Variable Coefficient KdV Equations and Waves in Elastic Tubes

RADU C. CASCAVAL Department of Mathematics, University of Missouri, Columbia, MO 65211 (radu@math.missouri.edu)

Dedicated to Jerry Goldstein and Rainer Nagel on the occasion of their 60th birthdays

ABSTRACT We present a simplified one-dimensional model for pulse wave propagation through fluid-filled tubes with elastic walls, which takes into account the elasticity of the wall as well as the tapering effect. The spatial dynamics in this model is governed by a variable coefficient KdV equation with conditions given at the inflow site. We discuss an existence theory for the associated evolution equation, based on a semilinear Hille-Yosida theory, which was previously developed for the classical KdV equation.

KEYWORDS KdV equation, elastic tubes, pressure wave.

1 INTRODUCTION

The study of pulse wave propagation in blood vessels constitutes a major component in the effort to better understand the dynamics of the circulatory system, both in normal and pathological conditions. The blood is a suspension of cells and other particles in plasma. Due to the complexity of blood rheology, a mathematical description of blood itself has not yet been completely formulated. Nevertheless, there have been many attempts of describing the dynamics in the circulatory system through mathematical models built on various simplifying assumptions, see [9], [16], [17], [19], [20], [21]. In the systemic circulation, the large vessels are approximated by tubes with thin, elastic walls, while the blood filling the vessels is considered as a continuum, incompressible fluid. For smaller vessels, the continuum assumption is no longer valid. The walls of the smaller arteries and arterioles become less elastic and consist of a muscular tissue with important role in controlling the arterial pressure.

The pressure wave, initiated in the ascending aorta by the heart, is propagated along the arterial tree until it reaches the smallest sites of the circulation, the capillaries. A major factor in the wave propagation is the elasticity of the wall. If the vessel walls were rigid, the wave motion would be in bulk, even in the pulsatile regime, with any disturbance at one end of the tube propagating with infinite speed along the tube. This is not the case when the vessels walls are compliant, as in the sistemic portion of the circulation. The compliance of the walls is reflected by the presence of a circumferential stress corresponding to a given amount of wall displacement from its initial configuration. This pressure is similar to the hydrostatic pressure which appears in the models of the water waves in shallow channels.

The heart plays the role of a wave maker in the systemic circulation. As an analogy, there are mathematical models of wave generation in water channels, described by Korteweg-de Vries equations in a quarter plane. The initial-boundary value problem has been studied extensively, including well-posedness and regularity of solutions (see, e.g., [1], [2], [3].) As noted in [2], laboratory coordinates are preferred for studying the initial-boundary value problem, since it restricts the (t, x) in the quarter plane. Nevertheless, in case of a variable depth channel, other coordinates were used, such as in [13] and [14]. It has been proven that with a particular choice of a moving frame, the equation describing the wave propagation over an uneven bottom is a variable coefficients Korteweg-de Vries equation. The evolution is described with respect to the spatial variable x rather than the physical time t. The advantage of such a description is that the inflow boundary condition becomes an initial condition of the evolution. This is similar to the case of wave propagation in fiber optics, where the nonlinear Schrödinger equation describes the spatial dynamics of the optical signal rather than its temporal evolution.

2 THE GOVERNING EQUATIONS

Throughout this paper we will assume that our fluid-filled tube has compliant walls and circular cross-sections, at rest and during deformations. We assume that the fluid is incompressible and inviscid, and therefore its motion is governed by the Euler equations. We also assume that the motion is axisymmetric, therefore $\bar{u} = \bar{u}(\bar{t}, \bar{x}, \bar{r})$ and same for $\bar{v}$, $\bar{p}$, while the circumferential component of the velocity is identically zero. Written in cylindrical coordinates, the equations of the fluid motion are

$$\bar{u}_{\bar{t}} + \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{r}} + \frac{1}{\rho}\bar{p}_{\bar{x}} = 0, \tag{2.1}$$

$$\bar{v}_{\bar{t}} + \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{r}} + \frac{1}{\rho}\bar{p}_{\bar{r}} = 0, \tag{2.2}$$

$$\bar{u}_{\bar{x}} + \bar{v}_{\bar{r}} + \frac{1}{\bar{r}}\bar{v} = 0, \tag{2.3}$$

whenever $0 \le \bar{r} \le \bar{r}^w(\bar{t}, \bar{x}) = \bar{r}_\mathrm{o}(\bar{x}) + \bar{\eta}(\bar{t}, \bar{x})$. Here $\bar{r}^w = \bar{r}^w(\bar{t}, \bar{x})$ is the inner radius of the tube, with $\bar{r}_\mathrm{o} = \bar{r}_\mathrm{o}(\bar{x})$ is the radius of the unstressed tube and $\bar{\eta} = \bar{\eta}(\bar{t}, \bar{x})$ the wall displacement from the unstressed position.

