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J. Math. Anal. Appl. 294 (2004) 113–121
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

$\mathcal{J}$-self-adjointness of a class of Dirac-type operators

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Received 17 December 2003
Available online 27 March 2004
Submitted by G. Hagedorn

Abstract

In this note, we prove that the maximally defined operator associated with the Dirac-type differential expression

$$M(Q)=i\begin{pmatrix} \frac{d}{dx}I_m & -Q \\ -Q^* & -\frac{d}{dx}I_m \end{pmatrix},$$

where Q represents a symmetric $m\times m$ matrix (i.e., $Q(x)^\top=Q(x)$ a.e.) with entries in $L^1_{\mathrm{loc}}(\mathbb{R})$, is $\mathcal{J}$-self-adjoint, where $\mathcal{J}$ is the antilinear conjugation defined by $\mathcal{J}=\sigma_1\mathcal{C}$, $\sigma_1=\left(\begin{smallmatrix} 0 & I_m \\ I_m & 0\end{smallmatrix}\right)$ and $\mathcal{C}(a_1,\dots,a_m,b_1,\dots,b_m)^\top=(\overline{a_1},\dots,\overline{a_m},\overline{b_1},\dots,\overline{b_m})^\top$. The differential expression $M(Q)$ is of significance as it appears in the Lax formulation of the non-abelian (matrix-valued) focusing nonlinear Schrödinger hierarchy of evolution equations.

Keywords: Dirac-type operator; J-self-adjointness

To set the stage for this note, we briefly mention the Lax pair and zero-curvature representations of the matrix-valued Ablowitz–Kaup–Newell–Segur (AKNS) equations and the special focusing and defocusing nonlinear Schrödinger (NLS) equations associated with it. Let $P=P(x,t)$ and $Q=Q(x,t)$ be smooth $m\times m$ matrices, $m\in\mathbb{N}$, and introduce the Lax pair of $2m\times 2m$ matrix-valued differential expressions

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.02.002

$$M(P,Q)=i\begin{pmatrix}\frac{d}{dx}I_m & -Q\\ P & -\frac{d}{dx}I_m\end{pmatrix}, \tag{1}$$

$$L(P,Q)=i\begin{pmatrix}\frac{d^2}{dx^2}I_m-\frac{1}{2}QP & -Q\frac{d}{dx}-\frac{1}{2}Q_x\\ P\frac{d}{dx}-\frac{1}{2}P_x & -\frac{d^2}{dx^2}I_m+\frac{1}{2}PQ\end{pmatrix} \tag{2}$$

and the $2m\times 2m$ zero-curvature matrices

$$U(z,P,Q)=\begin{pmatrix}-izI_m & Q\\ P & izI_m\end{pmatrix}, \tag{3}$$

$$V(z,P,Q)=\begin{pmatrix}-iz^2I_m-\frac{i}{2}QP & zQ+\frac{i}{2}Q_x\\ zP-\frac{i}{2}P_x & iz^2I_m+\frac{i}{2}PQ\end{pmatrix}, \tag{4}$$

where $z\in\mathbb{C}$ denotes a (spectral) parameter and I_m is the identity matrix in $\mathbb{C}^m$. Then the Lax equation

$$\frac{d}{dt}M-[L,M]=0 \tag{5}$$

is equivalent to the $m\times m$ matrix-valued AKNS system

$$\begin{aligned}&Q_t-\frac{i}{2}Q_{xx}+iQPQ=0,\\ &P_t+\frac{i}{2}P_{xx}-iPQP=0,\end{aligned} \tag{6}$$

where $[\cdot\,,\cdot]$ denotes the commutator symbol. Similarly, the zero-curvature equation

$$U_t-V_x+[U,V]=0 \tag{7}$$

is also equivalent to the $m\times m$ matrix-valued AKNS system (6). Two special cases of this formalism are of particular importance: The *focusing* NLS equation,

$$\text{focusing:}\quad Q_t-\frac{i}{2}Q_{xx}-iQQ^*Q=0, \tag{8}$$

obtained from (1)–(7) in the special case where $P=-Q^*$, and the *defocusing* NLS equation,

$$\text{defocusing:}\quad Q_t-\frac{i}{2}Q_{xx}+iQQ^*Q=0, \tag{9}$$

obtained from (1)–(7) in the special case where $P=Q^*$. Here Q^* denotes the adjoint (i.e., complex conjugate and transpose) matrix of Q.

