Moment of Inertia

PES 1160 Advanced Physics Lab I

Purpose of the experiment

- Learn to measure the moments of inertia of a rotating body.
- Learn how the moment of inertia changes with a respect to mass and location.
- To understand what rotational inertia is and how it depends on the size, shape, and mass distribution of an object, as well as the axis the object is rotated about.
- To dynamically measure the rotational inertia of a disk, bar, and ring.
- To verify the Parallel Axis Theorem.
- [FYI]

[FYI] If you toss a penny 10000 times, it will not be heads 5000 times, but more like 4950. The heads picture weighs more, so it is slightly more likely to end up on the bottom.

Background

Torque

Torque is the rotational analog to force. You should recall that a force is a push or a pull that produces an acceleration. Similarly, torque is an influence that produces an *angular acceleration*. To understand torque, let's compare Newton's 2$^{\underline{nd}}$ law to its rotational equivalent for a fixed axis:

"Force" term. Cause of acceleration.

$$\sum \vec{F} = m \; \vec{a}$$

$$\sum \vec{\tau} = I \; \vec{\alpha}$$

"Inertia" term. Resistance to acceleration.

"Acceleration" term.

So, by this comparison, we can see that torque is the cause of angular acceleration. The resistance to angular acceleration, called *rotational inertia* and has the symbol I, is the subject of this lab (see the section 'Rotational Inertia' below).

The fundamental mathematical expression of torque is:

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

where the '×' symbol is the *vector product* or *cross product*. If you know what a cross product is, you are ahead of the game. If not, a physical interpretation of the cross product follows below.

When you apply the cross product, the magnitude of the torque is shown to be:

$$\boxed{\tau = rF\sin\theta}$$

From this equation, you can see that the torque depends on the applied force (F), the distance the force is applied from the pivot (r), and the angle θ between F and r.

To clarify this mathematical relationship, let us take a look at the simple case of a wrench that is being used to loosen a bolt:

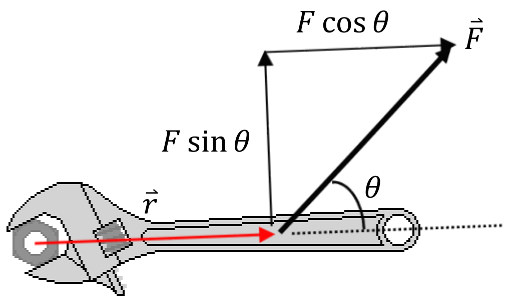


Figure 1: Applying a torque to loosen a bolt.

From this example you can see that the perpendicular component of the force ($F\sin\theta$) will cause the wrench to rotate counterclockwise. Whereas, the component of the force that is parallel to the wrench ($F\cos\theta$) will only pull the wrench off the bolt and will not contribute to the rotation. This is why the cross product is used in the torque equation: to only include those forces (or components of forces) directly responsible for rotation, and to eliminate those forces (or components) that do not contribute to the torque.

If you take another look at the expression of torque, it is proportional to the distance from the pivot point where the force is applied. Therefore, the same force applied at a greater distance will produce a greater torque. If the force is applied at the pivot point the object will not rotate. Have you ever tried to open a door from the hinge side? The door will not rotate open when pushed on the hinge. However, if you try to open the door from the handle side the door rotates open quite easily.

In summary, torque is based on not only the perpendicular component of the force applied, but also the distance from the pivot. Therefore, a larger torque can be applied, when the force remains constant, by simply increasing the distance from the pivot.

