

Contact process in a wedge

J. Theodore Cox* Nevena Marić†
Syracuse University *University of Missouri - St. Louis*

Rinaldo Schinazi‡
University of Colorado - Colorado Springs

February 19, 2010

Abstract

We prove that the supercritical one-dimensional contact process survives in certain wedge-like space-time regions, and that when it survives it couples with the unrestricted contact process started from its upper invariant measure. As an application we show that a type of weak coexistence is possible in the nearest-neighbor “grass-bushes-trees” successional model introduced in [5].

Key words: contact process, grass-bushes-trees

AMS Classification: Primary: 60K35; Secondary: 82B43

1 Introduction

The contact process of Harris (introduced in [7]) is a well known model of infection spread by contact. The one-dimensional model is a continuous time Markov process ξ_t on $\{0, 1\}^{\mathbb{Z}}$. For $x \in \mathbb{Z}$, $\xi_t(x) = 1$ means the individual at site x is infected at time t while $\xi_t(x) = 0$ means the individual is healthy. Infected individuals recover from their infection after an exponential time with mean 1, independently of everything else. Healthy individuals become infected at a rate proportional to the number of infected neighbors. Alternatively, individuals (1's) die at rate one and give birth onto neighboring empty sites (0's) at rate λ . If we let $n_i(x, \xi) = \sum_{y:|y-x|=1} 1\{\xi(y) = i\}$, and $\lambda \geq 0$ the infection parameter, then the transitions at x in state ξ are

$$1 \rightarrow 0 \text{ at rate } 1 \quad \text{and} \quad 0 \rightarrow 1 \text{ at rate } \lambda n_1(x, \xi). \quad (1)$$

When convenient we will identify $\xi \in \{0, 1\}^{\mathbb{Z}}$ with $\{x : \xi(x) = 1\}$, and use the notation $\|\xi\|_i = \sum_x 1\{\xi(x) = i\}$.

*Supported in part by NSF Grant No. 0803517

†Supported in part by NSF Grant No. 0803517

‡Supported in part by NSF Grant No. 0701396

Let ξ_t^0 denote the contact process with initial state $\xi_0^0 = \{0\}$. The critical value λ_c is defined by

$$\lambda_c = \inf\{\lambda \geq 0 : P(\xi_t^0 \neq \emptyset \text{ for all } t \geq 0) > 0\}. \quad (2)$$

It is well known that $0 < \lambda_c < \infty$, and that in the supercritical case $\lambda > \lambda_c$ there is a unique stationary distribution ν for ξ_t , called the upper invariant measure, with the property

$$\nu(\xi : \|\xi\|_1 = \infty) = 1.$$

There are also well-defined ‘‘edge speeds.’’ Let ξ_0^- (ξ_0^+) be the initial state given by $\xi_0^- = \mathbb{Z}^-$ ($\xi_0^+ = \mathbb{Z}^+$), and define the edge processes

$$r_t = \max\{x : \xi_t^-(x) = 1\} \text{ and } l_t = \min\{x : \xi_t^+(x) = 1\}. \quad (3)$$

There is a strictly increasing function $\alpha : (\lambda_c, \infty) \rightarrow (0, \infty)$ such that for $\lambda > \lambda_c$

$$\lim_{t \rightarrow \infty} \frac{r_t}{t} = \alpha(\lambda) \text{ and } \lim_{t \rightarrow \infty} \frac{l_t}{t} = -\alpha(\lambda) \text{ a.s.} \quad (4)$$

All of the above facts are contained in Chapter VI of [8] and Part I of [9]. See also Part I of [9] and the Bibliography there for many references to original papers.