In this note, we will restrict our attention to the focusing NLS case $P=-Q^*$. (See, e.g., [1, Section 3.3, Chapter 8], [13] and [14, Section 3.1] in which an inverse scattering approach is developed for the matrix NLS equation (8).) Actually, (6), (8), and (9) are just the first equations in an infinite hierarchy of nonlinear evolution equations (the non-abelian AKNS, and focusing and defocusing NLS hierarchies) but we will not further dwell on this point.

Recently, it has been proved in [3] that the 2×2 matrix-valued Lax differential expression

$$M(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ -\bar{q} & -\frac{d}{dx} \end{pmatrix} \tag{10}$$

corresponding to the scalar focusing NLS hierarchy defines, under the most general hypothesis $q \in L^1_{\text{loc}}(\mathbb{R})$ on the potential q, a $\mathcal{J}$-self-adjoint operator in $L^2(\mathbb{R})^2$, where J is the antilinear conjugation, $\mathcal{J} = \sigma_1 \mathcal{C}$, with $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\mathcal{C}$ is complex conjugation in $\mathbb{C}^2$. This is the direct analog of a recently proven fact in [4, Lemma 2.15] that the Dirac-type Lax differential expression in the defocusing nonlinear Schrödinger (NLS) case is always in the limit point case at $\pm\infty$. Equivalently, the maximally defined Dirac-type operator corresponding to the defocusing NLS case is always self-adjoint.

In this paper we present an extension of the result in [3], for $2m \times 2m$ matrix-valued Dirac-type differential expressions of the form

$$M(Q) = i \begin{pmatrix} \frac{d}{dx} I_m & -Q \\ -Q^* & -\frac{d}{dx} I_m \end{pmatrix} \tag{11}$$

associated with the non-abelian (matrix-valued) focusing NLS equation (8).

We will assume the following conditions on Q from now on ($A^\top$ denotes the transpose of the matrix A):

Hypothesis 1. Assume $Q \in L^1_{\text{loc}}(\mathbb{R})^{m \times m}$ satisfies

$$Q = Q^\top \quad \text{a.e.} \tag{12}$$

Next, we briefly recall some basic facts about $\mathcal{J}$-symmetric and $\mathcal{J}$-self-adjoint operators in a complex Hilbert space $\mathcal{H}$ (see, e.g., [5, Section III.5] and [6, p. 76]) with scalar product denoted by $(\cdot\,,\cdot)_{\mathcal{H}}$ (linear in the first and antilinear in the second place) and corresponding norm denoted by $\|\cdot\|_{\mathcal{H}}$. Let $\mathcal{J}$ be a conjugation operator in $\mathcal{H}$, that is, $\mathcal{J}$ is an antilinear involution satisfying

$$(\mathcal{J}u, v)_{\mathcal{H}} = (\mathcal{J}v, u)_{\mathcal{H}} \quad \text{for all } u, v \in \mathcal{H}, \quad \mathcal{J}^2 = I. \tag{13}$$

In particular,

$$(\mathcal{J}u, \mathcal{J}v)_{\mathcal{H}} = (v, u)_{\mathcal{H}}, \quad u, v \in \mathcal{H}. \tag{14}$$

A linear operator S in $\mathcal{H}$, with domain $\operatorname{dom}(S)$ dense in $\mathcal{H}$, is called $\mathcal{J}$-symmetric if

$$S \subseteq \mathcal{J} S^* \mathcal{J} \quad (\text{equivalently, if} \quad JSJ \subseteq S^*). \tag{15}$$

Clearly, (15) is equivalent to

$$(\mathcal{J}u, Sv)_{\mathcal{H}} = (\mathcal{J}Su, v)_{\mathcal{H}}, \quad u, v \in \operatorname{dom}(S). \tag{16}$$

Here S^* denotes the adjoint operator of S in $\mathcal{H}$. If S is $\mathcal{J}$-symmetric, so is its closure $\overline{S}$. The operator S is called $\mathcal{J}$-self-adjoint if

$$S = \mathcal{J} S^* \mathcal{J} \quad (\text{equivalently, if} \quad \mathcal{J} S \mathcal{J} = S^*). \tag{17}$$

