

# A note on branching random walks on finite sets

Thomas Mountford

EPFL-DMA, CH-1015 Lausanne Switzerland

e-mail: thomas.mountford@epfl.ch

Rinaldo B. Schinazi

University of Colorado

Colorado Springs CO 80933 USA

email: schinazi@math.uccs.edu

**Abstract.** *We show that a branching random walk that is supercritical on  $\mathbf{Z}^d$ , that is, starting with a single particle there is a positive probability that there will be particles at all times somewhere on  $\mathbf{Z}^d$  is also supercritical, on a rather strong sense, on a large enough finite ball of  $\mathbf{Z}^d$ . This implies that the critical value of branching random walks on finite balls converges to the critical value of branching random walks on  $\mathbf{Z}^d$  as the radius increases to infinity. Our main result also implies coexistence of an arbitrary finite number of species for an ecological model.*

## 1. Introduction and results.

Consider a branching random walk  $(\xi_t : t \geq 0)$  on  $\{\mathbf{Z}_+\}^{\mathbf{Z}^d} = \{0, 1, 2, \dots\}^{\mathbf{Z}^d}$ . Multiple indeed unbounded number of individuals are permitted at each site. More precisely, for  $x \in \mathbf{Z}^d$  and  $\xi$  in  $\{0, 1, 2, \dots\}^{\mathbf{Z}^d}$ ,

$\xi(x) = 0$  will represent a vacancy site  $x$  for configuration  $\xi$ ,  $\xi(x) = n > 0$  will represent the presence of  $n$  individuals at site  $x$ .

Individuals die at rate 1 and at a site new individuals are born according to the number of individuals that are present at neighboring sites. The system is a spin system

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Key words and phrases: branching random walks, coexistence, ecological model, spatial stochastic model, contact process

in that changes can occur at a single site at most for any time  $t$ ; this change must consist of a change in value (up or down) of precisely 1. For  $n > 0$ ,  $\xi(x) = n$ , the up rate is

$$c_+(x, \xi) = \lim_{t \rightarrow 0} \frac{P^\xi(\xi_t(x) = n + 1)}{t} = \lambda_1 \sum_{y \in \mathbf{Z}^d: x \sim y} \xi(y)/(2d),$$

where  $x \sim y$  means that  $y$  is one of the  $2d$  nearest neighbors of  $x$ . The down rate is

$$c_-(x, \xi) = \lim_{t \rightarrow 0} \frac{P^\xi(\xi_t(x) = n - 1)}{t} = n = \xi(x).$$

The process  $\xi_t$  can be constructed using Harris' graphical construction. See, for instance, Section 3 in Pemantle and Stacey (2001).

**Remark.** Since we are dealing with unbounded spins (that is, an unbounded number of individuals is possible at each site) the process will not be defined for all  $\xi_0$  but following e.g. methods of Kesten & Van den Berg (2000) one can show the existence of a non explosive process satisfying the above conditions for  $\xi_0(x)$  bounded over  $x$ .

Let  $|\xi_t| = \sum_{y \in \mathbf{Z}^d} \xi_t(y)$  be the number of particles of  $\xi_t$  at time  $t$ , for an initial configuration  $\xi_0$  such that  $|\xi_0| < \infty$ . Note that if  $|\xi_t| = n$

$$|\xi_t| \rightarrow n + 1 \text{ at rate } n\lambda_1$$

$$|\xi_t| \rightarrow n - 1 \text{ at rate } n.$$

That is, the process  $|\xi_t|$  is a continuous time (non spatial) branching process. Clearly its critical value is 1: starting with one individual, there is a positive probability that the process does not become extinct if and only if  $\lambda_1 > 1$ .

In this paper we are concerned with branching random walks *restricted* to a finite set: births from outside the finite set into the finite set are not permitted. Let  $|\cdot|$  denote the Euclidean norm on  $\mathbf{Z}^d$  and let

$$B_n = \{x \in \mathbf{Z}^d : |x| \leq n\}.$$

A branching random walk restricted to the set  $B_n$  is the Markov chain on  $\{0, 1, 2, \dots\}^{B_n}$  with transition rates, for  $x$  in  $B_n$ ,