The focus of this paper is a version of the contact process in which infection is restricted to certain space-time regions. In particular, we are interested in whether or not survival of the infection is possible in narrow, wedge-like regions, and, given survival, how does such a process behave. For $\mathcal{W} \subset \mathbb{Z} \times [0, \infty)$ define the space-time inhomogeneous \mathcal{W} -restricted contact process $\xi_t^{\mathcal{W}}$ as follows. First, set $\xi_t^{\mathcal{W}}(x) = 0$ for all $(x, t) \notin \mathcal{W}$. Second, for $(x, t) \in \mathcal{W}$, replace (1) with

$$1 \rightarrow 0 \text{ at rate } 1 \quad \text{and} \quad 0 \rightarrow 1 \text{ at rate } \lambda \sum_{y:|y-x|=1} \xi(y)1_{\mathcal{W}}(y, t), \quad (5)$$

so that infection spreads only between sites in the wedge. We will give an explicit *graphical construction* of $\xi_t^{\mathcal{W}}$ in Section 2. Our notation is slightly different from the standard one for which the superscript designates the initial state. For space-time regions \mathcal{W} , the notation $\xi_t^{\mathcal{W}}$ indicates the contact process with infection restricted to \mathcal{W} , and we will always explicitly state our choice of initial state $\xi_0^{\mathcal{W}}$.

For $0 < \alpha_l < \alpha_r < \infty$ and nonnegative integers $M \geq 0$ define the ‘‘wedges’’ $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$ by

$$\mathcal{W} = \{(x, t) \in \mathbb{Z} \times [0, \infty) : \alpha_l t \leq x \leq M + \alpha_r t\}. \quad (6)$$

In view of (4), we will impose the conditions

$$\lambda > \lambda_c \text{ and } 0 < \alpha_l < \alpha_r < \alpha(\lambda). \quad (7)$$

Our first result is that survival in wedges is possible.

Theorem 1. *Assume (7) holds, $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$, and $\xi_0^{\mathcal{W}} = [0, M] \cap \mathbb{Z}$. Then*

$$\lim_{M \rightarrow \infty} P(\xi_t^{\mathcal{W}} \neq \emptyset \text{ for all } t \geq 0) = 1. \quad (8)$$

When $\xi_t^{\mathcal{W}}$ survives it looks like the unrestricted contact process in equilibrium. To state this more precisely, let

$$r_t^{\mathcal{W}} = \max\{x : \xi_t^{\mathcal{W}}(x) = 1\} \text{ and } l_t^{\mathcal{W}} = \min\{x : \xi_t^{\mathcal{W}}(x) = 1\}, \quad (9)$$

and let $\hat{\xi}_t$ denote the contact process started in its upper invariant measure ν . (That is, $\hat{\xi}_0$ is random with law ν , and given $\hat{\xi}_0$, $\hat{\xi}_t$ makes transitions according to (1).)

Theorem 2. *Assume $M \geq 1$ and (7) holds. Let $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$ and assume $\xi_0^{\mathcal{W}} = [0, M] \cap \mathbb{Z}$. On the event $\{\xi_t^{\mathcal{W}} \neq \emptyset \text{ for all } t \geq 0\}$,*

$$\lim_{t \rightarrow \infty} \frac{r_t^{\mathcal{W}}}{t} = \alpha_r \text{ and } \lim_{t \rightarrow \infty} \frac{l_t^{\mathcal{W}}}{t} = \alpha_l \text{ a.s.} \quad (10)$$

Furthermore, $\xi_t^{\mathcal{W}}$ and $\hat{\xi}_t$ can be coupled so that on the event $\{\xi_t^{\mathcal{W}} \neq \emptyset \text{ for all } t \geq 0\}$,

$$\xi_t^{\mathcal{W}}(x) = \hat{\xi}_t(x) \text{ for all } x \in [l_t^{\mathcal{W}}, r_t^{\mathcal{W}}] \text{ for all large } t \text{ a.s.} \quad (11)$$

Theorem 2 is analogous to results for the unrestricted contact process, see Theorem VI.2.2 in [8].