Finally, a densely defined, closable operator T is called essentially $\mathcal{J}$-self-adjoint if its closure, $\overline{T}$, is $\mathcal{J}$-self-adjoint, that is, if

$$\overline{T} = \mathcal{J} T^* \mathcal{J}. \tag{18}$$

Next, assuming S to be $\mathcal{J}$-symmetric, one introduces the following inner product $(\cdot\,,\cdot)_*$ on $\mathrm{dom}(\mathcal{J} S^* \mathcal{J}) = \mathcal{J}\,\mathrm{dom}(S^*)$ according to [8] (see also [12]),

$$(u, v)_* = (\mathcal{J}u, \mathcal{J}v)_{\mathcal{H}} + (S^*\mathcal{J}u, S^*\mathcal{J}v)_{\mathcal{H}}, \quad u, v \in \mathrm{dom}(\mathcal{J} S^* \mathcal{J}), \tag{19}$$

which renders $\mathrm{dom}(\mathcal{J} S^* \mathcal{J})$ a Hilbert space. Then the following theorem holds ($I_{\mathcal{H}}$ denotes the identity operator in $\mathcal{H}$).

Theorem 2 (Race [12]). *Let S be a densely defined closed $\mathcal{J}$-symmetric operator. Then*

$$\mathrm{dom}(\mathcal{J} S^* \mathcal{J}) = \mathrm{dom}(S) \oplus_* \ker\big((S^*\mathcal{J})^2 + I_{\mathcal{H}}\big), \tag{20}$$

where $\oplus_*$ means the orthogonal direct sum with respect to the inner product $(\cdot\,,\cdot)_*$. In particular, a densely defined closed $\mathcal{J}$-symmetric operator S is $\mathcal{J}$-self-adjoint if and only if

$$\ker\big((S^*\mathcal{J})^2 + I_{\mathcal{H}}\big) = \{0\}. \tag{21}$$

Theorem 2 will be used to prove the principal result of this note that the (maximally defined) Dirac-type operator associated with the differential expression $M(Q)$ in (11) (relevant to the focusing matrix NLS equation (8)) is always $\mathcal{J}$-self-adjoint under most general conditions on the coefficient Q in Hypothesis 1 (see Theorem 4). This will be done by verifying a relation of the type (21).

To this end, it is convenient to introduce some standard notations to be used throughout the remainder of this paper. The Hilbert space $\mathcal{H}$ is chosen to be $L^2(\mathbb{R})^{2m} = L^2(\mathbb{R})^m \oplus L^2(\mathbb{R})^m$. The space of $m \times m$ matrices with entries in $L^1_{\mathrm{loc}}(\mathbb{R})$ is denoted by $L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$. An antilinear conjugation $\mathcal{J}$ in the complex Hilbert space $L^2(\mathbb{R})^{2m}$ is defined by

$$\mathcal{J} = \sigma_1 \mathcal{C}, \tag{22}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$
$$\mathcal{C}(a_1, \dots, a_m, b_1, \dots, b_m)^\top = (\overline{a_1}, \dots, \overline{a_m}, \overline{b_1}, \dots, \overline{b_m})^\top. \tag{23}$$

Given Hypothesis 1, we now introduce the following maximal and minimal Dirac-type operators in $L^2(\mathbb{R})^{2m}$ associated with the differential expression $M(Q)$:

$$D_{\max}(Q)F = M(Q)F, \tag{24}$$
$$F \in \mathrm{dom}\big(D_{\max}(Q)\big) = \big\{G \in L^2(\mathbb{R})^{2m} \,\big|\, G \in AC_{\mathrm{loc}}(\mathbb{R})^{2m},\ M(Q)G \in L^2(\mathbb{R})^{2m}\big\},$$
$$D_{\min}(Q)F = M(Q)F, \tag{25}$$
$$F \in \mathrm{dom}\big(D_{\min}(Q)\big) = \big\{G \in \mathrm{dom}\big(D_{\max}(Q)\big) \,\big|\, \mathrm{supp}(G) \text{ is compact}\big\}.$$

It follows by standard techniques (see, e.g., [10, Chapter 8] and [15]) that under Hypothesis 1, $D_{\min}(Q)$ is densely defined and closable in $L^2(\mathbb{R})^{2m}$ and $D_{\max}(Q)$ is a densely defined closed operator in $L^2(\mathbb{R})^{2m}$. Moreover one infers (see, e.g., [10, Lemma 8.6.2] and [15] in the analogous case of symmetric Dirac operators)

$$\overline{D_{\min}(Q)} = D_{\max}(-Q)^*, \quad \text{or equivalently,} \quad D_{\min}(Q)^* = D_{\max}(-Q). \tag{26}$$

The following result will be a crucial ingredient in the proof of Theorem 4, the principal result of this note.