To obtain the boundary conditions at the wall, we assume that the fluid velocity at the wall equals the velocity of the wall itself, (no-slip condition), i.e.

$$\begin{aligned}\bar{v}(\bar{t},\bar{x},\bar{r})\big|_{\bar{r}=\bar{r}^w(\bar{t},\bar{x})} &= \frac{d}{d\bar{t}}\,\bar{r}^w(\bar{t},\bar{x})\\ &= \bar{\eta}_{\bar{t}}(\bar{t},\bar{x}) + (\bar{r}_{\rm o}(\bar{x})+\bar{\eta}(\bar{t},\bar{x}))_{\bar{x}}\,\bar{u}(\bar{t},\bar{x},\bar{r})\big|_{\bar{r}=\bar{r}^w},\end{aligned}$$

or, ommiting the independent variables,

$$\bar{v} = \bar{\eta}_{\bar{t}} + (\bar{r}_{{\rm o}\bar{x}} + \bar{\eta}_{\bar{x}})\,\bar{u}, \tag{2.4}$$

whenever $\bar{r} = \bar{r}^w(\bar{t},\bar{x}) = \bar{r}_{\rm o}(\bar{x}) + \bar{\eta}(\bar{t},\bar{x})$. Note that, under the incompressibility assumption (2.8), the equation (2.4) is equivalent to

$$\frac{\partial \bar{A}}{\partial \bar{t}} + \frac{\partial \bar{Q}}{\partial \bar{x}} = 0, \tag{2.5}$$

where $\bar{A} = \pi(\bar{r}_{\rm o}(\bar{x}) + \bar{\eta}(\bar{t},\bar{x}))^2$ is the cross-sectional area and Q is the flux at site x and time t.

$$\bar{Q}(\bar{t},\bar{x}) = \pi\int_0^{\bar{r}^w(\bar{t},\bar{x})} s u(\bar{t},\bar{x},s)ds. \tag{2.6}$$

Note that (2.5) is precisely the equation of conservation of total mass of the fluid.

The wall motion is determined by the transmural pressure $\bar{p}^w = \bar{p} - \bar{p}_{\rm o}$ (difference between pressure exerted by the fluid particles "pushing" the wall and the atmospheric pressure) and the circumpherential stress, $S_{\theta\theta}$, which has the form

$$S_{\theta\theta} = \frac{\bar{E}_\sigma h}{\bar{r}_{\rm o}^2}\bar{\eta}.$$

Here $\bar{E}_\sigma = \frac{\bar{E}}{1-\sigma^2}$, $\bar{E}$ is the Young modulus of elasticity (may vary along the length of the tube), σ is the Poisson ratio of the elastic wall (usually one takes $\sigma = 1/2$, since the wall is considered incompressible), and h is the thickness of the tube. In reality the strain-stress relation for the arterial wall is nonlinear, which means that the Young modulus is dependent on the wall displacement as well as on the pulse frequency. Nevertheless, in this paper we neglect such effects and consider only linear elastic properties of the wall. We also neglect any gravitational effects.

If ρ^w denotes the density of the tube wall, then the equation of the wall motion is

$$\rho^w h\,\bar{\eta}_{\bar{t}\,\bar{t}} = \bar{p}^w - \frac{\bar{E}_\sigma(\bar{x})h}{\bar{r}_{\rm o}(\bar{x})^2}\bar{\eta}. \tag{2.8}$$

Note that we neglect the wall motion in the axial direction due to stretching.

Finally, the last restriction imposed is that the radial velocity vanishes in the center of the tube,

$$\bar{v}\big|_{\bar{r}=0} = 0, \tag{2.9}$$

follows from the axisymmetry of the physical system.

The first step in analyzing the system of equations (2.1)–(2.9) is to rewrite it in terms of non-dimensional variables. To this end, consider new variables (t, x, r), η as follows;

$$\bar{x} = \bar{\Lambda}\, x, \qquad \bar{r} = \bar{R}\, r, \qquad \bar{\eta} = \bar{A}\, \eta, \tag{2.10}$$

$$\bar{t} = \frac{\bar{\Lambda}}{\bar{c}_{\mathrm{MK}}}\, t, \tag{2.11}$$

where $\bar{\Lambda}$ is a typical wave length of the waves propagating in the tube, $\bar{R}$ is a typical radius of a cross-section of the tube, and $\bar{A}$ is a typical amplitude of the wall displacement from the unstressed position. The quantity $\bar{c}_{\mathrm{MK}} = \sqrt{\frac{\bar{E}h}{2\bar{R}\rho}}$ is the Moens-Korteweg velocity of a wave propagating along an elastic tube when all non-linear terms are neglected. Here ρ is the density of the fluid and $\bar{E}$ is the Young modulus of elasticity of the vessel wall.

