (1 - \omega h_k) (1 - \omega \tilde{h}_j) \\ &\leq (h_k + \tilde{h}_j - 2\omega h_k \tilde{h}_j) |x_0 - u| + \Lambda_{k,j} |\mathcal{B}u| + \Omega_{k,j,\varepsilon,\tilde{\varepsilon}} \end{aligned}$$

where the coefficient of  $|\mathcal{B}u|$  is

$$\begin{aligned} \Lambda_{k,j} := &\quad \tilde{h}_j (1 - \omega h_k) [(t_{k-1} - \tilde{t}_j)^2 + \varepsilon t_{k-1} + \tilde{\varepsilon} \tilde{t}_j]^{\frac{1}{2}} + h_k (1 - \omega \tilde{h}_j) [(t_k - \tilde{t}_{j-1})^2 \\ &\quad + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_{j-1}]^{\frac{1}{2}} \end{aligned}$$

and the remaining term is

$$\begin{aligned}\Omega_{k,j,\varepsilon,\tilde{\varepsilon}} := & \tilde{h}_j(1 - \omega h_k)(\varepsilon t_{k-1} + \tilde{\varepsilon} \tilde{t}_j) + h_k(1 - \omega \tilde{h}_j)(\varepsilon t_k + \tilde{\varepsilon} \tilde{t}_{j-1}) \\ & + (\varepsilon + \tilde{\varepsilon}) h_k \tilde{h}_j (1 - \omega h_k)(1 - \omega \tilde{h}_j).\end{aligned}$$

We estimate these two quantities. First,

$$\begin{aligned}\Lambda_{k,j} & \leq [\tilde{h}_j(1 - \omega h_k)^2 + h_k(1 - \omega \tilde{h}_j)^2]^{\frac{1}{2}} [(t_{k-1} - \tilde{t}_j)^2 + \varepsilon t_{k-1} + \tilde{\varepsilon} \tilde{t}_j \\ & \quad + (t_k - \tilde{t}_{j-1})^2 + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_{j-1}]^{\frac{1}{2}} \\ & \leq [\tilde{h}_j(1 - \omega h_k)^2 + h_k(1 - \omega \tilde{h}_j)^2]^{\frac{1}{2}} [(t_k - \tilde{t}_j)^2 + 2h_k(t_{k-1} - \tilde{t}_j) + h_k^2 + \varepsilon t_k \\ & \quad + \tilde{\varepsilon} \tilde{t}_j - \varepsilon h_k + (t_k - \tilde{t}_j)^2 + 2\tilde{h}_j(t_k - \tilde{t}_{j-1}) + \tilde{h}_j^2 + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j - \varepsilon \tilde{h}_j]^{\frac{1}{2}} \\ & \leq [\tilde{h}_j(1 - \omega h_k)^2 + h_k(1 - \omega \tilde{h}_j)^2]^{\frac{1}{2}} (h_k + \tilde{h}_j)^{\frac{1}{2}} [(t_k - \tilde{t}_j)^2 + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j]^{\frac{1}{2}} \\ & \leq (h_k + \tilde{h}_j - \omega h_k \tilde{h}_j) [(t_k - \tilde{t}_j)^2 + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j]^{\frac{1}{2}}.\end{aligned}$$

Next,

$$\begin{aligned}\Omega_{k,j,\varepsilon,\tilde{\varepsilon}} & = [\tilde{h}_j(1 - \omega h_k) + h_k(1 - \omega \tilde{h}_j)](\varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j) \\ & \quad - \varepsilon h_k \tilde{h}_j (1 - \omega h_k) - \tilde{\varepsilon} \tilde{h}_j h_k (1 - \omega \tilde{h}_j) + (\varepsilon + \tilde{\varepsilon}) h_k \tilde{h}_j (1 - \omega h_k)(1 - \omega \tilde{h}_j) \\ & \leq (h_k + \tilde{h}_j - 2\omega h_k \tilde{h}_j)(\varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j).\end{aligned}$$

Putting together all the previous estimates, we can derive the desired estimate

$$\begin{aligned}(h_k + \tilde{h}_j - \omega h_k \tilde{h}_j) a_{k,j} & \leq (h_k + \tilde{h}_j - \omega h_k \tilde{h}_j) |x_0 - u| + \\ & \quad (h_k + \tilde{h}_j - \omega h_k \tilde{h}_j) [(t_k - \tilde{t}_j)^2 + \varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j]^{\frac{1}{2}} |\mathcal{B}u| \\ & \quad + (h_k + \tilde{h}_j - \omega h_k \tilde{h}_j)(\varepsilon t_k + \tilde{\varepsilon} \tilde{t}_j),\end{aligned}$$

which is nothing but (3.35).

This completes the proof of Lemma 3.6.  $\square$

Now we are in the position to prove the convergence result announced above, that is, given  $\varepsilon = \varepsilon_n \rightarrow 0$ , the sequence of approximate solutions  $u_n(t) = u_{\varepsilon_n}(t)$  constructed in (3.32) converges to a continuous function  $u : [0, T] \rightarrow C_R$ , which turns out to be a mild solution for the initial value problem (3.31).

Indeed, for each  $n$ , we choose  $(x_k^{(n)}, h_k^{(n)})_k \in \Gamma_{x_0, \varepsilon_n}$  and denote  $t_k^{(n)} = \sum_{i=1}^k h_i^{(n)}$ . Let  $u \in \mathcal{D}(\mathcal{B})$  arbitrary but fixed. Fix also  $t \in [0, T]$ . Let  $m, n \in \mathbb{N}$ . One can find  $k$  and  $j$  such that  $t \in [t_{k-1}^{(n)}, t_k^{(n)}) \cap [t_{j-1}^{(m)}, t_j^{(m)})$ . We then apply Lemma 3.6 with  $\varepsilon = \varepsilon_n, \tilde{\varepsilon} = \varepsilon_m$  :

$$\begin{aligned}|u_n(t) - u_m(t)| & \leq e^{\eta \omega (t_k^{(n)} + t_j^{(m)})} \{ |x_0 - u| + \\ & \quad [(t_k^{(n)} - t_j^{(m)})^2 + \varepsilon_n t_k^{(n)} + \varepsilon_m t_j^{(m)}]^{\frac{1}{2}} |\mathcal{B}u| + \varepsilon_n t_k^{(n)} + \varepsilon_m t_j^{(m)} \}.\end{aligned}$$

Letting  $n, m \rightarrow \infty$ , we get

$$\limsup_{n, m \rightarrow \infty} |u_n(t) - u_m(t)| \leq e^{2\eta \omega t} |x_0 - u|, \text{ for all } u \in \mathcal{D}(\mathcal{B}).$$

Because  $\mathcal{D}(\mathcal{B}) = \mathcal{D}(A) \cap C_R$  is dense in  $C_R$  we obtain

$$\lim_{n, m \rightarrow \infty} |u_n(t) - u_m(t)| = 0, \text{ uniformly for } t \in [0, T].$$

This implies that  $(u_n)_n$  is convergent to a function  $u : [0, T] \rightarrow C_R$  :

$$\lim_{n \rightarrow \infty} u_n(t) = u(t), \text{ uniformly for } t \in [0, T].$$

Reasoning in the same way, but for  $t \neq s$ , we get

$$(3.37) \quad |u(t) - u(s)| \leq e^{2\omega T} [|x_0 - v| + |t - s| |\mathcal{B}v|], \text{ for all } v \in \mathcal{D}(\mathcal{B}).$$

Because  $\mathcal{D}(\mathcal{B})$  is dense in  $C_R$ , it follows from the above inequality that  $u : [0, T] \rightarrow C_R$  is continuous.

Define  $S(t)x_0 := u(t)$ . It is important to note at this point that, when  $x_0 \in \mathcal{D}(\mathcal{B}) = \mathcal{D}(A) \cap C_R$ , we have more than continuity. Choosing  $v = x_0$  in the inequality (3.37),

$$(3.38) \quad |u(t) - u(s)| \leq e^{2\omega T} |t - s| |Ax_0|,$$

for  $t, s \in [0, T]$ , i.e.  $u$  is Lipschitz continuous on  $[0, T]$ , for  $x_0 \in \mathcal{D}(\mathcal{B})$ .