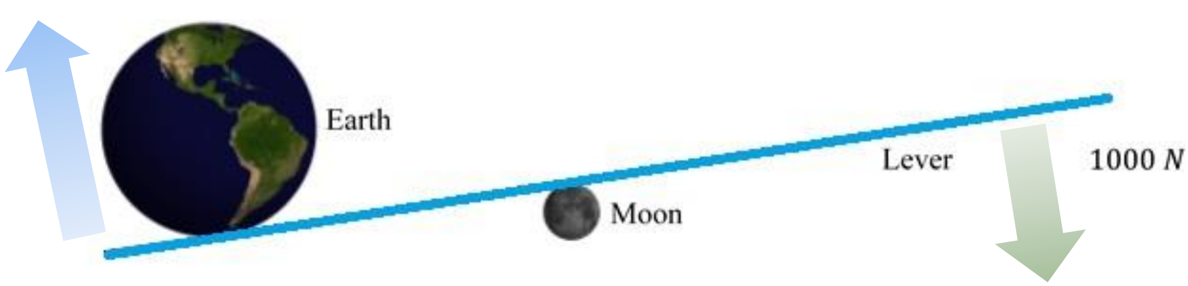


Figure 2: “Give me a long enough lever and a place to put it and I can move the world.” -Archimedes

As we have seen, rotational inertia is the resistance to angular acceleration in a similar sense to the way mass is the resistance to linear acceleration. You may see another term for rotational inertia, called *moment of inertia*. These two terms are used interchangeably. However, the term moment of inertia is unfortunate because the word “moment” invokes the notion of time. As we shall see, rotational inertia does not (directly anyway) depend on time. Rather, it is a function of the size, shape and mass of the object, as well as the distribution of mass about the rotational axis.

Consider a point mass undergoing uniform circular motion of radius r:

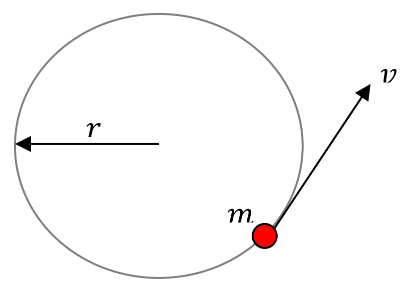


Figure 3: Point mass in uniform circular motion.

The object’s kinetic energy is given by:

$$K = \frac{1}{2} mv^2$$

where *m* is the mass of the object and *v* is its speed around the circle. Using the relationship between linear speed v and angular speed ω.

$$v = \omega r$$

we can rewrite the kinetic energy:

$$K = \frac{1}{2} m(\omega r)^2$$

Next, simply move the parentheses:

$$K = \frac{1}{2} \underbrace{(mr^2)}_{I} \omega^2$$

We now define the rotational inertia as the term in parentheses. So:

$$K = \frac{1}{2} I \omega^2 \qquad \text{Rotational Kinetic Energy}$$

$$\text{where: } I = m r^2$$

I is the rotational inertia of a point mass. As you can see, the rotational inertia of a point mass depends only on its mass m and its distance r from the axis it is rotating about. Now, let me emphasize something about that last calculation. I didn't really change anything about the kinetic energy. All I did was *rewrite* the kinetic energy in terms of other variables. The physical quantity remains unchanged. The equations $K = \frac{1}{2} mv^2$ and $K = \frac{1}{2} I \omega^2$ are exactly the same thing for the point mass, just written in linear (m and v) and rotational (I and ω) term respectively.

Point masses are fine. However, there are many objects that aren't point masses. So, what if we want to know the rotational inertia of something other than a point mass, such as a disk, ring or ball? The answer is simple. All we need to do is build the other objects (disk, ring, ball, etc.) out of a suitable arrangement of point masses. Then, we add up the inertial contributions of each point mass to find the total rotational inertia of the object. In doing so, we find that objects typically have a rotational inertia whose mathematical form looks like:

$$I = CMR^2$$

where C is a constant that depends on the shape of the object and the axis the object is rotated about (different axes have different rotational inertias). Additionally, R is usually some measure of the size of the object, such as the radius of a disk or sphere. Again, M is the mass of the object. Therefore, the rotational inertia depends on:

1.) The mass of the object (M).
2.) The size of the object (R).
3.) The shape of the object (C).
4.) The axis the object is rotated about (C and R).