Let $\varepsilon = \frac{\bar{A}}{\bar{R}}$ and $\delta = \frac{\bar{R}}{\bar{\Lambda}}$. In the sequel we assume that the following long-wave hypothesis holds true:

$$\varepsilon << 1, \quad \delta^2 = k\varepsilon, \quad k = O(1). \tag{2.12}$$

In vivo, the ratios ε and δ vary considerably, depending on the vessel type; thus the subsequent analysis and the model derived herein are valid only on scales compatible with those satisfying (2.12). As a typical example of an artery under consideration is the brachial artery, with typical radius $\bar{R} = 0.3$ mm and ratio $\varepsilon = 0.1$ and $\delta = 0.4$.

Rescaling the axial and radial velocity and the pressure,

$$\bar{u} = \varepsilon \bar{c}_{\mathrm{MK}} u, \qquad \bar{v} = \varepsilon \bar{c}_{\mathrm{MK}} \delta v, \tag{2.13}$$

$$\bar{p} - \bar{p}_{\mathrm{yo}} = \varepsilon \rho \bar{c}_{\mathrm{MK}}^2\, p, \tag{2.14}$$

then the non-dimensional variables $u = u(t, x, r)$, $v = v(t, x, r)$ and $p = p(t, x, r)$ satisfy the system

$$u_t + \varepsilon u\, u_x + \varepsilon v\, u_r + p_x = 0, \tag{2.15}$$

$$\delta^2 \left[v_t + \varepsilon u\, v_x + \varepsilon v\, v_r \right] + p_r = 0, \tag{2.16}$$

$$u_x + v_r + \frac{1}{r} v = 0, \tag{2.17}$$

in the region $0 \le r \le r^w(t, x) = r_{\mathrm{o}}(x) + \varepsilon \eta(t, x)$, with the boundary conditions

$$v = \eta_t + r_{\mathrm{o}x}(x) u + \varepsilon \eta_x u, \tag{2.18}$$

$$\frac{\rho^w h}{\rho R} \delta^2\, \eta_{tt} = p^w - 2\frac{E(x)}{r_{\mathrm{o}}(x)^2} \eta, \tag{2.19}$$

whenever $r = r^w(t, x) = r_{\mathrm{o}}(x) + \varepsilon \eta(t, x)$. Here $p^w(t, x) = p(t, x, r^w(t, x))$ and $E(x) = \frac{\bar{E}_\sigma(x)}{\bar{E}}$ is non-dimensional quantity depending on the axial position x. In the case of a tube with same elasticity throughout its length, one can choose $E(x) \equiv 1$. Finally, the axisymmetry assumption translates into

$$v = 0, \qquad \text{whenever } r = 0. \tag{2.20}$$

Next, consider new independent variables τ and ξ, similar to those used in [13], to account for the variable "landscape" (radius and elasticity) in which the wave propagation occurs, namely

$$\tau = \varepsilon x, \quad \xi = G(x;\varepsilon) - t. \tag{2.21}$$

The choice for $G(x;\varepsilon)$ will be indicated below. The change of variables implies

$$\partial_t = -\partial_\xi, \tag{2.22}$$
$$\partial_x = \varepsilon\partial_\tau + g(\tau)\partial_\xi, \tag{2.23}$$

where $g = \frac{dG}{dx}$. Thus, the new set of equations satisfied by the velocity, pressure and displacement in terms of the new variables, $u = u(\tau,\xi,r)$, $v = v(\tau,\xi,r)$, $p = p(\tau,\xi,r)$ and $\eta = \eta(\tau,\xi)$, is

$$-u_\xi + \varepsilon u\left(\varepsilon u_\tau + g(\tau)u_\xi\right) + \varepsilon v u_r + \varepsilon p_\tau + g(\tau)p_\xi = 0, \tag{2.24}$$
$$k\varepsilon\left[-v_\xi + \varepsilon u\left(\varepsilon v_\tau + g(\tau)v_\xi\right) + \varepsilon v v_r\right] + p_r = 0, \tag{2.25}$$
$$\varepsilon u_\tau + g(\tau)u_\xi + v_r + \frac{1}{r}v = 0, \tag{2.26}$$