$$\bar{c}_+(x, \xi) = \lambda_1 \sum_{y \in B_n: x \sim y} \xi(y)/(2d)$$

and

$$\bar{c}_-(x, \xi) = c_-(x, \xi).$$

**Theorem 1.** If  $\lambda_1 > 1$  then there exists an integer  $n$  such that the branching random walk restricted to  $B_n$  survives in the following (strong) sense: there exists a function  $f_n$  on  $B_n$  such that for any  $\alpha > 0$  there exists  $N = N(\alpha, n)$  such that

$$\text{if } \xi_0(x) > Nf_n(x), \quad \forall x \in B_n,$$

then with probability at least  $1 - \alpha$  we have for any  $\delta \in (0, 1)$

$$\xi_t(x) > N(1 - \delta)f_n(x)e^{(\lambda_1 - 1)t/2}, \quad \forall x \in B_n, \quad \text{and } \forall t > 0.$$

Theorem 1 is concerned with the behavior of a branching random walk restricted to a finite set when the unrestricted branching random walk is supercritical. A dual point of view is to examine the local behavior of unrestricted branching random walks. This has been done for continuous space branching random walks, see, for instance, Englander and Kyprianou (2004) or Englander and Pinsky (1999) and the references there.

We now turn to two applications of Theorem 1.

It is easy to see by the attractiveness of the systems (see e.g. Liggett(1985)) that the branching random walk restricted to  $B_n$  has a critical value  $\lambda_c^n$  such that, starting with a single particle, this process gets extinct with probability 1 for  $\lambda_1$  below  $\lambda_c^n$  and becomes extinct with probability strictly less than 1 for  $\lambda_1$  above  $\lambda_c^n$ . It is also not difficult to show that  $\lambda_c^n$  is larger than 1 (the critical value of the unrestricted branching random walk) and is finite but an exact computation seems out of the question. This is so because the birth rate of a particle depends on where the particle is: near the boundary or inside  $B_n$ . For this process (unlike what happens for the unrestricted process) the critical value depends on the geometry of the space on which the process is restricted. However, as a direct consequence of Theorem 1 we get

**Corollary 1.** *The critical value  $\lambda_c^n$  of the branching random walk on  $B_n$  converges to the critical value of the branching random walk on  $\mathbf{Z}^d$  as  $n \rightarrow \infty$ .*

*Proof of Corollary 1.*

Take any  $\lambda_1 > 1$ , according to Theorem 1 there exists  $n_0$  such that there is a positive probability for the branching random walk restricted to  $B_{n_0}$  to survive (using the Markov property of the process it is easy to see that if the process may survive starting from a particular finite distribution it may also survive starting from any non empty finite distribution). Thus, for any  $\lambda_1 > 1$  there is  $n_0$  such that  $\lambda_c^{n_0} \leq \lambda_1$ . Since the sequence  $(\lambda_c^n)_{n \geq 1}$  is also decreasing and bounded below by 1 we get that it converges to 1 as  $n$  goes to infinity. This completes the proof of Corollary 1.

Note that Liggett (1999) has computed asymptotics for  $\lambda_c^n$  (as  $n$  goes to infinity) for branching random walks on finite trees but even in that case an exact computation of  $\lambda_c^n$  seems impossible.

Consider now a model in which  $\nu$  species compete for space. Each species gives birth and dies according to a branching random walk. Species  $i$  has birth rate  $\lambda_i$  and death rate 1 (we could take different death rates as well). There is no bound on the number of individuals per site but we have at most one species per site. That is, birth attempts on sites that are already colonized by another species are suppressed. This process can be viewed as a process  $(\underline{\xi}_t = (\xi_t^1, \xi_t^2 \dots \xi_t^\nu) : t \geq 0)$  where  $\xi^i(x)$  gives the number of individuals of type  $i$  present at position  $x$ . The prohibition of multiple species at the same site implies for the process that for each  $t \geq 0$ ,  $x \in \mathbf{Z}^d$  and distinct  $i, j \in \{1, 2 \dots \nu\}$ ,  $\xi_t^i(x)\xi_t^j(x) = 0$ . As before the process is a spin system and if  $\xi^i(x) = n \geq 0$  and  $\sum_{j \neq i} \xi^j(x) = 0$

$$c_+(x, \xi, i) = \lim_{t \rightarrow 0} \frac{P^\xi(\xi_t^i(x) = n + 1)}{t} = \lambda_1 \sum_{y \in \mathbf{Z}^d: x \sim y} \xi^i(y)/(2d),$$

where  $x \sim y$  means that  $y$  is one of the  $2d$  nearest neighbors of  $x$ . The down rate (for  $\xi^i(x) = n > 0$ ) is

$$c_-(x, \xi, i) = \lim_{t \rightarrow 0} \frac{P^\xi(\xi_t^i(x) = n - 1)}{t} = n = \xi^i(x).$$