Please notice that the rotational inertia does **not** depend on whether or not the object is actually rotating. That is, it is independent of how fast the object is rotating. Rotational inertia is mostly a *geometric quantity* that depends on sizes, shapes and distances, and not on motion.

In this lab, you will dynamically determine the rotational inertia of several objects and compare your answer to what the rotational inertia should be based on the mass, size, shape and rotational axis.

The *Parallel-Axis Theorem* is a simple relationship between the rotational inertia about an axis through an object's center of mass (called the center-of-mass axis) and any other axis that is parallel to the center-of-mass axis. The diagram below shows this configuration:

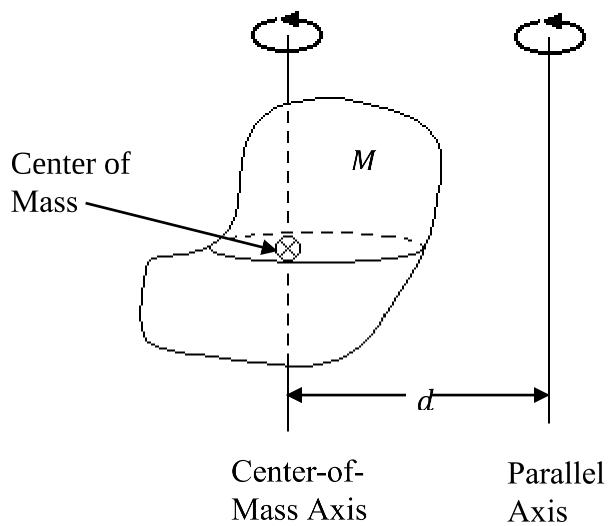


Figure 3: The Parallel Axis Theorem.

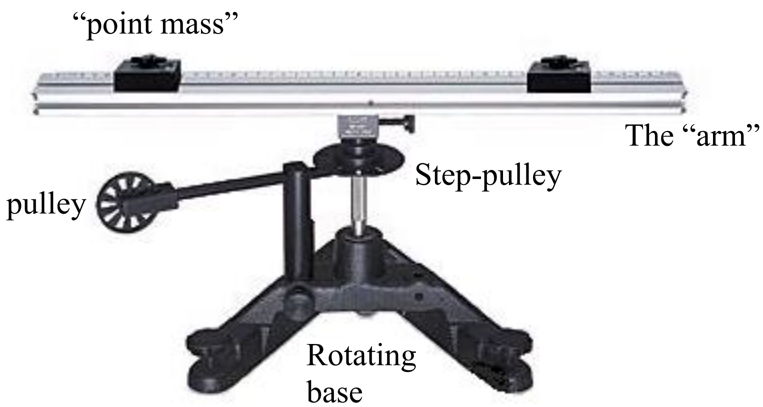
The relationship between the rotational inertia about the center-of-mass axis and the parallel axis is simple:

$$I_{PARALLEL} = I_{CM} + Md^2 \qquad \text{(Parallel Axis Theorem)}$$

While the size, shape and mass of the object have not changed, the axis about which the object is rotated has changed. When rotating the object about the parallel axis, the object is further on average from the rotational axis than it is when it is rotating about the center-of-mass axis. Consequently, every part of the object would be moving around in a bigger circle on average, and this situation results in a larger rotational inertia. Mathematically, this is because when $d = 0$, the rotational inertia is as small as possible.