Theorem 3. *Assume Hypothesis* 1. *Let $N(Q)$ be the following* (*formally self-adjoint*) *differential expression*

$$N(Q) = i\begin{pmatrix} \frac{d}{dx}I_m & -Q \\ Q^* & \frac{d}{dx}I_m \end{pmatrix} \tag{27}$$

and denote by $\widetilde{D}_{\max}(Q)$ the maximally defined Dirac-type operator in $L^2(\mathbb{R})^{2m}$ associated with $N(Q)$,

$$\widetilde{D}_{\max}(Q)F = N(Q)F, \tag{28}$$
$$F \in \operatorname{dom}\big(\widetilde{D}_{\max}(Q)\big) = \big\{G \in L^2(\mathbb{R})^{2m} \bigm| G \in AC_{\mathrm{loc}}(\mathbb{R})^{2m},\ N(Q)G \in L^2(\mathbb{R})^{2m}\big\}.$$

Then,

(i) *The following identity holds*:

$$M(-Q)M(Q) = N(Q)^2. \tag{29}$$

(ii) *Let $U_Q = U_Q(x)$ satisfy the initial value problem*

$$U_Q' = \begin{pmatrix} 0 & Q \\ -Q^* & 0 \end{pmatrix} U_Q, \quad U_Q(0) = I_{2m}. \tag{30}$$

Then $\{U_Q(x)\}_{x\in\mathbb{R}}$ is a family of unitary matrices in $\mathbb{C}^{2m}$ with entries in $AC_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$U_Q^{-1}N(Q)U_Q = i\frac{d}{dx}I_{2m}. \tag{31}$$

(iii) *Let $\mathcal{U}_Q$ denote the multiplication operator with $U_Q(\cdot)$ on $L^2(\mathbb{R})^{2m}$. Then $\widetilde{D}_{\max}(Q)$ is unitarily equivalent to the maximally defined operator in $L^2(\mathbb{R})^{2m}$ associated with the differential expression $i\frac{d}{dx}I_{2m}$,*

$$\mathcal{U}_Q^{-1}\widetilde{D}_{\max}(Q)\mathcal{U}_Q = \left(i\frac{d}{dx}I_{2m}\right)_{\max}, \tag{32}$$
$$\operatorname{dom}\left(\left(i\frac{d}{dx}I_{2m}\right)_{\max}\right) = H^{1,2}(\mathbb{R})^{2m}$$
$$= \big\{F \in L^2(\mathbb{R})^{2m} \bigm| F \in AC_{\mathrm{loc}}(\mathbb{R})^{2m},\ F' \in L^2(\mathbb{R})^{2m}\big\}.$$

Moreover,

$$\mathcal{U}_Q^{-1} D_{\max}(-Q) D_{\max}(Q) \mathcal{U}_Q = \left(-\frac{d^2}{dx^2} I_{2m}\right)_{\max}, \tag{33}$$
$$\operatorname{dom}\left(\left(-\frac{d^2}{dx^2} I_{2m}\right)_{\max}\right) = H^{2,2}(\mathbb{R})^{2m}$$
$$= \left\{F \in L^2(\mathbb{R})^{2m} \;\middle|\; F, F' \in AC_{\mathrm{loc}}(\mathbb{R})^{2m}, F', F'' \in L^2(\mathbb{R})^{2m}\right\}.$$

Proof. That $N(Q)$ is formally self-adjoint and $M(-Q)M(Q) = N(Q)^2$, as stated in (i), is an elementary matrix calculation.

To prove (ii), we note that the initial value problem (30) is well-posed in the sense of Carathéodory since $Q \in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$ (cf., e.g., [7, Lemma IX.2.2]) with a solution matrix U_Q with entries in $AC_{\mathrm{loc}}(\mathbb{R})$. Moreover, for each $x \in \mathbb{R}$, $U_Q(x)$ is a unitary matrix in $\mathbb{C}^{2m}$, since $U_Q' = -B(Q)U_Q$, with $B(Q) = \left(\begin{smallmatrix} 0 & -Q \\ Q^* & 0 \end{smallmatrix}\right)$ being skew-adjoint. Thus, the entries $U_{Q,j,k}$, $1 \leqslant j,k \leqslant 2m$ of U_Q (as well as those of U_Q^{-1}) actually satisfy

$$U_{Q,j,k} \in AC_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 1 \leqslant j,k \leqslant 2m. \tag{34}$$