Remark 3. *It is not difficult to extend the argument used in the proof of Theorem VI.3.33 in [8] to show that for $a < b$, $|\hat{\xi}_t \cap [at, bt]|/t \rightarrow (b - a)\rho(\lambda)$ a.s. as $t \rightarrow \infty$, where $\rho(\lambda) = \nu(\xi : \xi(0) = 1)$. Therefore Theorem 2 implies that if $\alpha_l < a < b < \alpha_r$, then when $\xi_t^{\mathcal{W}}$ survives, $|\xi_t^{\mathcal{W}} \cap [at, bt]|/t \rightarrow (b - a)\rho(\lambda)$ a.s.*

Theorem 1 can be used to obtain information about the ‘‘grass-bushes-trees’’ model (GBT) of [5]. In this model sites are either empty (0), occupied by a bush (1) or occupied by a tree (2). Both 1’s and 2’s turn to 0’s at rate one. The 2’s give birth at rate λ_2 on top of 1’s and 0’s. The 1’s give birth at rate λ_1 on top of 0’s only, and hence are at a disadvantage compared to 2’s. The state space for the process is $\{0, 1, 2\}^{\mathbb{Z}}$, and the nearest-neighbor version of the model makes transitions at x in state ζ

$$0 \rightarrow \begin{cases} 1 & \text{at rate } \lambda_1 n_1(x, \zeta) \\ 2 & \text{at rate } \lambda_2 n_2(x, \zeta) \end{cases} \quad 1 \rightarrow \begin{cases} 0 & \text{at rate } 1 \\ 2 & \text{at rate } \lambda_2 n_2(x, \zeta) \end{cases} \quad 2 \rightarrow 0 \text{ at rate } 1. \quad (12)$$

A natural question to ask is whether or not coexistence of 1’s and 2’s is possible. It was shown in [5] that coexistence is possible for a non-nearest neighbor version of the model and appropriate λ_i , where coexistence meant that ζ_t had a stationary distribution μ such that

$$\mu\left(\{\zeta : \|\zeta\|_i = \infty \text{ for } i = 1, 2\}\right) = 1. \quad (13)$$

It was also shown in [5] that there is no stationary distribution satisfying (13) in the nearest-neighbor case for *any* choice of the λ_i . Moreover, if there are infinitely many 2’s initially then for each site there is a last time at which a 1 can be present. Nevertheless, it is a consequence of Theorem 1 and the construction used in its proof that a form of weak coexistence is possible, even starting from a single 1 and infinitely many 2’s.

Corollary 4. *Let ζ_t be the GBT process with initial state ζ_0 , where $\zeta_0(x) = 2$ for $x < 0$, $\zeta_0(0) = 1$ and $\zeta_0(x) = 0$ for $x > 0$. For all $\lambda_c < \lambda_2 < \lambda_1$,*

$$P\left(\lim_{t \rightarrow \infty} \|\zeta_t\|_1 = \infty\right) > 0. \quad (14)$$

The 2's spread to the right at rate $\alpha(\lambda_2)$, ignoring the 1's, while the 1's try to spread to the right at the faster rate $\alpha(\lambda_1)$. The 1's will be killed by 2's invading from the left, but Theorem 1 shows that they can survive with positive probability by moving off to the right in the space-time region free of 2's.

Remark 5. (1) *Corollary 4 is proved by showing that the set of 1's in ζ_t dominates an appropriate wedge-restricted contact process with positive probability. By working harder one can use this idea and Theorem 2 to obtain more information about the set of 1's in ζ_t , but we will not pursue this here.* (2) *Our model is a stochastic process in an inhomogeneous environment. There are numerous papers in the physics literature regarding such models, see for instance [11], [2] and [10] and more recently in the mathematical biology literature, see [1]. Closer to this paper is the work on non-oriented percolation in various subsets of \mathbb{Z}^d that has been studied in [6] and in [3], but as far as we are aware our results on oriented percolation are new.*

In Section 2 we give the standard graphical construction due to Harris, then prove Theorem 1 in Section 3, Theorem 2 in Section 4, and Corollary 4 in Section 5.