What is left to prove is that, for any  $x_0 \in C_R$ ,  $u(t)$  satisfies

$$(3.39) \quad u(t) = T(t)x_0 + \int_0^t T(t-s)B(u(s))ds,$$

$$(3.40) \quad \varphi(u(t)) \leq m(t, \varphi(x_0)), \text{ for } t \in [0, T].$$

Let  $u_n(t)$  be the approximate solution constructed above and let  $f \in \mathcal{D}(A^*)$ . If we denote  $\epsilon_k^{(n)} := x_k^{(n)} - x_{k-1}^{(n)} - h_k^{(n)}(Ax_k^{(n)} + B(x_k^{(n)}))$ , then we know (see  $(\mathcal{H})$ ) that  $|\epsilon_k^{(n)}| \leq \varepsilon_n h_k^{(n)}$ . Then

$$\begin{aligned} \langle x_k^{(n)}, f \rangle &= \langle x_{k-1}^{(n)}, f \rangle + h_k^{(n)} \langle x_k^{(n)}, A^* f \rangle + h_k^{(n)} \langle B(x_k^{(n)}), f \rangle + \langle \epsilon_k^{(n)}, f \rangle \\ &= \langle x_{k-1}^{(n)}, f \rangle + \int_{t_{k-1}^{(n)}}^{t_k^{(n)}} [\langle u_n(s), A^* f \rangle + \langle B(u_n(s)), f \rangle + \langle \epsilon_n(s), f \rangle] ds, \end{aligned}$$

which implies

$$\langle u_n(t), f \rangle = \langle x_0, f \rangle + \int_0^{t_k^{(n)}} [\langle u_n(s), A^* f \rangle + \langle B(u_n(s)), f \rangle + \langle \epsilon_n(s), f \rangle] ds$$

for  $t \in [t_{k-1}^{(n)}, t_k^{(n)}]$ . Letting  $n \rightarrow \infty$ , we deduce

$$(3.41) \quad \langle u(t), f \rangle = \langle x_0, f \rangle + \int_0^t [\langle u(s), A^* f \rangle + \langle B(u(s)), f \rangle] ds.$$

In the formulas above,  $\epsilon_n(t) = \epsilon_k^{(n)}$  for  $t \in [t_{k-1}^{(n)}, t_k^{(n)}]$  and  $\left| \int_0^{t_k^{(n)}} \epsilon_n(s) ds \right| \leq \varepsilon_n t_k^{(n)} \rightarrow 0$  as  $n \rightarrow \infty$ . The integrand in (3.41) is continuous in  $s$ , because  $u$  is continuous in  $X$  and, consequently,  $B(u)$  weakly continuous in  $X$  (see (3.2)). Therefore  $\langle u(t), f \rangle$  is differentiable in  $t$  and

$$(3.42) \quad \frac{d}{dt} \langle u(t), f \rangle = \langle u(t), A^* f \rangle + \langle B(u(t)), f \rangle.$$

Let  $\tilde{u}(t) := T(t)x_0 + \int_0^t T(t-s)B(u(s))ds$ . We want to prove that  $u(t) \equiv \tilde{u}(t)$ .

We have, for  $f \in \mathcal{D}(A^*)$ ,

$$\langle \tilde{u}(t), f \rangle = \langle T(t)x_0, f \rangle + \int_0^t \langle T(t-s)B(u(s)), f \rangle ds;$$



thus

$$\begin{aligned} \frac{d}{dt} \langle \tilde{u}(t), f \rangle &= \left\langle \frac{d}{dt} T(t)x_0, f \right\rangle + \int_0^t \left\langle \frac{d}{dt} T(t-s)B(u(s)), f \right\rangle ds + \langle B(u(t)), f \rangle \\ &= \langle u(t), A^* f \rangle + \langle B(u(t)), f \rangle \\ &= \frac{d}{dt} \langle u(t), f \rangle. \end{aligned}$$

This shows that  $\langle \tilde{u}(t) - u(t), f \rangle = \langle \tilde{u}(0) - u(0), f \rangle = 0$  for all  $f \in D(A^*)$ , and consequently (3.39) holds, i.e.

$$(3.43) \quad u(t) = T(t)x_0 + \int_0^t T(t-s)B(u(s))ds.$$

To prove inequality (3.40), let  $u_n = u_{\varepsilon_n}$  be given the approximate solutions. We know from the estimate (3.19) that

$$\varphi(u_n(t)) \leq m_{\eta g_{\varepsilon_n}}(\varphi(x_0), t) = m_{g_{\varepsilon_n}}(\eta\varphi(x_0), \eta t)$$

whenever  $\varepsilon_n < \frac{\eta-1}{\eta M}$  for some constant  $M$ , independent of  $n$ . Thus, for sequences  $\eta \rightarrow 1$  and  $n \rightarrow \infty$ , we infer that

$$(3.44) \quad \varphi(u(t)) \leq \liminf_{n \rightarrow \infty} \varphi(u_n(t)) \leq m_g(\varphi(x_0), t).$$

i.e. (3.13) holds. Here we used the lower semicontinuity of  $\varphi$ .

Note that all the above results hold true for all  $x_0 \in C = \bigcup_{r>0} C_r$ . To complete the proof of Theorem 3.2, it remains to prove the uniqueness of the solution  $u(t) = S(t)x_0$ , with  $x_0 \in C$ . This is the claim of the following

**Proposition 3.7.** *The mild solution of the Cauchy problem*

$$(3.45) \quad \begin{aligned} u'(t) &= Au(t) + B(u(t)), t \geq 0, \\ u(0) &= x_0 \in C, \end{aligned}$$

that is, the strongly continuous function  $u$  satisfying (3.43) and (3.44), is unique.

*Proof.* Let  $x_0, y_0 \in C_r$  and  $T > 0$ . Denote  $R := m(r, T)$ .

Assuming there are two functions,  $u(t)$  and  $v(t)$ , both satisfying (3.43), we have

$$u(t), v(t) \in C_R, \text{ for all } t \in [0, T].$$

As  $B - \Omega I$  is dissipative operator on  $C_R$  with some constant  $\Omega$ , we can apply the standard techniques to obtain that

$$\|u(t) - v(t)\| \leq e^{\Omega t} \|x_0 - y_0\|, \text{ for } t \in [0, T].$$

This clearly implies the uniqueness of the Cauchy problem (3.45).

The proof of Theorem 3.2 is now complete.

*Remark 3.8.* In Theorem 3.2, only the lower-semicontinuity of the functional  $\varphi$  was needed. The converse of Theorem 3.2 holds in the special case when the set  $C$  and the functional(s)  $\varphi : X \rightarrow \mathbb{R}$  is (are) convex. This was proved in [61] for the case of a single functional and the result was already quoted in Section 1.2.

*Remark 3.9.* Theorem 3.2 can be generalized by considering  $X$  a general Banach space and  $N$  functionals  $\varphi_1, \varphi_2, \dots, \varphi_N : X \rightarrow \overline{\mathbb{R}}$ , instead of just one functional. In this case  $C = \bigcap_i \mathcal{D}(\varphi_i)$ . This is the result announced in Section 1.3. The proof follows the same lines as in the Hilbert space case, with the inner product replaced by the semi-inner product (in the sense of Lumer) that was defined in (3.9). We omit the details, which are tedious but routine.

*Remark 3.10.* Our proof of the abstract theorem is related to the ideas of Kobayashi [55], who was inspired by the historic Crandall-Liggett paper [30]. L.C. Evans ([33], [34]) extended Kobayashi's construction from the context of  $du/dt = Au$  to the time-dependent operator context of  $du/dt = A(t)u$ . This enables us to generalize our abstract result to  $du/dt = A(t)u + B(t)u$ , and to, for instance,

$$u_t - \alpha(t)Mu_x + F(t, u)_x = 0,$$

a nonstationary version of (NDE). Such variable coefficient dispersive equations appear in various physical applications such as modeling pressure waves in fluid-filled tubes with elastic walls [25].

**3.4. Continuous dependence of solutions on the data.** In this section we show that, under appropriate assumptions, the solution  $u = u(t)$  of the Cauchy problem

$$\begin{aligned} (\text{ACP}_0) \quad & \frac{du}{dt} = Au + B(u) \\ & u(0) = u_0 \in C \end{aligned}$$

depends continuously on the initial data  $u_0$  and on the operators  $A$  and  $B$ . We refer to the beginning of the chapter for the detailed assumptions made regarding the operators  $A$  and  $B$ , the functional  $\varphi$ , the set  $C$  and the function  $g$ . The solution  $u$  is assumed to satisfy the inequality

$$\varphi(u(t)) \leq m(\varphi(u_0), t),$$

for all  $t$  for which  $m = m(\alpha, t)$  is defined. Here  $m$  is the maximal solution of the initial value problem  $m'(r) = g(m(r))$ ,  $m(0) = \alpha$ . Recall that  $g > 0$  is a  $C^1$  function such that the hypothesis  $(\mathcal{H})$  holds.

Consider, for each  $n = 0, 1, 2, \dots$ , the Cauchy problem

$$\begin{aligned} (\text{ACP}_n) \quad & \frac{du}{dt} = A_n u + B_n(u), \\ & u(0) = u_n^0 \in C_n, \end{aligned}$$

where  $A_n, B_n, C_n$  and  $\varphi_n$  are as in Theorem 3.2, with  $\omega_0$  (for  $A_n$ ) independent of  $n$ . We assume that there exist positive  $C^1$  functions  $g_n$ , for  $n \geq 0$ , such that the hypothesis  $(\mathcal{H})$  holds for each  $n \geq 0$ , when  $A$  is replaced with  $A_n$ ,  $B$  with  $B_n$  and  $g$  with  $g_n$ . Here we have one functional  $\varphi_n$  for each  $n$ . Similarly, we could have  $N$  such functionals, where  $N$  is independent of  $n$ .