If the initial configuration has individuals of all  $\nu$  species it is easy to see that at time 1, say, there is a positive probability that  $\nu$  balls of a given radius in  $\mathbf{Z}^d$  are occupied each by a single species. Moreover, there is a positive probability that each species will occupy

a ball with a radius and a number of individuals per site large enough to apply Theorem 1. Since there is a positive probability that every site of each colonized ball will be occupied forever by the same species there is a positive probability that all  $\nu$  species will coexist forever. This proves the following

**Corollary 2.** *Consider an ecological model with  $\nu$  species where each species gives birth and dies according to a branching random walk. Let the birth rates be  $\lambda_i > 1$ ,  $1 \leq i \leq \nu$  and the death rates be 1. Each site may be occupied by at most one species. For any initial configuration containing all  $\nu$  species there is a positive probability that all species will coexist.*

Note that coexistence occurs even if some birth rates are much larger than the others. This is in sharp contrast with a model in which there is a limit of one individual per site. For such a model, it has been shown that two species may coexist if and only if  $\lambda_1 = \lambda_2$  and  $d \geq 3$ , see Neuhauser (1992).

## 2. Proof of Theorem 1.

We will use coupling arguments as well as some simple quasi stationary properties of random walks. Our starting point is the existence of quasi stationary distributions (defined as an eigenvector corresponding to the largest eigenvalue of the transition matrix) for the simple random walk on a finite connected subset of  $\mathbf{Z}^d$  with Dirichlet boundary conditions (the random walk is killed on exiting the set). The largest eigenvalue for the quasi stationary distribution tends to 1 as the finite set tends pointwise to  $\mathbf{Z}^d$ , in particular

**Lemma 1.** *For all  $a > 0$  there exists an integer  $N_0$  so that the largest eigenvalue of the subprobability matrix for the simple random walk on  $B_{N_0}$  with Dirichlet boundary condition is greater than  $1 - a$ .*

**Proof of Lemma 1** We reference Aldous and Fill (2003), chapter three, section 6.5 for details on quasistationary distributions. We consider the sub Markov chain obtained by killing the simple random walk, starting in  $B_n$  when it leaves  $B_n$ . For this Markov chain the sub probability matrix  $P^n$  is given simply by

$$P_{ij}^n = \frac{1}{2d} \text{ for } i, j \text{ neighbours in } B_n;$$

$= 0$  otherwise. There is a quasistationary distribution  $f_i^n$  for  $i \in B_n$  which is an eigenvector for  $P^n$  corresponding to  $\mu(n)$  the largest eigenvalue of this matrix. That is for each  $i \in B_n$

$$f_i^n = \mu(n) \sum_j P_{ji}^n f_j^n = \frac{\mu(n)}{2d} \sum_j f_j^n,$$

where in both cases the summation is over  $j$  in  $B_n$  that are neighbors to site  $i$ .

The eigenvalue  $\mu(n)$  is endowed with the following probabilistic meaning

$$\forall i \in B_n \quad P^i(\tau_n \geq N) \sim (\mu(n))^N \tag{1}$$

where  $\tau_n$  is the death time for the subMarkov chain (or equivalently the quitting time of  $B_n$  for the unrestricted simple random walk).

Here  $\sim$  means that the ratio of the two quantities tends to a finite, strictly positive constant as  $N$  tends to infinity. We will use Donsker's invariance principle.