2 The graphical representation

For $x \in \mathbb{Z}$ let $\{T_n^x : n \geq 1\}$ be the arrival times of a Poisson process with rate 1, and for all pairs of nearest-neighbor sites x, y let $\{B_n^{x,y} : n \geq 1\}$ be the arrival times of a Poisson process with rate λ . The Poisson processes $T^x, B^{x,y}$, $x, y \in \mathbb{Z}$, are all independent. At the times T_n^x we put a δ at site x to indicate a death at x , and at the times $B_n^{x,y}$ we draw an arrow from x to y , indicating that a 1 at x will give birth to a 1 at y . For $0 \leq s < t$ and sites x, y we say that there is an active path up from (x, s) to (y, t) if there is a sequence of times $t_0 = s \leq t_1 < t_2 < \dots < t_n \leq t_{n+1} = t$ and a sequence of sites $x_0 = x, x_1, \dots, x_n = y$ such that

1. if $n \geq 1$, then for $i = 1, 2, \dots, n$, $|x_i - x_{i-1}| = 1$ and there is an arrow from x_{i-1} to x_i at time t_i
2. for $i = 0, \dots, n$, the time segments $\{x_i\} \times [t_i, t_{i+1}]$ do not contain any δ 's

By default there is always an active path up from (y, t) to (y, t) . For a space-time region $\mathcal{W} \subset \mathbb{Z} \times [0, \infty)$ we define $\xi_t^{\mathcal{W}}$, the contact process restricted to \mathcal{W} , as follows. Given an initial state $\xi_0 \subset \{x : (x, 0) \in \mathcal{W}\}$, set $\xi_t(y) = 0$ for all $(y, t) \notin \mathcal{W}$. If there is a site x with $\xi_0(x) = 1$ and an active path up from $(x, 0)$ to (y, t) lying entirely in \mathcal{W} set $\xi_t^{\mathcal{W}}(y) = 1$, otherwise set $\xi_t^{\mathcal{W}}(y) = 0$. For $\mathcal{W} = \mathbb{Z} \times [0, \infty)$ we will write ξ_t and refer to it as the unrestricted process.

We may also construct the GBT process ζ_t with the above Poisson processes and the help of some additional independent coin flips. Fix $\lambda_c < \lambda_2 < \lambda_1$, and suppose

$\lambda = \lambda_1$ in the construction just given. Independently of everything else, label the arrows determined by the B_n^{xy} with a “1-only” sign with probability $(\lambda_1 - \lambda_2)/\lambda_1$. Call an active path up from (x, s) to (y, t) a 2-path if none of its arrows are 1-only arrows. Given ζ_0 , we may now construct ζ_t as follows. (i) For all $t > 0$ and $x \in \mathbb{Z}$, put $\zeta_t(x) = 2$ if for some site y with $\zeta_0(y) = 2$ there is an active 2-path up from $(y, 0)$ to (x, t) . (ii) If we have not already set $\zeta_t(x) = 2$, put $\zeta_t(x) = 1$ if for some site y with $\zeta_0(y) = 1$ there is an active path up from $(y, 0)$ to (x, t) with the property that no vertical segments in the path contain a point (z, u) such that $\zeta_u(z) = 2$. (iii) For all other (x, t) , put $\zeta_t(x) = 0$. A little thought shows that ζ_t is the GBT process with the rates given in (12). The process of 2’s is a contact process with infection parameter λ_2 , and in the absence of 2’s, the process of 1’s is a contact process with infection parameter λ_1 .

3 Proof of Theorem 1

The space-time regions \mathcal{Y}_{jk} . We will modify somewhat the standard approach of constructing a mapping from appropriate space-time regions of the construction just given to an oriented-percolation model with the property that survival of the percolation process implies survival of the contact process. We will call the regions \mathcal{Y}_{jk} , they will be defined using the parallelograms of Section VI.3 of [8]. We will make use of the following notation: for $x \in \mathbb{R}^2$ and $A \subset \mathbb{R}^2$, $x + A = \{x + a : a \in A\}$, the translate of A by x .