We will prove that, under certain hypotheses, listed below, the sequence of solutions  $(u_n)_{n \geq 1}$  of the problem  $(\text{ACP}_n)$  converges to the solution  $u(t)$  of the problem  $(\text{ACP}_0)$ .

Thus, we make the following assumptions.

- (i)  $(I - \lambda A_n)^{-1}h \rightarrow (I - \lambda A_0)^{-1}h$  for all  $h \in X$ ,  $0 < \lambda < \lambda_0$  independent of  $n$ .

- (ii)  $\lim C_n \supset C_0$  in the sense that, for all  $y_0 \in C_0$ , there exist  $y_n \in C_n$ , such that  $y_n \rightarrow y_0$ .  
 (iii) For all  $y_0 \in C_0$ ,  $y_n \in C_n$  such that  $y_n \rightarrow y_0$  it follows that  $B_n y_n \rightarrow B_0 y_0$ .  
 (iv)  $\varphi_n \rightarrow \varphi_0$  and  $g_n \rightarrow g_0$  uniformly on compact sets in  $\mathbb{R}$ .

We now introduce the space:

$$\mathcal{X} = c(X) = \{\bar{x} = (x_n)_{n=0}^\infty | x_n \in X, x_n \rightarrow x_0\}$$

of convergent sequences in  $X$ , which is a Banach space when equipped with the sup norm:

$$\|\bar{x}\|_{\mathcal{X}} = \sup_n \|x_n\|_X.$$

Following an idea presented in [42], (see also [39]), we convert the problem of continuous dependence of solutions into a problem of existence of solutions to an abstract Cauchy problem defined on the Banach space  $\mathcal{X}$ . Here is the strategy.

Define the operator  $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \rightarrow \mathcal{X}$  by

$$\mathcal{A}\bar{x} = \bar{y}, \text{ for } \bar{x} = (x_n), \bar{y} = (y_n) \in \mathcal{X} \text{ if and only if } A_n x_n = y_n \text{ for all } n.$$

Then  $\mathcal{A} - \omega_0 \mathcal{I}$  is densely defined and m-dissipative on  $\mathcal{X}$  (see [40]), hence it generates a  $C_0$ -semigroup of operators  $\{\mathcal{T}(t) : \mathcal{X} \rightarrow \mathcal{X}\}_{t \geq 0}$ . It is easy to see that, for  $\bar{x} = (x_n)_{n \geq 0}$  in  $\mathcal{X}$ ,  $\mathcal{T}(t)\bar{x} = (T_n(x_n))_{n \geq 0}$ , where  $\{T_n(t) : X \rightarrow X\}_{t \geq 0}$  has generator  $A_n$ .

The operator  $\mathcal{B} : \mathcal{C} \rightarrow \mathcal{X}$ , is defined on  $\mathcal{C} = \{\bar{x} \in \mathcal{X} | x_n \in C_n \text{ for all } n \geq 0\}$  by

$$\mathcal{B}(\bar{x}) = \bar{y} \text{ for } \bar{x} \in \mathcal{C}, \bar{y} \in \mathcal{X} \text{ if and only if } B_n(x_n) = y_n \rightarrow y_0 = B_0(x_0).$$

Concerning the operator  $\mathcal{B}$ , we make the following assumptions:

- (v) for  $(\bar{x}^m)_m \subset \mathcal{C}_\alpha = \{\bar{u} \in \mathcal{X} | \varphi_n(u_n) \leq \alpha, \text{ for all } n\}$ ,  $\bar{x}^m \rightarrow \bar{x}^0$  as  $m \rightarrow \infty$  implies  $B_n x_n^m \rightarrow B_n x_n^0$  as  $m \rightarrow \infty$ , uniformly in  $n$ , and  
 (vi) for each  $\alpha > 0$ , the dissipativity constant  $\omega_\alpha$  in (3.3) for  $B_n$  is independent of  $n$ .

With the operators  $\mathcal{A}, \mathcal{B}$  and the set  $\mathcal{C}$  satisfying the hypothesis (i)-(vi) above, we can formulate the Cauchy problem on  $\mathcal{X}$

$$(3.46) \quad \begin{aligned} \frac{d\bar{u}}{dt} &= \mathcal{A}\bar{u} + \mathcal{B}(\bar{u}) \\ \bar{u}(0) &= \bar{u}^0 \in \mathcal{C}. \end{aligned}$$

The following theorem holds

**Theorem 3.11.** *With the assumptions made above, the Cauchy problem (3.46) has a unique mild solution  $\bar{u}(t) = S(t)\bar{u}^0$  satisfying*

$$(3.47) \quad S(t)\bar{u}^0 = \mathcal{T}(t)\bar{u}^0 + \int_0^t \mathcal{T}(t-s)\mathcal{B}(S(s)\bar{u}^0)ds,$$

$$(3.48) \quad \tilde{\varphi}(S(t)u^0) \leq \tilde{m}(\tilde{\varphi}(u^0), t).$$

Here  $\tilde{\varphi} : \mathcal{C} \rightarrow \overline{\mathbb{R}}$  is defined by  $\tilde{\varphi}(\bar{x}) = \sup_n \varphi_n(x_n)$  and  $\tilde{m}(\alpha, t) = \sup_n m_n(\alpha, t)$ .

Note that the variation of constants formula (3.47) is equivalent to the sequence of formulas written for each component  $n \geq 0$ :

$$S_n(t)u_n^0 = T_n(t)u_n^0 + \int_0^t T_n(t-s)B_n(S_n(s)u_n^0)ds.$$

Thus the existence of the solution  $\bar{u}(t) = S(t)\bar{u}^0 \in \mathcal{X}$  at time  $t > 0$  implies that  $u_n(t) = S_n(t)u_n^0$  converges to  $u(t) = S(t)u^0$ , because of the structure of the space  $\mathcal{X}$ .

*Proof.* The proof of Theorem 3.11 consists in applying the abstract theorem to the semilinear problem (3.46) on the space  $\mathcal{X}$ . To be more precise, we obtain the solution  $\bar{u}(t) \in \mathcal{X}$  as a limit of approximate solutions  $\bar{u}^\varepsilon(t) \in \mathcal{X}$ , constructed in the same manner as the step functions  $u^\varepsilon(t) \in X$  (see section 3.1), i.e. by solving the resolvent equation.

In fact this can be done on each component. If  $\bar{v} = (v_n)_{n \geq 0} \in \mathcal{C}$  and  $\varepsilon > 0$ , there exists  $\lambda_0$  (independent of  $n$ ) such that, according to  $(\mathcal{H})$ , for  $0 < \lambda < \lambda_0$ , we can find  $(u_{n,\lambda})_{n \geq 0}$  satisfying, for each  $n \geq 0$ ,

$$(3.49) \quad u_{n,\lambda} - \lambda A_n u_{n,\lambda} - \lambda B_n(u_{n,\lambda}) = v_n,$$

$$(3.50) \quad \varphi_n(u_{n,\lambda}) - \varphi_n(v_n) \leq \lambda g_n(u_{n,\lambda}) + \lambda \varepsilon.$$

Thus, for fixed  $n > 0$ , subtracting the equality (3.49) from same equality corresponding to  $n = 0$  and then multiplying both sides by  $u_{n,\lambda} - u_{0,\lambda}$ , we obtain

$$\|u_{n,\lambda} - u_{0,\lambda}\|^2 \leq \lambda \omega \|u_{n,\lambda} - u_{0,\lambda}\|^2 + \langle u_{n,\lambda} - u_{0,\lambda}, v_n - v_0 \rangle$$

for some  $\omega$  depending on  $\tilde{\varphi}(\bar{v})$ . We used here also the quasi-dissipativity of  $B_n$ . Consequently,

$$\|u_{n,\lambda} - u_{0,\lambda}\| \leq \lambda \omega \|u_{n,\lambda} - u_{0,\lambda}\| + \|v_n - v_0\|$$

and, for  $\lambda$  sufficiently small, we obtain  $u_{n,\lambda} \rightarrow u_{0,\lambda}$  as  $n \rightarrow \infty$ , i.e.  $\bar{u}_\lambda = (u_{n,\lambda})_{n \geq 0} \in \mathcal{C}$ . The inequality

$$(3.51) \quad \tilde{\varphi}(\bar{u}_\lambda) \leq \tilde{m}(\tilde{\varphi}(\bar{v}), \lambda)$$

follows from (3.50), using the same argument as in Section 3.1.