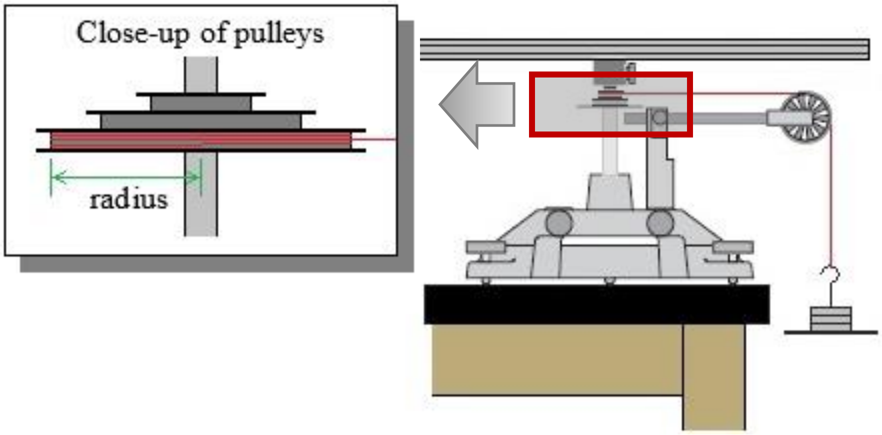
In Part IV of this lab, you will verify the Parallel Axis Theorem in the case of a ring whose center has been offset from the rotational center.

We can now simplify our discussion somewhat by focusing on the specific configuration we will be using today: an arm rotating on an axle. The setup can be changed by moving masses to different intervals along the length of the arm.



The apparatus is made up of several parts. A flat bar located on the top connected to a block which is screwed to an axle with a step pulley. Masses can then be placed anywhere along the length of the arm to change the inertial setup. From what you know from lecture class you could combine the equations of all the cylinders and bars that make up the apparatus and come up with an equation for the moment of inertia of this object! The point of this lab is to test the theory not to beat it to death! Let's then simply measure the moment of inertia of this "arm". Later we will test the theory by applying point-masses along the arm (a **much** easier calculation) and measure how the inertia changed then compare the experimental data to the expected result.

The "arm" is rotated by a mass attached to a string that is then wrapped around the pulley. When the mass is released the arm will rotate. The mass will fall at a rate based on the moment inertia of the arm. Change the moment of inertia of the system (by adding and/or repositioning masses) on the arm and the "hanging mass" will fall at a different rate.



The reason the arm rotates is because the hanging mass applies a torque on the pulley.

$$\vec{\mathbf{T}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

Since the force (F) is tangential to the pulley the cross product can be simplified to

$$T = r\,F$$

Using the rotational analog to Newton's Second Law we get:

$$T = I\,\alpha$$

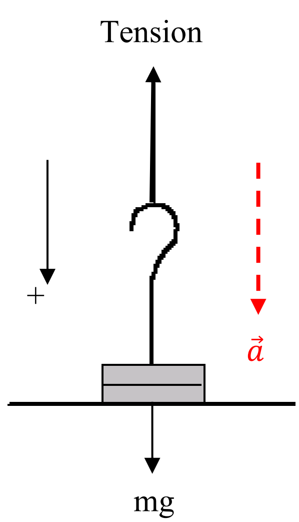
where I is the moment of inertia (of whatever is being spun) and α is the angular acceleration. The angular acceleration of the pulley is related to the linear acceleration (a) of the hanging mass by:

$$a = r\,\alpha$$

where r is the radius of the pulley.

Putting all this together we get a new expression for the Torque:

$$T = \frac{I\,a}{r}$$



If we now switch to an analysis of the hanging mass we can get some more information. To avoid confusion between the tension in the string and the torque on the pulley I will write out Tension instead of using T. From Newton's Laws we get:

$$-\text{Tension} + mg = ma$$

In this case the tension in the string is the same force acting on the surface of the pulley. The torque (T) can be written as:

$$\text{Torque} = r\,F$$

$$\text{Torque} = r\,(\text{Tension})$$

Therefore, when we put everything together we get:

$$r(mg - ma) = I\alpha$$

Solving for I we get:

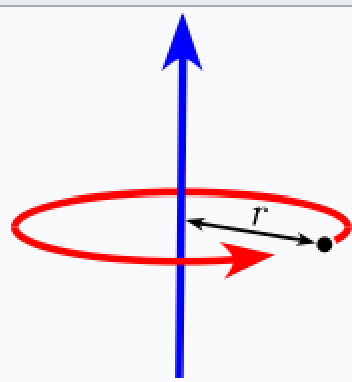
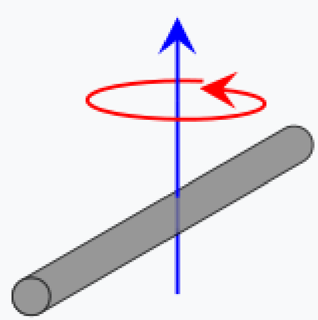
$$I = \frac{mr}{\alpha}(g - a) \text{ since } a = r\alpha \text{ then,}$$

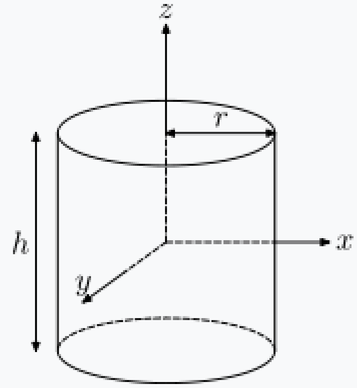
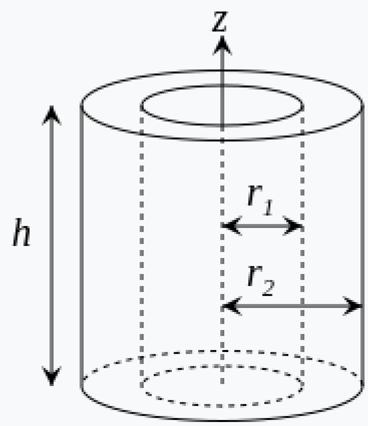
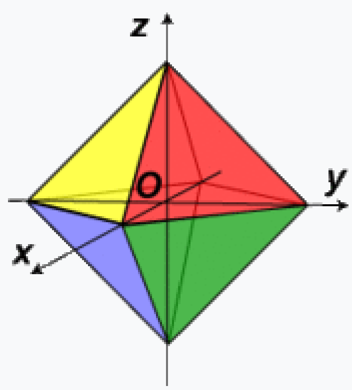
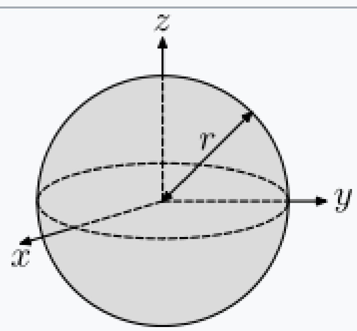
$$I = \frac{r}{\alpha}mg - mr^2 \qquad \textbf{equation 1}$$

We now have the moment of inertia in terms of things we can measure: hanging mass (m), radius of the pulley (r) and the angular acceleration of the platform (α).

Moment of inertia of some common shapes

The moment of inertia has been predetermined for some common geometric shapes:

Description	Figure	Moment(s) of inertia
Point mass *m* at a distance *r* from the axis of rotation.		$I = mr^2$
Rod of length *L* and mass *m*, rotating about its center.		$I_{center} = \frac{1}{12}mL^2$

Solid cylinder of radius r, height h and mass m.		$I_z = \frac{1}{2}mr^2$ $I_x = I_y = \frac{1}{12}m(3r^2 + h^2)$
Thick-walled cylindrical tube, of inner radius r_1, outer radius r_2, length h and mass m.		$I_z = \frac{1}{2}m({r_1}^2 + r_2^2)$ $I_x = I_y = \frac{1}{12}m(3({r_1}^2 + r_2^2) + h^2)$
Regular octahedron of side s and mass m		$I_{hollow} = \frac{1}{6}ms^2$ $I_{solid} = \frac{1}{10}ms^2$
Solid sphere (ball) of radius r and mass m.		$I = \frac{2}{5}mr^2$