whenever $0 \le r \le r_{\mathrm{o}}(\tau) + \varepsilon\eta(\tau,\xi)$, together with

$$v = -\eta_\xi + \varepsilon r_{\mathrm{o}\tau} u + \varepsilon\left(\varepsilon\eta_\tau + g(\tau)\eta_\xi\right)u, \tag{2.27}$$
$$\gamma k\varepsilon\eta_{\xi\xi} = p^w - 2\frac{E(\tau)}{r_{\mathrm{o}}(\tau)^2}\eta. \tag{2.28}$$

whenever $r = r_{\mathrm{o}}(\tau) + \varepsilon\eta(\tau,\xi)$. For convenience, the notation $\gamma = \frac{\rho^w h}{\rho R}$ was introduced. In addition,

$$v = 0, \text{ when } r = 0. \tag{2.29}$$

A standard method of solving the system (2.24)–(2.29) is to look for solutions η, u and v (similarly for p) in the form

$$\eta(\tau,\xi;\varepsilon) = \eta_0(\tau,\xi) + \varepsilon\eta_1(\tau,\xi) + o(\varepsilon), \tag{2.30}$$
$$u(\tau,\xi,r;\varepsilon) = u_0(\tau,\xi,r) + \varepsilon u_1(\tau,\xi,r) + o(\varepsilon), \tag{2.31}$$
$$v(\tau,\xi,r;\varepsilon) = v_0(\tau,\xi,r) + \varepsilon v_1(\tau,\xi,r) + o(\varepsilon). \tag{2.32}$$

The zero-order terms must satisfy the system

$$\left\{\begin{array}{ll} -u_{0\xi} + g(\tau)p_{0\xi} = 0 & \\ p_{0r} = 0 & \qquad 0 \le r \le r_{\mathrm{o}}(\tau) \\ g(\tau)u_{0\xi} + v_{0r} + \frac{1}{r}v_0 = 0 & \end{array}\right. \tag{2.33}$$

and

$$\left\{\begin{array}{ll} v_0 = -\eta_{0\xi} & \qquad r = r_{\mathrm{o}}(\tau) \\ 0 = p_0 - 2\frac{E(\tau)}{r_{\mathrm{o}}(\tau)^2}\eta_0. & \end{array}\right. \tag{2.34}$$

In addition, $v_0 = 0$ when $r = 0$. From (2.33) it follows that p_0 is independent of r,

$$p_0(\tau,\xi,r) = p_0^w(\tau,\xi) = 2\frac{E(\tau)}{r_o(\tau)^2}\eta_0, \qquad \text{for all } 0 \le r \le r_o(\tau). \tag{2.35}$$

After a few computations, one obtains

$$v_0 = -rg(\tau)^2\frac{E(\tau)}{r_o(\tau)^2}\eta_{0\xi}, \qquad \text{for all } 0 \le r \le r_o(\tau). \tag{2.36}$$

Note that v_0 depends linearly on r. To satisfy the condition (2.34) at $r = r_o(\tau)$, the following must hold

$$g(\tau) = \frac{r_o(\tau)^{1/2}}{E(\tau)^{1/2}}, \tag{2.37}$$

which is equivalent to choosing (in (2.21))

$$G(x;\varepsilon) = \int_0^x \frac{r_o(\varepsilon x')^{1/2}}{E(\varepsilon x')^{1/2}}dx' = \frac{1}{\varepsilon}\int_0^{\varepsilon x}\frac{r_o(y)^{1/2}}{E(y)^{1/2}}dy. \tag{2.38}$$

It is worth mentioning that in the case of an uniform tube (same radius and elasticity along its length), then $G(x;\varepsilon) = gx$ for some constant g, which means that our choice of the variable ξ in (2.21) represents the moving frame (to the right) with constant speed $c = 1/g$.

Thus far, the following relations hold true for the zero-order terms;

$$v_0 = -\frac{r}{r_o(\tau)}\eta_{0\xi}, \tag{2.39}$$

$$p_0 = 2\frac{E(\tau)}{r_o(\tau)^2}\eta_0, \tag{2.40}$$

$$u_{0\xi} = 2\frac{E(\tau)^{1/2}}{r_o(\tau)^{3/2}}\eta_{0\xi}. \tag{2.41}$$

We note that $u_{0\xi}$ is also independent of r, although u_0 is not.