The convergence of the approximate solutions  $\bar{u}^\varepsilon(t)$  to the mild solution  $\bar{u}(t)$  satisfying (3.47) is the result of estimates analogue to (3.33). Iterating inequality (3.51) we obtain the desired estimate (3.48). This concludes the proof of Theorem 3.11.  $\square$

#### 4. NONLINEAR DISPERSIVE EQUATIONS

We will fit our concrete nonlinear dispersive equation in the general framework of the semilinear Hille-Yosida theory presented in the preceding section. This will be done in a sequence of lemmas.

**4.1. Properties of the operators  $A$  and  $B$ .** Let  $Au = D^{2\beta}\partial u$  and  $B(u) = -\partial F(u)$  be the operators introduced in Section 2.2. The linear operator  $A$  is clearly a skew-adjoint operator on each Sobolev space  $H^s$ ,  $s \geq 0$ . For now we will regard these operators as acting on  $H^{2\beta} = H^{2\beta}(\mathbb{T})$ .

Then the following result holds true:

**Lemma 4.1.** (i) If  $w_n \rightarrow w$  in  $L^2(\mathbb{T})$  and  $\sup_n |w_n|_{H^{2\beta}} \leq M < +\infty$ , then  $(I - \lambda A)^{-1}B(w_n) \rightarrow (I - \lambda A)^{-1}B(w)$  in  $H^{2\beta}$ , for all real  $\lambda \neq 0$ .

(ii) If  $w_n \rightarrow w$  in  $L^2(\mathbb{T})$  and  $\sup_n |w_n|_{H^{2\beta+1}} \leq M < +\infty$ , then  $B(w_n) \rightarrow B(w)$  in  $L^2(\mathbb{T})$ .

*Proof.* We will use the following abstract result.

**Lemma 4.2.** *If  $A : D(A) \subset X \rightarrow X$  is a skew-adjoint operator on a Hilbert space, then*

*$f_n \rightarrow f$  in  $X$  implies  $(I - \lambda A)^{-1} f_n \rightarrow (I - \lambda A)^{-1} f$  in  $X$ , for all  $\lambda \in \mathbb{R} - \{0\}$ .*

This is obvious; since  $\mathbb{R} \setminus \{0\}$  is contained in the resolvent set of  $A$ , hence  $(I - \lambda A)^{-1}$  is a bounded linear operator for all  $\lambda \in \mathbb{R} \setminus \{0\}$  and the result follows.

Let us return to the proof of Lemma 4.1. Consider  $w_n \rightarrow w$  in  $L^2(\mathbb{T})$ , as  $n \rightarrow \infty$ , with  $\sup_n |w_n|_{H^{2\beta}} \leq M < +\infty$ . For any real  $\lambda \neq 0$ , we will show that  $(I - \lambda A)^{-1} B(w_n) \rightarrow (I - \lambda A)^{-1} B(w)$  in  $H^{2\beta}(\mathbb{T})$ .

To this end, rewrite

$$\begin{aligned} D^{2\beta} (I - \lambda A)^{-1} B(w_n) &= (I + \lambda D^{2\beta} \partial)^{-1} D^{2\beta} \partial F(w_n) \\ &= \frac{1}{\lambda} \left[ I - (I + \lambda D^{2\beta} \partial)^{-1} \right] F(w_n) \\ &= \frac{1}{\lambda} \left[ F(w_n) - (I + \lambda D^{2\beta} \partial)^{-1} F(w_n) \right]. \end{aligned}$$

To conclude the proof of (i), it is enough to show that  $F(w_n) \rightarrow F(w)$  in  $L^2(\mathbb{T})$ . This fact and the previous lemma would allow us to take the limit as  $n \rightarrow \infty$  in the previous calculations, and obtain the desired conclusion (i).

But

$$\begin{aligned} F(w_n) - F(w) &= \int_0^1 \frac{d}{d\theta} F(\theta w_n + (1 - \theta)w) d\theta \\ &= \int_0^1 F'(\theta w_n + (1 - \theta)w) d\theta (w_n - w) \end{aligned}$$

so that

$$\begin{aligned} |F(w_n) - F(w)|_{L^2} &\leq \left| \int_0^1 F'(\theta w_n + (1 - \theta)w) d\theta \right|_{\infty} |w_n - w|_{L^2} \\ &\leq c(M) |w_n - w|_{L^2} \\ &\rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

where  $M$  is a finite uniform bound for  $\|w_n\|_{L^\infty}$ , since  $\{w_n\}$  is bounded sequence in  $H^{2\beta}(\mathbb{T})$ , thus in  $L^\infty(\mathbb{T})$  ( $2\beta > \frac{1}{2}$  and inequality (2.8)).

To prove assertion (ii), take  $w_n \rightarrow w$  in  $L^2(\mathbb{T})$  and  $\sup_n |w_n|_{H^{2\beta+1}} \leq M < +\infty$ . Then ( $w_n$  and  $w$  are real valued)

$$\begin{aligned} |B(w_n) - B(w)|_{L^2}^2 &= |\partial F(w_n) - \partial F(w)|_{L^2}^2 \\ &= (F(w_n) - F(w), \partial^2 F(w_n) - \partial^2 F(w)) \\ &\leq |F(w_n) - F(w)| (|\partial^2 F(w_n)| + |\partial^2 F(w)|) \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Here we used the fact that  $\partial^2 F(w_n) = F''(w_n) (\partial w_n)^2 + F'(w_n) \partial^2 w_n$  is uniformly bounded in  $L^2(\mathbb{T})$  (since  $2\beta + 1 \geq 2$ ).  $\square$

**Lemma 4.3.** *(i) For each real  $\lambda \neq 0$ , the operators  $\pm \mathcal{A}_\lambda = \pm (I - \lambda A)^{-1} B$  are locally quasi-dissipative on  $H^{2\beta}(\mathbb{T})$ , i.e. for all  $r > 0$ , there exists  $\omega = \omega(r) \in \mathbb{R}$  such that*

$$\left| ((I - \lambda A)^{-1} B(v) - (I - \lambda A)^{-1} B(w), v - w)_{H^{2\beta}} \right| \leq \frac{\omega}{\lambda} |v - w|_{H^{2\beta}}^2$$

whenever  $|v|_{H^{2\beta}}, |w|_{H^{2\beta}} \leq r$ .

(ii)  $\pm B$  are locally quasi-dissipative in  $L^2(\mathbb{T})$  on bounded sets in  $H^{2\beta+1}(\mathbb{T})$ , i.e. for all  $r > 0$ , there exists  $\omega = \omega(r) \in \mathbb{R}$  such that

$$|(B(v) - B(w), v - w)_{L^2}| \leq \omega |v - w|_{L^2}^2$$

whenever  $|v|_{H^{2\beta+1}}, |w|_{H^{2\beta+1}} \leq r$ .

*Proof.* We shall make use of the identity  $\lambda A(I - \lambda A)^{-1} = I - (I - \lambda A)^{-1}$ .

Then

$$\begin{aligned} & |((I - \lambda A)^{-1}B(v) - (I - \lambda A)^{-1}B(w), v - w)_{H^{2\beta}}| = \\ & = |(D^{2\beta}(I - \lambda D^{2\beta}\partial)^{-1}\partial(F(v) - F(w)), D^{2\beta}(v - w))| \\ & = \left| \frac{1}{\lambda} (((I - \lambda D^{2\beta}\partial)^{-1} - 1)(F(v) - F(w)), D^{2\beta}(v - w)) \right| \\ & = \left| \frac{1}{\lambda} ((I - \lambda D^{2\beta}\partial)^{-1}(F(v) - F(w)), D^{2\beta}(v - w)) \right. \\ & \quad \left. - \frac{1}{\lambda} ((F(v) - F(w)), D^{2\beta}(v - w)) \right| \\ & \leq \frac{1}{\lambda} (|(I - \lambda D^{2\beta}\partial)^{-1}(F(v) - F(w))|_{L^2} + |F(v) - F(w)|_{\infty}) |D^{2\beta}(v - w)|_{L^2} \\ & \leq \frac{1}{\lambda} (|F(v) - F(w)|_{L^2} + |F(v) - F(w)|_{\infty}) |v - w|_{H^{2\beta}} \\ & \leq \frac{c_1(r)}{\lambda} |v - w|_{\infty} |v - w|_{H^{2\beta}} \quad (\text{since } |v|_{\infty}, |w|_{\infty} \leq c_0(r)) \\ & \leq \frac{c_2(r)}{\lambda} |v - w|_{H^{2\beta}}^2. \end{aligned}$$

The conclusion (i) follows from the fact that  $H^{2\beta}(\mathbb{T}) \subset L^{\infty}(\mathbb{T})$  continuously, since  $\beta \geq \frac{1}{2}$ .