(Since U_Q is a bounded matrix-valued operator of multiplication in $L^2(\mathbb{R})^{2m}$, its entries $U_{Q,j,k}$ are all in $L^\infty(\mathbb{R})$, as one readily verifies by studying scalar products of the form $(F_j, U_Q F_k)_{L^2(\mathbb{R})^{2m}} = (f_j, U_{Q,j,k} f_k)_{L^2(\mathbb{R})}$, $1 \leqslant j,k \leqslant 2m$, where $F_j = (0,\dots,0,f_j,0,\dots,0)^\top$ with $f_j \in L^2(\mathbb{R})$, $1 \leqslant j \leqslant 2m$.) Next, fix $F \in AC_{\mathrm{loc}}(\mathbb{R})^{2m}$, such that $\mathcal{U}_Q^{-1} F \in H^{1,2}(\mathbb{R})^{2m}$. Then

$$U_Q\left(i\frac{d}{dx} I_{2m}\right)U_Q^{-1} F = i\frac{d}{dx}F + iU_Q\frac{d}{dx}\left(U_Q^{-1}\right)F = i\frac{d}{dx}F + iU_Q\left(U_Q^{-1}B(Q)^*\right)F$$
$$= N(Q)F, \tag{35}$$

where we used the fact that $(U_Q^{-1})' = U_Q^{-1}B(Q)^*$. Thus, (ii) follows.

Moreover, by (34), the fact that U_Q is unitary in $\mathbb{C}^{2m}$, and by (35) one concludes $\operatorname{dom}(\widetilde{D}_{\max}(Q)) = \mathcal{U}_Q H^{1,2}(\mathbb{R})^{2m}$. This proves (32).

Clearly, (i) and (ii) yield the relation

$$U_Q^{-1} M(-Q)M(Q)U_Q = -\frac{d^2}{dx^2} I_{2m}.$$

Thus, (33) will follow once we prove the following facts:

$$\text{(i)} \quad U_Q F \in L^2(\mathbb{R})^{2m} \quad \text{if and only if} \quad F \in L^2(\mathbb{R})^{2m}, \tag{36}$$
$$\text{(ii)} \quad U_Q F \in AC_{\mathrm{loc}}(\mathbb{R})^{2m} \quad \text{if and only if} \quad F \in AC_{\mathrm{loc}}(\mathbb{R})^{2m}, \tag{37}$$
$$\text{(iii)} \quad M(Q)U_Q F \in L^2(\mathbb{R})^{2m} \quad \text{if and only if} \quad F' \in L^2(\mathbb{R})^{2m}, \tag{38}$$
$$\text{(iv)} \quad M(Q)U_Q F \in AC_{\mathrm{loc}}(\mathbb{R})^{2m} \quad \text{if and only if} \quad F' \in AC_{\mathrm{loc}}(\mathbb{R})^{2m}, \tag{39}$$
$$\text{(v)} \quad M(-Q)M(Q)U_Q F \in L^2(\mathbb{R})^{2m} \quad \text{if and only if} \quad F'' \in L^2(\mathbb{R})^{2m}. \tag{40}$$

Clearly (36) and (40) hold since U_Q is unitary in $\mathbb{C}^{2m}$. (37) is valid since

$$U_{Q,j,k}, U^{-1}_{Q,j,k} \in AC_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad j,k = 1,\dots,2m.$$

Next, for $F = (F_1^\top, F_2^\top)^\top$, $F_1, F_2 \in L^2(\mathbb{R})^m$, an explicit computation yields

$$M(Q)U_Q F = i \begin{pmatrix} U_Q^{(1)} F_1' + U_Q^{(2)} F_2' \\ -U_Q^{(3)} F_1' - U_Q^{(4)} F_2' \end{pmatrix}, \qquad F = \big(F_1^\top, F_2^\top\big)^\top, \tag{41}$$

where $U_Q^{(l)}$, $l = 1,2,3,4$, are blocks of the matrix U_Q,

$$U_Q = \begin{pmatrix} U_Q^{(1)} & U_Q^{(2)} \\ U_Q^{(3)} & U_Q^{(4)} \end{pmatrix}. \tag{42}$$

Introducing

$$V_Q = \sigma_3 U_Q \sigma_3 = \begin{pmatrix} U_Q^{(1)} & -U_Q^{(2)} \\ -U_Q^{(3)} & U_Q^{(4)} \end{pmatrix}, \tag{43}$$

one infers $V_{Q,j,k}, V^{-1}_{Q,j,k} \in AC_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $j,k = 1,\dots,2m$, and

$$V_Q^{-1} M(Q) U_Q F = i\big(F_1', -F_2'\big)^\top, \tag{44}$$

and hence (38) and (39) hold. This proves (33). □

The principal result of this note then reads as follows.