Consider a speed  $\frac{1}{d}$  Brownian motion  $(W_t : t \geq 0)$  starting at  $x_0$  of magnitude  $1/2$ . Let  $\sigma_a = \inf\{t > 0 : |W_t| = a\}$ , then (see e.g. Ito and MacKean (1965)) there exists  $c_d \in (0, \infty)$  so that independent of the particular  $x_0$ ,

$$P(\sigma_{1/3} < \sigma_1) = c_d.$$

For instance if  $d \geq 3$ ,  $c_d = \frac{2^{d-2}-1}{3^{d-2}-1}$ . Thus by path continuity and the isotropy of Brownian motion, there exists  $h_d > 0$  so that for all  $x_0$  of magnitude  $1/2$

$$P^{x_0}(\sigma_{1/3} < \sigma_1 \wedge h_d) > c_d/2.$$

By Donsker's invariance principle and a simple compactness argument we have that for  $n$  sufficiently large, uniformly over all initial positions  $x_0$  on  $\delta(B_{n/2})$ , the boundary of  $B_{n/2}$ , the probability that starting from  $x_0$  a simple random walk hits  $B_{n/3}$  before leaving  $B_n$  after time  $h_d n^2$  is at least  $c_d/2$ .

Thus (using repeatedly the Strong Markov property) for  $n$  sufficiently large the simple random walk starting at  $x_0$  on  $\delta(B_{n/2})$  will exit  $B_n$  after time  $n^2 h_d N$  with probability at least  $(c_d/2)^N$ .

This fact and (1) imply that

$$\mu(n) \geq (c_d/2)^{\frac{1}{n^2 h_d}} > 1 - a$$

for  $n$  sufficiently large. This completes the proof of Lemma 1.

We fix  $\epsilon = \lambda_1 - 1 > 0$ . Consider a simple branching process so that particles die at rate 1 and split in two at rate  $1 + \epsilon/2$ , alternatively  $(X_t : t \geq 0)$  is a birth & death process with 0 absorbing

$$q_{n,n+1} = n(1 + \epsilon/2)$$

$$q_{n,n-1} = n$$

It is well known that if  $X_0 = 1$ ,  $(X_t e^{-t\epsilon/2} : t \geq 0)$  is an  $L_2$  bounded *martingale*. Let this bound be  $K$ .

**Lemma 2.**  $\forall \delta > 0$ , we have

$$P(\sup_{t>0} |\frac{X_t}{X_0} e^{-\epsilon t/2} - 1| > \delta) < \frac{2}{\delta} \frac{K}{\sqrt{X_0}}.$$

*Proof of Lemma 2.*

Note that

$$M_t = \frac{X_t}{X_0} e^{-\epsilon t/2} - 1 \text{ is a martingale}$$

with  $M_0 = 0$ . Thus, for any  $T > 0$

$$P(\inf_{t \leq T} M_t \leq -\delta) \leq E(M_T^+)/\delta$$

see for instance (2.47) in Ethier and Kurtz (1986). Similarly, we have

$$P(\sup_{t \leq T} M_t \geq \delta) \leq E(M_T^+)/\delta.$$

Therefore,

$$P(\sup_{t \leq T} |M_t| \geq \delta) \leq 2E(M_T^2)^{1/2}/\delta.$$

We now compute

$$E(M_T^2) = \frac{1}{X_0^2} \|X_T e^{-\epsilon T/2} - X_0\|_2^2$$

where  $\|\cdot\|_2$  denotes the  $L_2$  norm. We write  $X_t$  as a sum of  $X_0$  i.i.d. processes, denoted by  $Y_t^{(i)}, 1 \leq i \leq X_0$ , having the same rates as  $X_t$  and with initial state 1. Thus,

$$E(M_T^2) = \frac{1}{X_0^2} \left\| \sum_{i=1}^{X_0} (Y_T^{(i)} e^{-\epsilon T/2} - 1) \right\|_2^2.$$

We use the independence of the  $Y_T^{(i)}, 1 \leq i \leq X_0$  to get

$$E(M_T^2) = \frac{1}{X_0^2} X_0 \|Y_T^{(1)} e^{-\epsilon T/2} - 1\|_2^2 \leq K^2/X_0.$$

This completes the proof of Lemma 2.