To prove (ii), let  $r > 0$  and  $u, v \in H^{2\beta+1}$  with  $\|u\|_{H^{2\beta+1}}, \|v\|_{H^{2\beta+1}} \leq r$ . We have

$$\begin{aligned} |(Bu - Bv, u - v)| & = |(F(u) - F(v), D(u - v))| \\ & = \left| \left( \int_0^1 F'(\theta u + (1 - \theta)v)(u - v) d\theta, \partial(u - v) \right) \right| \\ & = \left| \left( \int_0^1 F'(\theta u + (1 - \theta)v) d\theta, \frac{1}{2} \partial(u - v)^2 \right) \right| \\ & = \left| \left( \int_0^1 F''(\theta u + (1 - \theta)v)(\theta \partial u + (1 - \theta) \partial v) d\theta, \frac{1}{2} (u - v)^2 \right) \right| \\ & \leq \frac{1}{2} |F''(\theta u + (1 - \theta)v)|_{\infty} |\theta \partial u + (1 - \theta) \partial v|_{\infty} |u - v|^2 \\ & \leq \omega_2(r) |u - v|^2 \end{aligned}$$

where  $\omega(r) = \frac{r}{2} \sup\{|F''(w)|; w \in H^{2\beta+1}, |w|_{H^{2\beta+1}} \leq r\}$ .

This concludes the proof of Lemma 4.3.  $\square$

#### 4.2. Estimates for the functionals $\varphi_j$ .

**Lemma 4.4.** (i) For  $\alpha_0, \alpha_1 > 0$ , there exists  $\theta_1 = \theta_1(\alpha_0, \alpha_1)$  such that  $w \in V_1, \varphi_0(w) \leq \alpha_0, \varphi_1(w) \leq \alpha_1$  implies  $|w|_1 \leq \theta_1$ .

(ii) For  $\alpha_0, \alpha_1, \alpha_2 > 0$ , there exists  $\theta_2 = \theta_2(\alpha_0, \alpha_1)$  such that  $w \in V_2, \varphi_0(w) \leq \alpha_0, \varphi_1(w) \leq \alpha_1, \varphi_2(w) \leq \alpha_2$  implies  $|w|_2 \leq \theta_2$ .

(iii) For  $\alpha_0, \alpha_1, \alpha_2, \alpha_3 > 0$ , there exists  $\theta_3 = \theta_3(\alpha_0, \alpha_1)$  such that  $w \in V_3, \varphi_j(w) \leq \alpha_j, j = 0, 1, 2, 3$ , implies  $|w|_3 \leq \theta_3$ .

*Proof.* Recall the growth condition (C) imposed on  $F$ . We assumed that there exists  $p < 4\beta$  such that

$$(4.1) \quad \limsup_{|r| \rightarrow \infty} \frac{F'(r)}{|r|^p} < \infty.$$

Hence, there are constants  $C, C'$  such that

$$(4.2) \quad F'(r) \leq C|r|^p + C'.$$

Because  $F(0) = 0$  and  $G(0) = 0$ , where  $G(w) = \int_0^w F(\xi) d\xi$ , it follows that (with different constants  $K, K'$ )

$$G(r) \leq K|r|^{p+2} + K'|r|^2 \text{ for all } r \in \mathbb{R}.$$

Thus,

$$\begin{aligned} \int_{\mathbf{T}} G(w) dx &\leq K|w|_{L^{p+2}}^{p+2} + K'|w|^2 \\ &\leq K|w|_{\infty}^p |w|^2 + K'|w|^2 \\ &\leq C|w|^{p(1-\frac{1}{2\beta})+2} |D^\beta w|^{\frac{p}{2\beta}} + K'|w|^2 \\ &\leq \frac{4\beta-p}{4\beta p} \left( C|w|^{p+2-\frac{p}{2\beta}} \right)^{\frac{4\beta}{4\beta-p}} + \frac{1}{4} |D^\beta w|^2 + K'|w|^2. \end{aligned}$$

Here we used (2.8) and Young's Inequality

$$ab \leq \frac{(a)^r}{r} + \frac{b^s}{s}, \quad r = \frac{4\beta}{4\beta-p}, \quad s = \frac{4\beta}{p} \quad (r, s > 1, \frac{1}{r} + \frac{1}{s} = 1).$$

The assumption  $p < 4\beta$  was essential in the previous calculation (to guarantee that  $r, s > 1$ ). As a result of the previous estimates we get

$$\frac{1}{4} |D^\beta w|^2 \leq \bar{K} |w|^{\frac{2(p(2\beta-1)+4\beta)}{4\beta-p}} + K'|w|^2 + \varphi_1(w),$$

which implies (i).

To prove (ii), note that

$$\begin{aligned} -(F(w), D^{2\beta} w) &= (\partial_x F(w), \partial_x^{-1} D^{2\beta} w) \\ &= (F'(w), \partial_x^{-1} D^{2\beta} w \partial_x w) = \int F'(w) \partial_x^{-1} D^{2\beta} w \partial_x w \\ &\leq C \int |w|^p |\partial_x^{-1} D^{2\beta} w \partial_x w| + C' \int |\partial_x^{-1} D^{2\beta} w \partial_x w| \\ &\leq C |w|_{\infty}^p |D^\beta w|^2 + C' |D^\beta w|^2 \\ &\leq C |D^\beta w|^{p+2} + C' |D^\beta w|^2, \end{aligned}$$

and also

$$\begin{aligned} (I(u), 1) &= \int_{\mathbf{T}} I(u) \leq C_1 |u|^2 + C_2 |u|_{L^{2p+2}}^{2p+2} \\ &\leq C |D^\beta u|^{2p+2}. \end{aligned}$$

Thus, from the definition of  $\varphi_2$ ,

$$\begin{aligned} \frac{1}{2} |D^{2\beta} w|^2 &= \varphi_2(w) - \frac{4\beta+1}{4\beta+2} (F(w), D^{2\beta} w) - \frac{4\beta+1}{4\beta+2} \int_{\mathbb{T}} I(u) \\ &\leq \varphi_2(w) + C_1 |D^\beta w|^{p+2} + C' |D^\beta w|^2, \end{aligned}$$

i.e. (ii) holds.

The proof of (iii) is straightforward:

$$\begin{aligned} |D^{2\beta+1} w| &\leq \varphi_3(w) + |\partial F(w)| \leq \varphi_3(w) + |F'(w)|_\infty |\partial u| \\ &\leq \varphi_3(w) + C |w|_{2\beta}. \end{aligned}$$

□

We need one more result, which will be used in the proof of Theorem 4.7.

**Lemma 4.5.** *Let*

$$\Psi_\beta(z) := (D^{2\beta} (F'(z) \partial z) - F'(z) D^{2\beta} \partial z - (2\beta+1) \partial F'(z) D^{2\beta} z, D^{2\beta} z)$$

(i) *For  $\beta > \frac{1}{2}$  we have that, for all  $z \in H^{2\beta+1}$ ,*

$$(4.3) \quad \Psi_\beta(z) \leq C |Dz|_\infty |D^{2\beta} z|^2.$$

*where  $C$  depends only on a bound for  $|D^\beta z|$ .*

(ii) *For  $\beta = \frac{1}{2}$  or  $\beta = 1$ , one can improve the estimate above, in the sense that  $\Psi_1(z) \leq C_1 |Dz|^4$  when  $\beta = \frac{1}{2}$ , with  $C_1$  depending only on  $|D^{1/2} z|$ , and  $\Psi_2(z) \leq C |D^2 z|^{3/2}$  in the case  $\beta = 1$ , for some  $C_2$  depending only on  $|Dz|$ .*

*Remark 4.6.* This result can be reformulated as follows. One can construct a function  $g = g(r)$  (depending on  $\beta$ ), which, in general, has superquadratic growth as  $r \rightarrow \infty$ , such that

$$(4.4) \quad \Psi_\beta(z) \leq C g(\varphi_2(z)),$$

where  $C$  depends only on  $|D^\beta z|$ .

For  $\beta > \frac{3}{2}$ ,  $g$  can be chosen to be linear. For  $\beta = 1$ ,  $g$  can be chosen to be sublinear and for  $\beta = \frac{1}{2}$ ,  $g$  can be chosen to be quadratic. What is remarkable is that in the special case  $F'(u) = u$  and  $\beta = 1$  or  $\beta = \frac{1}{2}$  we get  $\Psi_\beta(z) = 0$ , for all  $z$ .

The proof of (i) relies on estimates obtained from the product rule and chain rule for fractional order derivatives, for which we refer to [38], [45], [53]. The improved estimates mentioned in (ii) are obtained in Section 2.3, so we do not reproduce them here.

**4.3. Resolvent equation.** With all these preparations, we are ready to apply the abstract theorem. Here is how the hypothesis  $(\mathcal{H})$  is translated in our context.