The first-order $O(\varepsilon)$ terms, evaluated for $r = r_o(\tau)$, obey the system of equations

$$\begin{cases} -u_{1\xi} + g(\tau)u_0u_{0\xi} + v_0u_{0r} + p_{0\tau} + g(\tau)p_{1\xi} = 0, \\ -kv_{0\xi} + p_{1r} = 0, \\ u_{0\tau} + g(\tau)u_{1\xi} + v_{1r} + \frac{1}{r}v_1 = 0 \end{cases} \qquad 0 \le r \le r_o(\tau) \tag{2.42}$$

and

$$\begin{cases} \eta_0 v_{0r} + v_1 = -\eta_{1\xi} + r_{o\tau}(\tau)u_0 + g(\tau)u_0\eta_{0\xi}, & r = r_o(\tau) \\ \gamma k\eta_{0\xi\xi} = p_1^o - 2\frac{E(\tau)}{r_o(\tau)^2}\eta_1. & \end{cases} \tag{2.43}$$

Additionally, $v_1 = 0$ when $r = 0$.

Our goal is to derive, from this system, equations for η_0 and u_0. First, (2.42) yields

$$p_{1r} = kv_{0\xi} = -k\frac{r}{r_o(\tau)}\eta_{0\xi\xi},$$

thus

$$p_1(\tau,\xi,r) = -k\frac{r^2}{2r_{\rm o}(\tau)}\eta_{0\xi\xi} + p_1(\tau,\xi,0), \tag{2.45}$$

which implies (setting $r = r_{\rm o}(\tau)$)

$$p_1^{\rm o}(\tau,\xi) = -k\frac{r_{\rm o}(\tau)}{2}\eta_{0\xi\xi} + p_1(\tau,\xi,0).$$

From (2.43) we can write

$$-\eta_1 = \frac{r_{\rm o}(\tau)^2}{2E(\tau)}\left[\gamma + \frac{r_{\rm o}(\tau)}{2}\right]k\eta_{0\xi\xi} - \frac{r_{\rm o}(\tau)^2}{2E(\tau)}p_1\big|_{r=0}.$$

and

$$\begin{aligned} v_1^{\rm o} =& \frac{1}{r_{\rm o}(\tau)}\eta_0\eta_{0\xi} + \frac{r_{\rm o}(\tau)^2}{2E(\tau)}\left[\gamma + \frac{r_{\rm o}(\tau)}{2}\right]k\eta_{0\xi\xi\xi} \\ & - \frac{r_{\rm o}(\tau)^2}{2E(\tau)}p_{1\xi}\big|_{r=0} + r_{{\rm o}\tau}(\tau)u_0^{\rm o} + \frac{r_{\rm o}(\tau)^{1/2}}{E(\tau)^{1/2}}u_0^{\rm o}\eta_{0\xi}. \end{aligned} \tag{2.48}$$

We now turn to (2.42). Eliminating $u_{1\xi}$ yields

$$\begin{aligned} \frac{1}{r}(rv_1)_r =& - u_{0\tau} - \frac{r_{\rm o}(\tau)}{E(\tau)}u_0u_{0\xi} + \frac{r}{r_{\rm o}(\tau)^{1/2}E(\tau)^{1/2}}u_{0r}\eta_{0\xi} \\ & - 2\frac{r_{\rm o}(\tau)^{1/2}}{E(\tau)^{1/2}}\left[\frac{E(\tau)}{r_{\rm o}(\tau)^2}\eta_0\right]_\tau - \frac{r_{\rm o}(\tau)}{E(\tau)}p_{1\xi}. \end{aligned} \tag{2.49}$$

Let

$$Q_0 = Q_0(\tau,\xi) = \int_0^{r_{\rm o}(\tau)} ru_0(\tau,\xi,r)dr \tag{2.50}$$

be the "zero-order" flux at "time" τ and "position" ξ. Solving (2.49) for v_1 and substituting $r = r_{\rm o}(\tau)$, ($v_1^{\rm o} = v_1\big|_{r=r_{\rm o}(\tau)}$), we obtain

$$\begin{aligned} v_1^{\rm o} =& - \frac{1}{r_{\rm o}(\tau)}Q_{0\tau} + r_{{\rm o}\tau}(\tau)u_0^{\rm o} - \frac{4}{r_{\rm o}(\tau)^{3/2}E(\tau)^{1/2}}Q_0\eta_{0\xi} \\ & + \frac{r_{\rm o}(\tau)^{1/2}}{E(\tau)^{1/2}}u_0^{\rm o}\eta_{0\xi} - \frac{r_{\rm o}(\tau)^{3/2}}{E(\tau)^{1/2}}\left[\frac{E(\tau)}{r_{\rm o}(\tau)^2}\eta_0\right]_\tau \\ & + k\frac{r_{\rm o}(\tau)^3}{8E(\tau)}\eta_{0\xi\xi\xi} - \frac{r_{\rm o}(\tau)^2}{2E(\tau)}p_{1\xi}\big|_{r=0}. \end{aligned} \tag{2.51}$$