Theorem 4. *Assume Hypothesis* 1. *Then the minimally defined Dirac-type operator $D_{\min}(Q)$ associated with the Lax differential expression*

$$M(Q) = i \begin{pmatrix} \frac{d}{dx} I_m & -Q \\ -Q^* & -\frac{d}{dx} I_m \end{pmatrix} \tag{45}$$

introduced in (25) *is essentially $\mathcal{J}$-self-adjoint in $L^2(\mathbb{R})^{2m}$, that is,*

$$\overline{D_{\min}(Q)} = \mathcal{J} D_{\min}(Q)^* \mathcal{J}, \tag{46}$$

where $\mathcal{J}$ is the conjugation defined in (22). *Moreover,*

$$\overline{D_{\min}(Q)} = D_{\max}(Q) \tag{47}$$

and hence $D_{\max}(Q)$ is $\mathcal{J}$-self-adjoint.

Proof. We first recall (cf. (26))

$$D_{\min}(Q)^* = D_{\max}(-Q) \tag{48}$$

and also note

$$\mathcal{J} D_{\max}(-Q) \mathcal{J} = D_{\max}\big(Q^\top\big) = D_{\max}(Q). \tag{49}$$

Here we employed the symmetry of Q (see (12)). Since $\overline{D_{\min}(Q)}$ is closed and $\mathcal{J}$-symmetric (this follows from (48) and (49)), its $\mathcal{J}$-self-adjointness is equivalent to showing that (cf. (21))

$$\ker\big(D_{\min}(Q)^*\mathcal{J}D_{\min}(Q)^*\mathcal{J}+I_{L^2(\mathbb{R})^{2m}}\big)=\ker\big(D_{\max}(-Q)D_{\max}(Q)+I_{L^2(\mathbb{R})^{2m}}\big)$$
$$=\{0\}. \tag{50}$$

Since $D_{\max}(-Q)D_{\max}(Q)$ is unitarily equivalent to $(-d^2I_{2m}/dx^2)_{\max}\geqslant 0$ by Theorem 3(iii), one concludes that

$$D_{\max}(-Q)D_{\max}(Q)\geqslant 0 \tag{51}$$

and hence (50) holds. The fact (47) now follows from (46) and (48),

$$\overline{D_{\min}(Q)}=\mathcal{J}D_{\min}(Q)^*\mathcal{J}=\mathcal{J}D_{\max}(-Q)\mathcal{J}=D_{\max}(Q). \qquad \square \tag{52}$$

As mentioned in the introductory paragraph, Theorem 4 in the $\mathcal{J}$-self-adjoint context can be viewed as an analog of [4, Lemma 2.15] in connection with self-adjoint Dirac-type operator relevant in the non-abelian (matrix-valued) defocusing nonlinear Schrödinger hierarchy (cf. also [9] for results of this type).

We conclude with a short remark. The special case where

$$Q=\begin{pmatrix} q_1 & 0 & \dots & 0\\ q_2 & 0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ q_m & 0 & \dots & 0\end{pmatrix}, \quad \text{or} \quad Q=(\mathbf{q}\quad \mathbf{0}), \tag{53}$$

is known as the vector NLS equation (cf. [2])

$$i\mathbf{q}_t+\frac{1}{2}\mathbf{q}_{xx}+\|\mathbf{q}\|^2\mathbf{q}=0, \tag{54}$$

a generalization of the well-known Manakov system [11] (for $m=2$). Here $\mathbf{q}=(q_1,\dots,q_m)^\top$, $(\|\mathbf{q}\|^2=\mathbf{q}^*\mathbf{q}=\sum_{j=1}^m|q_j|^2)$. Unfortunately, the methods applied in this note forced us to restrict our attention to symmetric matrices Q only (i.e., $Q=Q^\top$) and hence our current result does not apply to the vector NLS case.

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