**Theorem 4.7.** *Let  $v \in V_3$  satisfy  $\|v\|_{H^{2\beta+1}} \leq r, \varepsilon > 0$ . There exists  $\lambda_0 = \lambda_0(r, \varepsilon)$  such that, for all real  $|\lambda| < \lambda_0$ , there exists an unique  $u = u_\lambda \in V_3$  satisfying*

$$(4.5) \quad u - \lambda D^{2\beta} \partial u + \lambda \partial F(u) = v,$$



$$(4.6) \quad \varphi_0(u) \leq \varphi_0(v) + |\lambda| \varepsilon,$$

$$(4.7) \quad \varphi_1(u) \leq \varphi_1(v) + |\lambda| \varepsilon,$$

$$(4.8) \quad \varphi_2(u) \leq \varphi_2(v) + |\lambda|(g(\varphi_2(u)) + \varepsilon),$$

$$(4.9) \quad \varphi_3(u) \leq \frac{\varphi_3(v)}{1 - \omega_0 |\lambda|}.$$

Here  $g$  is the function satisfying (4.4), depending on  $\beta$  and the nonlinearity  $F$ , and  $\omega_0$  can be chosen to depend only on a bound for  $|Dv|_{L^\infty}$ .

*Proof.* Fix  $r > 0$  and  $\varepsilon > 0$  arbitrarily. Let  $v \in V_3$ , with  $|v| < r$ , and choose  $\alpha_0, \alpha_1$  be positive numbers such that  $\varphi_0(v) + \varepsilon < \alpha_0, \varphi_1(v) + \varepsilon < \alpha_1$ . By Lemma 4.4, there exists  $\theta_1 = \theta_1(\alpha_0, \alpha_1)$  such that for all  $z \in H^\beta$ , with  $\varphi_0(z) \leq \alpha_0, \varphi_1(z) \leq \alpha_1$  implies  $|z|_1 \leq \theta_1$ .

For the function  $g$  defined by (4.4), let  $m_g(t, \alpha)$  be the maximal solution of the initial value problem (3.11). Choose

$$(4.10) \quad \alpha_2 \geq m_{g_\varepsilon}(\tau, \varphi_2(v)).$$

Here  $\tau$  is sufficiently small (depending only on  $\varphi_2(v)$ ) such that the right hand side of (4.10) is finite.

From Lemma 4.4 we conclude that there exists  $\theta_2 = \theta_2(\alpha_0, \alpha_1, \alpha_2)$  such that, for all  $z \in H^{2\beta}$ , with  $\varphi_0(z) \leq \alpha_0, \varphi_1(z) \leq \alpha_1, \varphi_2(z) \leq \alpha_2$  implies  $|z|_2 \leq \theta_2$ .

For later purposes, let

$$(4.11) \quad \rho = \sup\{|F''(w)\partial w|, w \in V_2, |w|_2 \leq \theta_2\}$$

and

$$(4.12) \quad \sigma_1 = \sup\{|F'(w)\partial w|, w \in V_2, |w|_2 \leq \theta_2\},$$

$$(4.13) \quad \sigma_2 = \sup\{|D^{2\beta}F(w)|, w \in V_2, |w|_2 \leq \theta_2\},$$

$$(4.14) \quad \sigma_3 = \sup\{|D^{2\beta+1}F(w)|, w \in V_3, |w|_3 \leq \theta_3\},$$

where we choose  $\theta_3 = |v|_3 + 2\sigma_1$ .

Because  $F$  is assumed to be smooth enough (at least in  $C^{2\beta+1}$ ), there exists  $\delta = \delta(|v|_3, \varepsilon) > 0$  such that, for all  $w \in V_3, |v - w| \leq \delta$  and  $|w|_j \leq \max\{\theta_j, |v|_j + \sigma_{j+1}\}$ ,  $j = 0, 1, 2$ , the following hold true:

$$(4.15) \quad |F(v) - F(w)|\theta_3 < \frac{\varepsilon}{2},$$

$$(4.16) \quad |D^{2\beta}F(v) - D^{2\beta}F(w)|\theta_3 < \varepsilon_1,$$

$$(4.17) \quad |F'(v) - F'(w)|_\infty \theta_3 (|v|_2 + \sigma_3) < \varepsilon_2,$$

$$(4.18) \quad |I'(v) - I'(w)|\theta_3 < \varepsilon_3,$$

where  $\varepsilon_j$  are chosen such that  $3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3 < \frac{\varepsilon}{2}$ . Let  $K = \sup\{|g'(r)|, r < \theta_3\}$  and denote  $\lambda_0 = \min\{\tau, \frac{\delta}{\theta_3}, \frac{1}{\rho}, \frac{1}{K}\}$ . Consider  $\lambda \in (-\lambda_0, \lambda_0), \lambda \neq 0$  be arbitrary but fixed.

Consider the set

$$(4.19) \quad K = \{w \in V_3 \mid |w - v| \leq |\lambda| \theta_3, |w|_j \leq \theta_j \text{ for all } j = 0, 1, 2, 3\}.$$

Note that  $K$  is a compact convex set in  $L^2(\mathbb{T})$ .

We seek fixed points for the operator  $\Gamma : K \rightarrow X$  defined by

$$\Gamma w := (I - \lambda A)^{-1}(v + \lambda B(w)).$$

By linearity of  $A$ ,

$$\Gamma w = (I - \lambda A)^{-1}v + \lambda(I - \lambda A)^{-1}B(w).$$

As we saw in Lemma 4.1, the operator  $\lambda(I - \lambda A)^{-1}B$  is  $L^2$ -continuous on bounded sets in  $H^{2\beta}$ , thus  $\Gamma$  is a continuous operator on  $K$ . In order to apply the Schauder-Tichonov fixed point principle, we have to ensure that  $\Gamma$  leaves  $K$  invariant, i.e.

$$(4.20) \quad \Gamma(K) \subset K.$$

We now prove (4.20). With  $v \in H^{2\beta+1}(\mathbb{T})$  fixed, let  $w \in K$  be arbitrary. Denote  $z = \Gamma w$ . We will show that  $z \in K$ . Since

$$(4.21) \quad z - \lambda D^{2\beta} \partial z = v - \lambda \partial F(w),$$

we obtain

$$\begin{aligned} |z - v|^2 &= (\lambda D^{2\beta} \partial z, z - v) - (\lambda \partial F(w), z - v) \\ &= \lambda (D^{2\beta} \partial v, z - v) - \lambda (\partial F(w), z - v) \\ &\leq |\lambda| (|D^{2\beta} \partial v| + |\partial F(w)|) |z - v| \\ &\leq |\lambda| (|v|_3 + \sigma_1) |z - v|, \end{aligned}$$

or

$$|z - v| \leq |\lambda| (|v|_3 + \sigma_1) \leq |\lambda| \theta_3.$$

From (4.21) we also obtain

$$\begin{aligned} |z|_3 &= |D^{2\beta} \partial z| \leq \frac{1}{|\lambda|} |z - v| + |\partial F(w)| \\ &\leq |v|_3 + \sigma_1 + \sigma_1 \leq \theta_3. \end{aligned}$$

In order to conclude that  $z \in K$  we only need to show

$$|z| \leq \theta_0, |D^\beta z| \leq \theta_1, |D^{2\beta} z| \leq \theta_2,$$

or, in view of Lemma 4.4, it is enough to prove

$$(4.22) \quad \varphi_0(z) \leq \alpha_0,$$

$$(4.23) \quad \varphi_1(z) \leq \alpha_1,$$

$$(4.24) \quad \varphi_2(z) \leq \alpha_2.$$

From (4.21), we obtain

$$\begin{aligned} |z|^2 &\leq (v, z) - (\lambda \partial F(w), z) \\ &= (v, z) - \lambda (\partial F(w) - \partial F(z), z) \\ &\leq |v| |z| + |\lambda| |\partial F(w) - \partial F(z)| |z| \\ &\leq (|v| + |\lambda| \varepsilon) |z|, \end{aligned}$$

i.e.  $|z| \leq |v| + |\lambda| \varepsilon$ . We conclude that  $\varphi_0(z) \leq \varphi_0(v) + |\lambda| \varepsilon < \varphi_0(v) + \varepsilon < \alpha_0$ .