Comparing this with the expression (2.48) we obtain

$$\begin{aligned} & \frac{E(\tau)^{1/2}}{r_{\rm o}(\tau)^{1/2}}\eta_{0\tau} + \frac{r_{\rm o}(\tau)^{3/2}}{E(\tau)^{1/2}}\left[\frac{E}{r_{\rm o}(\tau)^2}\right]_\tau\eta_0 + \frac{1}{r_{\rm o}(\tau)}\eta_0\eta_{0\xi} + k\frac{r_{\rm o}(\tau)^2}{2E(\tau)}\left[\gamma + \frac{r_{\rm o}(\tau)}{4}\right]\eta_{0\xi\xi\xi} \\ & + \frac{1}{r_{\rm o}(\tau)}Q_{0\tau} + \frac{4}{r_{\rm o}(\tau)^{3/2}E(\tau)^{1/2}}Q_0\eta_{0\xi} = 0. \end{aligned} \tag{2.52}$$

Now, since $u_{0\xi}$ is independent of r (see (2.41)), we have

$$Q_{0\xi} = \int_0^{r_o(\tau)} rdr\, u_{0\xi} = E(\tau)^{1/2} r_o(\tau)^{1/2} \eta_{0\xi},$$

therefore

$$Q_0(\tau,\xi) = E(\tau)^{1/2} r_o(\tau)^{1/2} \eta_0(\tau,\xi) + f(\tau), \tag{2.54}$$

where $f(\tau)$ is independent of ξ. This is an important observation; it allows us to determine $f(\tau)$ from the initial state of the tube, by setting $t = 0$ or, equivalently, $\xi = G(x;\varepsilon)$. Once f is known, the equation governing the τ-evolution of the wall displacement reads

$$\begin{aligned} \eta_{0\tau} &+ \frac{3}{4}\left[\frac{E_\tau(\tau)}{E(\tau)} - \frac{r_{o\tau}(\tau)}{r_o(\tau)}\right]\eta_0 + \frac{5}{2}\frac{1}{r_o(\tau)^{1/2}E(\tau)^{1/2}}\eta_0\eta_{0\xi} \\ &+ \frac{k}{2}\frac{r_o(\tau)^{5/2}}{E(\tau)^{3/2}}\left(\gamma + \frac{r_o}{4}\right)\eta_{0\xi\xi\xi} + \frac{1}{2r_o(\tau)^{1/2}E(\tau)^{1/2}}f'(\tau) + \frac{2}{r_o(\tau)E(\tau)}f(\tau)\eta_{0\xi} = 0. \end{aligned} \tag{2.55}$$

This is a variable coefficient Korteweg-de Vries equation with a forcing term (both in wave amplitude and in wave velocity) which depends on the initial state of the tube.

Remark 2.1. (i) As noted earlier, if one considers that the tube is initially in a quiescent state (zero flux), then in the above equation one has $f(\tau) = 0$, for all τ.

(ii) In the case of an uniformly elastic tube, when the elasticity does not change along the tube one can take without loss of generality $E(\tau) \equiv 1$ and thus obtain

$$\eta_{0\tau} - \frac{3}{4}\frac{r_{o\tau}(\tau)}{r_o(\tau)}\eta_0 + \frac{5}{2r_o(\tau)^{1/2}}\eta_0\eta_{0\xi} + \frac{k}{2}r_o(\tau)^{5/2}\left(\gamma + \frac{r_o(\tau)}{4}\right)\eta_{0\xi\xi\xi} = 0. \tag{2.56}$$

If, in addition, the radius of the unstressed tube remains constant, then $r_o(\tau) \equiv r_o$ and the equation reads

$$\eta_{0\tau} + \sigma\eta_0\eta_{0\xi} + \kappa\eta_{0\xi\xi\xi} = 0, \tag{2.57}$$

with $\sigma = \frac{5}{2{r_o}^{1/2}}$, $\kappa = \frac{k}{2}{r_o}^{5/2}\left(\gamma + \frac{r_o}{4}\right)$. This is the classical (constant coeficient) KdV equation, derived in the context of elastic tubes in [7], [19], [20].