Let \mathcal{L} be the lattice $\mathcal{L} = \{(j, k) \in \mathbb{Z}^2 : k \geq 0 \text{ and } j + k \text{ is even}\}$ with norm $\|(j, k)\| = 1/2(|j| + |k|)$. Fix $0 < \beta < \alpha/3$ and $M > 0$ so that $M\beta/2$ and $M\alpha$ are integers. Later we will set $\alpha = \alpha(\lambda)$ and take β small. For $(j, k) \in \mathcal{L}$, L_{jk} and R_{jk} are the “large” space-time parallelograms in $\mathbb{Z} \times [0, \infty)$ given by:

$$L_{jk} = (Mj(\alpha - \beta), Mk) + L_{00}, \quad R_{jk} = (Mj(\alpha - \beta), Mk) + R_{00}$$

where

$$L_{00} = \{(x, t) \in \mathbb{Z} \times [0, M(1 + \beta/\alpha)] : M\beta/2 \leq x + \alpha t \leq 3M\beta/2\}$$

$$R_{00} = \{(x, t) \in \mathbb{Z} \times [0, M(1 + \beta/\alpha)] : -3M\beta/2 \leq x - \alpha t \leq -M\beta/2\}.$$

Observe that $L_{jk}(R_{jk})$ is simply a translation of $L_{00}(R_{00})$ by the vector $(Mj(\alpha - \beta), Mk)$. We will also need the “small” parallelograms

$$L_{jk}^{small} = (Mj(\alpha - \beta), Mk) + L_{00}^{small}, \quad R_{jk}^{small} = (Mj(\alpha - \beta), Mk) + R_{00}^{small}$$

where

$$L_{00}^{small} = \{(x, t) \in \mathbb{Z} \times [0, M\frac{3\beta}{2\alpha}] : M\beta/2 \leq x + \alpha t \leq 3M\beta/2\}$$

$$R_{00}^{small} = \{(x, t) \in \mathbb{Z} \times [0, M\frac{3\beta}{2\alpha}] : -3M\beta/2 \leq x - \alpha t \leq -M\beta/2\}.$$

It is important to note that $L_{00}^{small} \subset L_{00}$, $R_{00}^{small} \subset R_{00}$, and

$$R_{jk} \cap L_{jk} = R_{jk} \cap L_{jk}^{small} = R_{jk}^{small} \cap L_{jk},$$

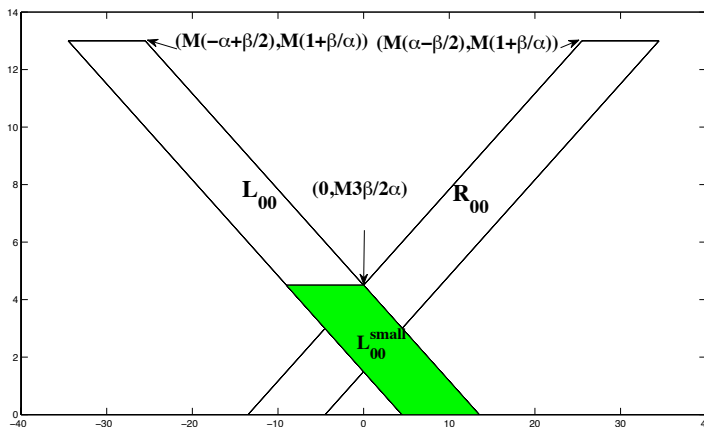


Figure 1: Large parallelograms L_{00} and R_{00} . The shaded region is L_{00}^{small} .

as shown in Figure 1.