Next, multiply (4.21) by  $D^{2\beta} z$  to obtain

$$\begin{aligned} |D^\beta z|^2 &= (D^{2\beta} z, z) = (D^{2\beta} z, v) - \lambda (D^{2\beta} z, \partial F(w)) \\ &= (D^\beta z, D^\beta v) + \lambda (D^{2\beta} \partial z, F(w)) \\ &\leq \frac{1}{2} |D^\beta z|^2 + \frac{1}{2} |D^\beta v|^2 + \lambda (D^{2\beta} \partial z, F(w)). \end{aligned}$$

Now we can estimate

$$\begin{aligned}
\frac{1}{\lambda} [\varphi_1(z) - \varphi_1(v)] &= \frac{1}{\lambda} \left[ \frac{1}{2} |D^{2\beta} z|^2 - \frac{1}{2} |D^{2\beta} v|^2 - \int_T (G(z) - G(v)) dx \right] \\
&\leq \frac{1}{\lambda} \left[ \lambda (D^{2\beta} \partial z, F(w)) - \int_T (G(z) - G(v)) dx \right] \\
&= (D^{2\beta} \partial z, F(w)) - \frac{1}{\lambda} \int_T \left( \int_0^1 F(\tau z + (1-\tau)v) d\tau \right) (z-v) dx \\
&= \frac{1}{\lambda} (z-v - \lambda \partial F(w), F(w)) - \frac{1}{\lambda} (z-v, F(v)) \\
&\quad - \frac{1}{\lambda} \int_T \left( \int_0^1 F(\tau z + (1-\tau)v) d\tau - F(v) \right) (z-v) dx \\
&\leq \frac{1}{|\lambda|} |z-v| |F(w) - F(v)| \\
&\quad + \frac{1}{|\lambda|} \int_T \left| \int_0^1 [F(\tau z + (1-\tau)v) - F(v)] d\tau \right| |z-v| dx \\
&\leq \frac{1}{|\lambda|} \frac{\varepsilon}{2\theta_3} |\lambda| \theta_3 + \frac{1}{|\lambda|} \frac{\varepsilon}{2\theta_3} |\lambda| \theta_3 = \varepsilon.
\end{aligned}$$

Thus we conclude

$$\varphi_1(z) \leq \varphi_1(v) + \varepsilon |\lambda| < \alpha_1.$$

The only remaining estimate is for  $\varphi_2(z)$ . Consider  $\gamma$  be an arbitrary number and let

$$\varphi(z) := \frac{1}{2} |D^{2\beta} z|^2 - \gamma (F(z), D^{2\beta} z) + \gamma (I(z), 1)$$

(we will obtain the value  $\gamma = \frac{4\beta+1}{4\beta+2}$  to be the useful one, thus  $\varphi = \varphi_2$ ).

Rewrite (4.21) in the form

$$z - v = \lambda D^{2\beta} \partial z - \lambda \partial F(w).$$

Then,

$$\begin{aligned}
|D^{2\beta} z|^2 &= (D^{2\beta} z, D^{2\beta} v) + (D^{2\beta} z, D^{2\beta} z - D^{2\beta} v) \\
&= (D^{2\beta} z, D^{2\beta} v) - (D^{2\beta} \partial z, \partial^{-1} D^{2\beta} (z-v)) \\
&= (D^{2\beta} z, D^{2\beta} v) - \frac{1}{\lambda} (z-v + \lambda \partial F(w), \partial^{-1} D^{2\beta} (z-v)) \\
&= (D^{2\beta} z, D^{2\beta} v) - (\partial F(w), \partial^{-1} D^{2\beta} (z-v)) \\
&= (D^{2\beta} z, D^{2\beta} v) + (D^{2\beta} F(w), (z-v)) \\
&= (D^{2\beta} z, D^{2\beta} v) + (D^{2\beta} F(w), \lambda D^{2\beta} \partial z - \lambda \partial F(w)) \\
&= (D^{2\beta} z, D^{2\beta} v) + \lambda (D^{2\beta} F(w), D^{2\beta} \partial z) \\
&\leq \frac{1}{2} |D^{2\beta} z|^2 + \frac{1}{2} |D^{2\beta} v|^2 + \lambda (D^{2\beta} F(z), D^{2\beta} \partial z) + |\lambda| \varepsilon_1.
\end{aligned}$$

where  $\varepsilon_1$  is as in (4.16). Hence

$$(4.25) \quad \frac{1}{2} |D^{2\beta} z|^2 - \frac{1}{2} |D^{2\beta} v|^2 \leq \lambda (D^{2\beta} F(z), D^{2\beta} \partial z) + |\lambda| \varepsilon_1.$$

On the other hand,

$$\begin{aligned}
& (F(z), D^{2\beta}z) - (F(v), D^{2\beta}v) = \\
&= (F(z) - F(v), D^{2\beta}z) + (F(v), D^{2\beta}z - D^{2\beta}v) \\
&= \left( \int_0^1 F'(\tau z + (1-\tau)v) d\tau D^{2\beta}z, (z-v) \right) + (D^{2\beta}F(v), z-v) \\
&\geq (F'(z)D^{2\beta}z, z-v) - |\lambda|\varepsilon_2 - |D^{2\beta}F(v) - D^{2\beta}F(w)| |z-v| \\
&\quad + (D^{2\beta}F(w), z-v) \\
&\geq (F'(z)D^{2\beta}z, \lambda D^{2\beta}\partial z - \lambda\partial F(w)) + (D^{2\beta}F(w), \lambda D^{2\beta}\partial z) - |\lambda|(\varepsilon_1 + \varepsilon_2) \\
&\geq \lambda (F'(z)D^{2\beta}z, D^{2\beta}\partial z) - \lambda (F'(z)D^{2\beta}z, F'(z)\partial z) \\
&\quad + (\lambda D^{2\beta}F(z), D^{2\beta}\partial z) - |\lambda|(2\varepsilon_1 + 2\varepsilon_2),
\end{aligned}$$

where  $\varepsilon_2$  is as in (4.17).

Also,

$$\begin{aligned}
(I(z), 1) - (I(v), 1) &= \int_{\mathbb{T}} (I(z) - I(v)) dx \\
&= \int_{\mathbb{T}} \int_0^1 I'(\tau z + (1-\tau)v) d\tau (z-v) dx \\
&= \left( \int_0^1 I'(\tau z + (1-\tau)v) d\tau, \lambda D^{2\beta}\partial z - \lambda\partial F(w) \right) \\
&= -\lambda \left( \int_0^1 I''(z_\tau) (\partial z_\tau) d\tau, D^{2\beta}z - F(w) \right), \\
&\quad \text{where } z_\tau = \tau z + (1-\tau)v, \\
&\leq -\lambda (I''(z)\partial z, D^{2\beta}z - F(w)) + |\lambda|\varepsilon_3 \\
&\leq -\lambda (I''(z)\partial z, D^{2\beta}z) + \lambda (I''(z)\partial z, F(z)) + |\lambda|\left(\frac{\varepsilon}{2} + \varepsilon_3\right) \\
&= -\lambda (F'(z)\partial z, F'(z)D^{2\beta}z) + |\lambda|\left(\frac{\varepsilon}{2} + \varepsilon_3\right),
\end{aligned}$$

where  $\varepsilon_3$  is as in (4.18).

Putting all the above estimates together, we obtain

$$\begin{aligned}
\frac{1}{\lambda} (\varphi(z) - \varphi(v)) &\leq \\
&\leq (D^{2\beta}F(z), D^{2\beta}\partial z) - \gamma (F'(z)D^{2\beta}z, D^{2\beta}\partial z) + \gamma (F'(z)D^{2\beta}z, F'(z)\partial z) \\
&\quad - \gamma (D^{2\beta}F(z), D^{2\beta}\partial z) - \gamma (F'(z)\partial z, F'(z)D^{2\beta}z) \\
&\quad + (1+2\gamma)\varepsilon_1 + 2\gamma\varepsilon_2 + \gamma\varepsilon_3 + \gamma\frac{\varepsilon}{2} \\
&\leq (1-\gamma) (D^{2\beta}F(z), D^{2\beta}\partial z) - \gamma (F'(z)D^{2\beta}z, D^{2\beta}\partial z) + \varepsilon \\
&= -(1-\gamma) (D^{2\beta}\partial F(z), D^{2\beta}z) - \gamma (F'(z)D^{2\beta}\partial z, D^{2\beta}z) + \varepsilon.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \frac{1}{\lambda} (\varphi(z) - \varphi(v)) \leq \\
& \leq (\gamma - 1) [(D^{2\beta} \partial F(z), D^{2\beta} z) - (F'(z) D^{2\beta} \partial z, D^{2\beta} z)] \\
& \quad - (F'(z) D^{2\beta} z, \partial D^{2\beta} z) + \varepsilon \\
& = (\gamma - 1) [(D^{2\beta} \partial F(z), D^{2\beta} z) - (F'(z) D^{2\beta} \partial z, D^{2\beta} z)] \\
& \quad - \frac{1}{2} (F'(z), \partial (D^{2\beta} z)^2) + \varepsilon. \\
& = (\gamma - 1) [(D^{2\beta} \partial F(z), D^{2\beta} z) - (F'(z) D^{2\beta} \partial z, D^{2\beta} z)] \\
& \quad + \frac{1}{2} (\partial F'(z) D^{2\beta} z, D^{2\beta} z) + \varepsilon \\
& = (\gamma - 1) (D^{2\beta} (F'(z) \partial z) - F'(z) D^{2\beta} \partial z - (2\beta + 1) \partial F'(z) D^{2\beta} z, D^{2\beta} z) + \varepsilon,
\end{aligned}$$

provided that

$$-(\gamma - 1)(2\beta + 1) = \frac{1}{2}$$

which is satisfied precisely for  $\gamma = \frac{4\beta+1}{4\beta+2}$ , when  $\varphi = \varphi_2$ .