(iii) Another special case worth mentioning is when the elasticity of the tube is proportional to the radius of the cross-section (without loss of generality, taking $E(\tau) = r_o(\tau)$.)

3 INITIAL VALUE PROBLEM

In this section we will restrict our attention to the equation (2.55) with the forcing term $f(\tau) = 0$, and show that under appropriate conditions, the corresponding initial value problem is well-posed. We will consider that the walls are thin compared to the radius of the tube and thus we can take $\gamma = 0$. (this is only for simplicity of the formulas.)

Let q be defined by

$$\eta_0(\tau,\xi) = \frac{r_o(\tau)^{3/4}}{E(\tau)^{3/4}} q(\tau,\xi), \tag{3.1}$$

so that q satisfies the equation

$$q_\tau + \frac{5}{2}\frac{r_o(\tau)^{1/4}}{E(\tau)^{5/4}} q q_\xi + \frac{k}{8}\frac{r_o(\tau)^{7/2}}{E(\tau)^{3/2}} q_{\xi\xi\xi} = 0. \tag{3.2}$$

Introducing the new "time" variable τ' satisfying

$$\frac{d\tau'}{d\tau} = \frac{5}{2}\frac{r_o(\tau)^{1/4}}{E(\tau)^{5/4}}, \tag{3.3}$$

we finally obtain,

$$q_{\tau'} + q q_\xi + \frac{k}{20}\frac{r_o(\tau')^{13/4}}{E(\tau')^{1/4}} q_{\xi\xi\xi} = 0. \tag{3.4}$$

Denote the time dependent coefficient, appearing in the equation above, by $\alpha(\tau') = \frac{k}{20}\frac{r_o(\tau')^{13/4}}{E(\tau')^{1/4}}$. Then one has the variable coefficient KdV equation (we write, for simplicity, τ instead of τ')

$$q_\tau + q q_\xi + \alpha(\tau) q_{\xi\xi\xi} = 0. \tag{3.5}$$

Often "constant coefficient" problems are easier to solve, but spatially dependent coefficients appear more often in models coming from the physical world, when the phenomenon under consideration occurs on a variable landscape. The coefficient $\alpha(\tau)$ in (3.5) looks like a real valued function of time alone, but it is actually spatially dependent, since τ is a variable that comes from rescaling x. This way of writing the problem leads to significant simplifications.

We assume that the original system is periodic in time (i.e. laboratory time t) with period T, that is we assume the system is in basal condition. Therefore the same is true for the system described in the new coordinate ξ. Under this assumption, all solutions of the equation (3.5) are $\xi-$periodic

$$q(\tau,\xi) = q(\tau,\xi+T), \text{ for all } \tau,\xi. \tag{3.6}$$

The initial data for (3.5) corresponds to $\tau = 0$, which, in laboratory coordinates, means $x = 0$, i.e. the inflow condition.

$$q(\tau = 0,\xi) = q_0(\xi) \; [= q_0(-t, x = 0)]. \tag{3.7}$$

One can verify that the following functional is invariant under the flow governed by equation (3.5).

$$\varphi_0(\tau,q) = \frac{1}{2}\int_{\mathbb{T}} q(\tau,\xi)^2 d\xi \tag{3.8}$$

Consider also the functionals

$$\varphi_1(\tau,q) = \frac{1}{2}\int_{\mathbb{T}} \left[\alpha(\tau) q_\xi(\tau,\xi)^2 - \frac{1}{3} q(\tau,\xi)^3\right] d\xi \tag{3.9}$$

and

$$\varphi_2(\tau,q)=\frac{1}{2}\int_{\mathbb{T}}\left[\alpha(\tau)q_{\xi\xi}(\tau,\xi)^2+\beta(\tau)q(\tau,\xi)^2q_{\xi\xi}(\tau,\xi)+\frac{\beta(\tau)}{6\alpha(\tau)}q(\tau,\xi)^4\right]\,d\xi. \tag{3.10}$$

For our purposes, the choice of the funtion $\beta(t)$ will be

$$\beta(\tau)=\frac{\alpha(\tau)+4}{\alpha(\tau)+5}.$$

A formal computation (assuming that $q=q(\tau,\xi)$ is a solution of the initial-value problem (3.5)–(3.7) and all derivatives of q involved exist), yields

$$\frac{d}{d\tau}\varphi_1(\tau,q)=\frac{1}{2}\alpha'(\tau)\int_{\mathbb{T}}q_\xi^2 \tag{3.11}$$

and, similarly,

$$\frac{d}{d\tau}\varphi_2(\tau,q)=\frac{1}{2}\alpha'(\tau)\int_{\mathbb{T}}q_{\xi\xi}^2+\gamma'(\tau)\int_{\mathbb{T}}q^2q_{\xi\xi}+\left(\frac{\gamma(\tau)}{6\alpha(\tau)}\right)'\int_{\mathbb{T}}q^4. \tag{3.12}$$

Thus a sufficient condition for controlling the growth of the functionals φ_1 and φ_2 is

$$\alpha'(\tau)\le 0. \tag{3.13}$$

The main result of this paper is a generalization of the results of [5] and [6], to the nonautonomous case.