Pick  $a > 0$  so that

$$\frac{(1+\epsilon)(1-a)}{1+\epsilon/2} > 1.$$

We choose  $N_0$  satisfying Lemma 1 for the  $a$  above. We now go back to the eigenfunction,  $f \equiv f_{N_0}$  corresponding to the largest eigenvalue,  $\mu(n)$ , of the subprobability matrix for the simple random walk, with Dirichlet boundary conditions, on  $B \equiv B_{N_0}$ . By Perron-Frobenius  $f$  is strictly positive on  $B$ . Thus, it has a minimum value  $m > 0$  and for all  $x$  in  $B$  and  $N_1$

$$\frac{f(x)N_1 + 1}{f(x)N_1} \leq \frac{mN_1 + 1}{mN_1}.$$

Pick  $\delta > 0$  small enough for

$$\frac{(1+\epsilon)(1-a)}{1+\epsilon/2} \frac{1-\delta}{1+\delta} > 1.$$

Then, there are integers  $N_1$  large enough to have

$$\frac{(1+\epsilon)(1-a)}{1+\epsilon/2} \frac{1-\delta}{1+\delta} > \frac{mN_1 + 1}{mN_1}.$$

**Lemma 3.** *There exists a system of identically distributed birth and death processes with rates  $q_{i,j}$  denoted by  $(X_t^x : t \geq 0)_{x \in B}$  and such that if for all  $x$  in  $B$ ,  $X_0^x = \xi_0(x) = \lceil f(x)N_1 \rceil$  then the following coupling holds*

$$X_t^x \leq \xi_t(x) \text{ for all } t \leq \tau \text{ and for all } x \in B$$



where

$$\tau = \inf\{s : \exists x \in B, \frac{X_s^x}{X_0^x} e^{-\epsilon s/2} \notin (1 - \delta, 1 + \delta)\}$$

*Proof of Lemma 3.*

We now construct  $(X_t^x : t \geq 0)_{x \in B}$  from the process  $(\xi_t(x) : t \geq 0)_{x \in B}$ , providing a coupling of the two processes.

Let  $(Y_t^x(n) : t \geq 0)_{x \in B, n \geq 1}$  be independent Poisson processes, independent of  $(\xi_t(x) : t \geq 0)_{x \in B}$  and such that  $Y_t^x(n)$  has rate  $n$ . If there is a death at  $t$  for  $\xi_t(x)$  and if  $X_t^x \leq \xi_t(x)$  then there is a death at  $t$  for  $X_t^x$  with probability

$$\frac{X_t^x}{\xi_t(x)}.$$

If  $X_t^x = n > \xi_t(x)$  and there is a birth at time  $t$  for the Poisson process  $Y_t^x(n)$  then there is a death at  $t$  for  $X_t^x$ .

For births we do something similar. Let  $(Z_t^x(n) : t \geq 0)_{x \in B, n \geq 1}$  be independent Poisson processes, independent of  $(\xi_t(x) : t \geq 0)_{x \in B}$  and such that  $Z_t^x(n)$  has rate  $n(1 + \epsilon/2)$ .

If there is a birth at  $x$  at  $t$  for  $\xi_t(x)$  and if  $(1 + \epsilon/2)X_{t-}^x \leq \lambda_1 \sum_{y \sim x} \xi_t(y)/(2d)$  there is a birth at the same time for  $X_t^x$  with probability

$$\frac{2d(1 + \epsilon/2)X_{t-}^x}{\lambda_1 \sum_{y \sim x} \xi_t(y)}$$

If  $(1 + \epsilon/2)X_{t-}^x = n > \lambda_1 \sum_{y \sim x} \xi_t(y)/(2d)$  and there is a birth at time  $t$  for the process  $Z_t^x(n)$  then there is a birth at the same time for  $X_t^x$ .

The condition  $\xi_t(x) \geq X_t^x \forall x \in B$  can evidently never be violated by a death (recall that for all  $x$  in  $B$   $X_0^x = \xi_0(x)$ ), so it remains to check that for  $t < \tau$  the domination relation holds good for births as well.

Assume that  $t < \tau$  and  $x \in B$ , the flip rate upwards for  $X_t^x$  is  $(1 + \epsilon/2)X_{t-}^x$ , while the flip rate upwards for  $\xi_t^x$  at time  $t$  is

$$(1 + \epsilon) \sum_{y \sim x} \xi_{t-}(y)/(2d).$$

By hypothesis  $\xi_{t-}(y) \geq X_{t-}^y$  for each relevant  $y$  and so this flip rate exceeds

$$(1 + \epsilon) \sum_{y \sim x} X_{t-}^y / (2d).$$

By the fact that  $t < \tau$  this is more than

$$\begin{aligned} & (1 + \epsilon) \sum_{y \sim x} [f(y)N_1] e^{t\epsilon/2} (1 - \delta) / (2d) \\ & \geq (1 + \epsilon) \left[ \sum_{y \sim x} f(y)N_1 \right] e^{t\epsilon/2} (1 - \delta) / (2d) \end{aligned}$$

and, by Lemma 1, this is more than

$$(1 + \epsilon) [(1 - a)2df(x)N_1] e^{t\epsilon/2} (1 - \delta) / (2d) \geq (1 + \epsilon)(1 - a)2df(x)N_1 e^{t\epsilon/2} (1 - \delta) / (2d).$$