Using Lemma 4.5 we get the estimate

$$\frac{1}{\lambda} (\varphi_2(z) - \varphi_2(v)) \leq g(\varphi_2(z)) + \varepsilon.$$

This implies (using (4.10) in conjunction with (3.20))

$$(4.26) \quad \varphi_2(z) \leq m_{g_\varepsilon}(\lambda, \varphi_2(v)) < \alpha_2.$$

This concludes the proof of (4.20). The Schauder-Tichonov theorem applied to  $\Gamma : K \rightarrow K$  gives us the desired fixed point,  $u = \Gamma u$ , so that

$$(4.27) \quad u - \lambda D^{2\beta} \partial u - \lambda \partial F(u) = v.$$

The estimates for  $\varphi_j(z)$  in terms of  $\varphi_j(v)$  imply (4.6),(4.7),(4.8).

What remains to be proven is the estimate (4.9) for  $\varphi_3(z)$ . We will make use of the dissipativity of  $A + B_R - \omega_R$ , for some  $\omega_R \in \mathbb{R}$ .

For  $v$  and  $u$  as above we know that  $\varphi_j(u) \leq \alpha_j, j = 0, 1, 2$ . Then

$$\begin{aligned}
\omega_R |u - v|^2 & \geq (Au + B(u) - Av - B(v), u - v) \\
& = (D^{2\beta} \partial u - \partial F(u) - D^{2\beta} \partial v + \partial F(v), u - v) \\
& = \lambda |D^{2\beta} \partial u - \partial F(u)|^2 + (D^{2\beta} \partial v + \partial F(v), u - v) \\
& \geq \lambda \varphi_3(u) - (\varphi_3(v))^{1/2} |u - v|.
\end{aligned}$$

Thus

$$\lambda \varphi_3(u) \leq \varphi_3(v) |u - v| + \omega_R |u - v|^2.$$

But  $|u - v| = |\lambda| \varphi_3(u)$ , so we conclude

$$\varphi_3(u) \leq \varphi_3(v) + \lambda \omega_R \varphi_3(u),$$

or, equivalently,

$$\varphi_3(u) \leq \frac{\varphi_3(v)}{1 - \lambda \omega_R}.$$

This completes the proof of Theorem 4.5.  $\square$

Having all the hypothesis in place, we can apply the Abstract Theorem 3.1 and obtain the global well-posedness for the Cauchy Problem (NDE) in the space  $H^s(\mathbb{T})$ , where  $s = \max\{2\beta, \frac{3}{2} + \varepsilon\}$ , for some  $\varepsilon > 0$  as in Theorem 1.2.

**4.4. Uniform bounds for the solutions in the  $H^\beta$  norm.** In the rest of this chapter we make a few remarks about sufficient conditions that will ensure the uniform boundedness of the solutions in the  $H^\beta$ -norm. As we saw in this chapter, the global existence in time of solutions is intimately related to the existence of a uniform bound for the  $H^\beta$ -norm.

If condition (C) holds, that is  $p < 4\beta$ , then we saw in Lemma 4.4 that the invariance of the  $\varphi_1$  along solutions implies the uniform boundedness of  $|u(t)|_{H^\beta}$ . Thus, the interesting case remains when  $p \geq 4\beta$ . For ease of exposition, we will assume in what follows that the nonlinearity is  $F'(u) = u^p$ . Then the following holds true.

**Theorem 4.8.** *(Sufficient conditions for uniform boundedness of solutions in  $H^\beta$ )* Let  $p \geq 4\beta$ ,  $q^* = p - 4\beta + 2 (\geq 2)$ , and  $u(t)$  be a solution of the initial value problem

$$\begin{aligned} u_t - D^{2\beta}\partial u + u^p\partial u &= 0 \\ u(0) &= u_0. \end{aligned}$$

(i) If  $|u(t)|_{L^q}$  uniformly bounded in  $t$  (for some  $q > q^*$ ), then  $|u(t)|_{H^\beta}$  is uniformly bounded in  $t$ .

(ii) There exists a constant  $C = C(p)$  such that if  $|u(t)|_{L^{q^*}} < C$  for all  $t$ , then  $|u(t)|_{H^\beta}$  is uniformly bounded in  $t$ .

In particular, if  $p = 4\beta$  then there exists a constant  $C = C(p) > 0$  such that  $|u_0|_{L^2} < C$  implies  $|u(t)|_{H^\beta}$  is uniformly bounded in  $t$ .

*Proof.* Recall the invariance of  $\varphi_1$  along solutions, which holds even for the case  $p \geq 4\beta$ :

$$\begin{aligned} \varphi_1(u) &= \frac{1}{2} \int |D^\beta u|^2 - c_p \int u^{p+2}, \quad (\text{where } c_p = \frac{1}{(p+1)(p+2)}) \\ &= \varphi_1(u_0). \end{aligned}$$

Here and in the subsequent calculations  $u = u(t)$ . To prove (i), assume there exists  $q > p - 4\beta + 2 \geq 2$  such that  $|u|_{L^q}$  is uniformly bounded:

$$|u(t)|_{L^q} \leq C \text{ for all } t.$$

With the convention  $u = u(t)$ , we have, for all  $t$ ,

$$\begin{aligned} |D^\beta u|_{L^2}^2 &= \int |D^\beta u|^2 \\ &= 2c_p \int u^{p+2} + 2\varphi_1(u_0) \\ &\leq 2c_p |u|_\infty^{p+2-q} |u|_{L^q}^q + 2\varphi_1(u_0) \\ &\leq 2c_p \left( C |u|_{L^2}^{1-\frac{1}{2\beta}} |D^\beta u|_{L^2}^{\frac{1}{2\beta}} \right)^{p+2-q} |u|_{L^q}^q + 2\varphi_1(u_0) \\ &\leq C_p |u|_{L^2}^{\left(1-\frac{1}{2\beta}\right)(p+2-q)} |D^\beta u|_{L^2}^{\frac{p+2-q}{2\beta}} |u|_{L^q}^q + 2\varphi_1(u_0) \\ &\leq C_1 |D^\beta u|_{L^2}^{\frac{p+2-q}{2\beta}} + C_2, \end{aligned}$$

where  $C_1, C_2$  are independent of  $t$ , but may depend on  $p$ . It is now clear that, since  $\frac{p+2-q}{2\beta} < 2$ ,  $|D^\beta u|_{L^2}$  is uniformly bounded. Thus,  $|u(t)|_{H^\beta}$  is uniformly bounded for all  $t$ . This completes the proof of (i).

To show (ii) we redo the calculation above, this time for  $q^* = p - 4\beta + 2 \geq 2$ . We obtain, as in (4.28),

$$\begin{aligned} |D^\beta u|_{L^2}^2 &\leq C_p |u|_{L^2}^{(1-\frac{1}{2\beta})(p+2-q^*)} |D^\beta u|_{L^2}^{\frac{p+2-q^*}{2\beta}} |u|_{L^{q^*}}^{q^*} + 2\varphi_1(u_0). \\ &= C_p |u|_{L^2}^{4\beta-2} |D^\beta u|_{L^2}^2 |u|_{L^{p-4\beta+2}}^{p-4\beta+2} + 2\varphi_1(u_0). \end{aligned}$$

We are in the periodic case, where  $|u|_{L^2} \leq c |u|_{L^{p-4\beta+2}}$ , thus we obtain

$$|D^\beta u|_{L^2}^2 \leq C(p) |D^\beta u|_{L^2}^2 |u|_{L^{p-4\beta+2}}^p + 2\varphi_1(u_0).$$

From this, we clearly get a bound on  $|D^\beta u|_{L^2}$  when  $C(p) |u|_{L^{p-4\beta+2}}^p < 1$ , i.e. when  $|u|_{L^{q^*}} < C(p)^{-\frac{1}{p}}$  we obtain  $|D^\beta u| \leq \text{const}(p, u_0)$ .

In the special case  $p = 4\beta$ , ( $q^* = 2$ ), we have the  $|u(t)|_{L^2} = |u_0|_{L^2}$  for all times, thus the  $(L^{q^*} = L^2)$  - upper bound for the initial data guarantees the uniform bound for  $|u(t)|_{H^\beta}$ .

This concludes the proof of the proposition.  $\square$

The bounds presented above may not be optimal. It is known, at least in some special cases, like (KdV), (BO), that there exist solitary waves  $\psi(x - ct)$ , defined for all times, with any prescribed speed  $c > 0$ . The remarkable property of these solitary waves in the case  $p = 4\beta$  is that the  $L^2$ -norm is independent of their speed. This value of the norm is the largest possible bound for general initial data which guarantees the global existence in time of the solution. Thus, the optimality of these bounds is very much related with the open question of whether smooth solutions blow-up in the  $H^\beta$  space in finite or infinite time.

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