We can now define the new objects \mathcal{Y}_{jk} which will be used to construct our oriented percolation process. As is the case with the parallelograms, the \mathcal{Y}_{jk} will be certain translates of \mathcal{Y}_{00} , and depend on two fixed integers ℓ, d which satisfy $\ell \geq 2$ and $d \geq 0$ with $\ell > d$ (the dependence on ℓ, d will be suppressed from the notation). We will form \mathcal{Y}_{00} by sticking together ℓ big right parallelograms, connected with appropriate small left parallelograms, and then two branches of d and $d+1$ big left parallelograms connected by small right parallelograms. Figure 2 shows examples of \mathcal{Y}_{00} with parameters $\ell = 5$ and $d = 0, 1, 2$. It seems simplest to define \mathcal{Y}_{00} in stages, beginning with $\mathcal{Y}_{00}^0 = R_{00}$.

1. Attach ℓ big right parallelograms with ℓ small parallelograms to connect them:

$$\mathcal{Y}_{00}^1 = \mathcal{Y}_{00}^0 \cup \left(\bigcup_{i=1}^{\ell} (R_{ii} \cup L_{ii}^{small}) \right).$$

2. Attach one big left parallelogram: $\mathcal{Y}_{00}^2 = \mathcal{Y}_{00}^1 \cup L_{\ell, \ell}$.

3. If $d = 0$ set $\mathcal{Y}_{00} = \mathcal{Y}_{00}^2$. If $d \geq 1$, attach another big left parallelogram:

$$\mathcal{Y}_{00}^3 = \mathcal{Y}_{00}^2 \cup L_{\ell+1, \ell+1}.$$

4. If $d = 1$, attach another big left and small right parallelogram:

$$\mathcal{Y}_{00}^4 = \mathcal{Y}_{00}^3 \cup (L_{\ell-1, \ell+1} \cup R_{\ell-1, \ell+1}^{small})$$

and set $\mathcal{Y}_{00} = \mathcal{Y}_{00}^4$. If $d \geq 2$, attach two branches, to reach “height” $\ell + d + 1$, of big left parallelograms with small right parallelograms as connectors:

$$\mathcal{Y}_{00}^4 = \mathcal{Y}_{00}^3 \cup \left(\bigcup_{i=0}^{d-1} (L_{\ell-i, \ell+i} \cup R_{\ell-i, \ell+i}^{small}) \cup (L_{\ell+1-i, \ell+1+i} \cup R_{\ell+1-i, \ell+1+i}^{small}) \right).$$

5. If $d \geq 2$, attach a final big left parallelogram and small right parallelogram:

$$\mathcal{Y}_{00}^5 = \mathcal{Y}_{00}^4 \cup L_{\ell-d, \ell+d} \cup R_{\ell-d, \ell+d}^{small}$$

and put $\mathcal{Y}_{00} = \mathcal{Y}_{00}^5$.

Having defined \mathcal{Y}_{00} we set

$$\mathcal{Y}_{jk} = (M[k(\ell - d) + j](\alpha - \beta), Mk(\ell + d + 1)) + \mathcal{Y}_{00}, (j, k) \in \mathcal{L}.$$

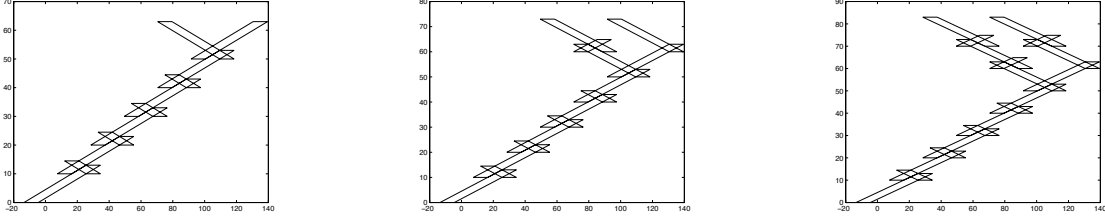


Figure 2: \mathcal{Y}_{00} with $\ell = 5$, $d = 0, 1, 2$.