Recall that  $N_1$  has been chosen so that

$$\frac{(1 + \epsilon)(1 - a)}{1 + \epsilon/2} \frac{1 - \delta}{1 + \delta} > \frac{mN_1 + 1}{mN_1} \geq \frac{f(x)N_1 + 1}{f(x)N_1}.$$

Thus, for all  $x$  in  $B$

$$(1 + \epsilon)(1 - a)f(x)N_1 e^{t\epsilon/2} (1 - \delta) \geq (1 + \epsilon/2)(1 + f(x)N_1) e^{t\epsilon/2} (1 + \delta).$$

This, in turn is more than

$$(1 + \epsilon/2)[f(x)N_1] e^{t\epsilon/2} (1 + \delta) \geq (1 + \epsilon/2)X_{t-}^x$$

where the last inequality comes again from the fact that  $t \leq \tau$ . This shows that the domination conditions cannot be violated for  $t < \tau$  and concludes the proof of Lemma 3.

We now conclude the proof of Theorem 1. Assume that  $\xi_0(x) = [f(x)N_1]$  for every  $x \in B$ . Let  $\mathcal{A}$  be the event

$$\mathcal{A} = \{\exists t > 0, \exists x \in B, \xi_t(x) < (1 - \delta)[f(x)N_1] e^{t\epsilon/2}\}.$$

Note that, by Lemma 3, the intersection of the events  $\{\tau = \infty\}$  and  $\mathcal{A}$  is empty. Thus,

$$P(\mathcal{A}) \leq P(\tau < \infty) \leq \sum_{x \in B} \frac{2}{\delta} \frac{K}{\sqrt{[f(x)N_1]}}$$

where the second inequality comes from Lemma 2. Since  $f$  is strictly positive on  $B$  we may pick  $N_1$  large enough so that

$$\min_{x \in B} [f(x)N_1] \geq 4 \frac{K^2}{\delta^2 \alpha^2} |B|^2$$

and

$$P(\mathcal{A}) \leq \alpha.$$

This concludes the proof of Theorem 1 for  $\delta$  small enough but this implies the theorem for every  $\delta \in (0, 1)$ .

**Acknowledgment.** R.B.S. thanks Enrique Andjel for numerous helpful discussions. We also thank an anonymous referee whose careful reading helped improve the presentation of the paper.

### References.

D. Aldous and J. Fill (2003) in Reversible Markov chains and random walks on graphs, available <http://www.stat.berkeley.edu/users/aldous/>.

J.Englander and A.E.Kyprianou (2004) Local extinction versus local exponential growth for spatial branching processes. *Annals of Probability*, to appear.

J. Englander and R. Pinsky (1999) On the construction and support properties of measure-valued diffusions on  $D \subset \mathbf{R}^d$  with spatially dependant branching. *Ann. Probab.* **27** (2), 684-730.

S.N.Ethier and T.G.Kurtz (1986) *Markov Processes, Characterization and convergence*, Wiley Interscience.

K. Ito and H.P. McKean (1965) *Diffusion Processes and their sample paths*, Springer-Verlag, 1965.

H. Kesten and J. Van den Berg (2000) Asymptotic density in a coalescing random walk model, *Annals of Probability*, **28**, 1, 303-352.

T.M. Liggett (1985) *Interacting Particle Systems*, Springer-Verlag.

T.M.Liggett (1999) Branching random walks on finite trees. In *Perplexing problems in probability: papers in honor of Harry Kesten* (M.Bramson and R.Durrett, eds), 315-330. Birkhauser, Boston.

C.Neuhauser (1992) Ergodic Theorems for the multitype contact process. Probability Theory and Related Fields, **91**, 467-506.

R. Pemantle and A. M. Stacey (2001), The Branching Random Walk and Contact Process on Galton-Watson and Nonhomogeneous Trees. Ann. Probab. 29, no. 4, 1563-1590.