Theorem 3.1. *Assume that $\alpha=\alpha(\tau)\in W^{1,\infty}$ is defined on some (possibly infinite) interval $\tau\in[0,L]$. Then the initial value problem* (3.5)–(3.7) *is well-posed in the space $H^2(\mathbb{T})$. If, in addition, $q_0\in H^3(\mathbb{T})$, then $q(\tau,\xi)$ is a classical solution. Moreover, if $\alpha(\tau)$ is positive, nonincreasing function, then the solution exists globally in time τ.*

The semilinear Hille-Yosida theory presented in [5] and [6] finds a stongly continuous semigroup solution to the autonomous problem

$$\frac{dq}{d\tau}=\mathcal{A}q,\quad q(0)=f, \tag{3.14}$$

in circumstances in which the Crandall-Liggett theory (and various others as well) does not apply. For the nonautonomous case,

$$\frac{dq}{d\tau}=\mathcal{A}(\tau)u,\quad q(s)=f, \tag{3.15}$$

$\mathcal{A}(\tau)$ can be approximated on $[s,\tau]$ by $\mathcal{A}_\pi(\tau)=\mathcal{A}(\tilde{\tau}_i)$ on $[\tau_{i-1},\tau_i)$, for a given partition π, $s=\tau_0<\tau_1<\ldots<\tau_n=\tau$, and a choice of $\tilde{\tau}_i\in[\tau_{i-1},\tau_i)$. Thus

$$q_\pi(\tau)=\prod_{j=1}^{n}T(\tau_j-\tau_{j-1};A(\tilde{\tau}_j))\;f, \tag{3.16}$$

where $T(s;\mathcal{A}(\tau))$ is the semigroup generated by $\mathcal{A}(\tau)$ and initial time s, and the product is ordered so that $\prod_{j=1}^{n} S_j f$ means $S_n S_{n-1} \dots S_2 S_1 f$. Finally one shows that the solution of the IVP in the nonautonomous case is

$$q(t) = \lim q_\pi(t)$$

where the limit is taken as the mesh of π $(= \max_j(\tau_j - \tau_{j-1}))$ tends to zero.

The proof of our theorem relies upon writing the IVP as the abstract Cauchy problem (on the Hilbert space $\mathcal{H} = L^2(\mathbb{T})$)

$$\frac{dq}{d\tau}(\tau) = \alpha(\tau)Aq(\tau) + B(q(\tau)) \; [=: \mathcal{A}(\tau)q]\,, \tag{3.17}$$

$$q(0) = q_0 \in \mathcal{H}, \tag{3.18}$$

where $A = -\partial_\xi^3$ generates a C_0 semigroup on $\mathcal{H}$ and $B(u) = -u\partial_\xi u$ is a nonlinear operator satisfying a local quasi-dissipative condition. "Local" here means that the quasi-dissipative constant depends on the level set of the functionals φ_0, φ_1 and φ_2. The estimates on the growth of these functionals along solutions follow from (3.11) and (3.12), using Holder inequalities. More precisely, for all τ,

$$\varphi_0(\tau, q) = \varphi_0(0, q_0), \tag{3.19}$$

$$\varphi_1(\tau, q) \leq \varphi_1(0, q_0), \tag{3.20}$$

$$\varphi_2(\tau, q) \leq e^{\omega\tau}\varphi_2(0, q_0), \tag{3.21}$$

$$\tag{3.22}$$

where $\omega = \omega(\|q_0\|_{H^1(\mathbb{T})})$. The last inequality allows us to extend solution for all "times" τ, thus obtaining global solutions. The solution itself is obtained as a limit of approximate solutions obtained by discretizing the "time". Thus it will be sufficient to solve (for q) the resolvent equation

$$q - \lambda\alpha Aq = \lambda B(q) + p \tag{3.23}$$

for every $p \in \mathcal{H}$, and for sufficiently small $\lambda > 0$. This can be done via a fixed point argument similar to the one presented in [5].

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