The percolation variables U_{jk} . Given ℓ, d and the objects \mathcal{Y}_{jk} , let \mathcal{O}_{jk} be the event that for every parallelogram \mathcal{P} in \mathcal{Y}_{jk} there is an active path in the graphical representation of the contact process which stays entirely in \mathcal{P} and connects some point in the bottom edge of \mathcal{P} to some point in the top edge of \mathcal{P} . Thus on \mathcal{O}_{jk} there is some point in the bottom edge of \mathcal{Y}_{jk} with the property that there are active paths in \mathcal{Y}_{jk} connecting this point to the top edge of every parallelogram in \mathcal{Y}_{jk} , and in particular to the top edges of the two top parallelograms \mathcal{Y}_{jk} . This means that on \mathcal{O}_{jk} there is a point in the bottom edge of \mathcal{Y}_{jk} and active paths in \mathcal{Y}_{jk} connecting this point to the bottom edges of both $\mathcal{Y}_{j-1, k+1}$ and $\mathcal{Y}_{j+1, k+1}$.

It is a consequence of Lemma VI.3.17 in [8] that $P(\mathcal{O}_{00})$ is close to 1 for large M .

Lemma 6. For $0 < \beta < \alpha/3$ and fixed ℓ, d , $\lim_{M \rightarrow \infty} P(\mathcal{O}_{00}) = 1$.

Proof: As in [8] let \mathcal{E}_{jk} be the event that there is an active path in the graphical representation of the contact process which goes from the bottom edge of R_{jk} to the top edge, always staying entirely within R_{jk} , and also that there is an active path from the bottom edge of L_{jk} to the top edge, always staying entirely within L_{jk} . It is clear that the probability of connecting the bottom edge of a small parallelogram to its top edge by an active path staying in the parallelogram is bounded below by $P(\mathcal{E}_{00})$. By Lemma 3.17 in [8], for $0 < \beta < \alpha/3$, $\lim_{M \rightarrow \infty} P(\mathcal{E}_{00}) = 1$. In the construction of \mathcal{Y}_{00} there are most $h = 2\ell + 4d$ (if $d \geq 1$) or $h = 2\ell + 1$ (if $d = 0$) parallelograms used. It follows from positive correlations that $P(\mathcal{O}_{00}) \geq P(\mathcal{E}_{00})^h$, and thus $\lim_{M \rightarrow \infty} P(\mathcal{O}_{00}) = 1$ ■

For $(j, k) \in \mathcal{L}$ let $U_{jk} = 1_{\mathcal{O}_{jk}}$. Then $P(U_{jk} = 1) = P(\mathcal{O}_{00})$ does not depend on (j, k) . Furthermore, the U_{jk} are 1-dependent, meaning that if $I \subset \mathcal{L}$ is such that

$\|(j, k) - (j', k')\| > 1$ for all $(j, k) \neq (j', k') \in I$, then the $U_{jk}, (j, k) \in I$ are independent. This is because the corresponding space-time regions $\mathcal{Y}_{jk}, \mathcal{Y}_{j'k'}$ are disjoint. Using the U_{jk} we may construct a 1-dependent oriented percolation process in the usual way. A *path* in \mathcal{L} is a sequence $(j_1, k_1), \dots, (j_n, k_n)$ of points of \mathcal{L} which satisfies $k_{i+1} = k_i + 1$ and $j_{i+1} = j_i \pm 1$ for all $1 \leq i \leq n - 1$. The path is said to be *open* if $U_{j_i, k_i} = 1$ for each $1 \leq i \leq n - 1$. It is clear from the properties of the \mathcal{O}_{jk} that if $(j_1, k_1), \dots, (j_n, k_n)$ is an open path in \mathcal{L} then there must be an active path in the graphical representation from the bottom edge of \mathcal{Y}_{j_1, k_1} to the bottom edge of \mathcal{Y}_{j_n, k_n} .

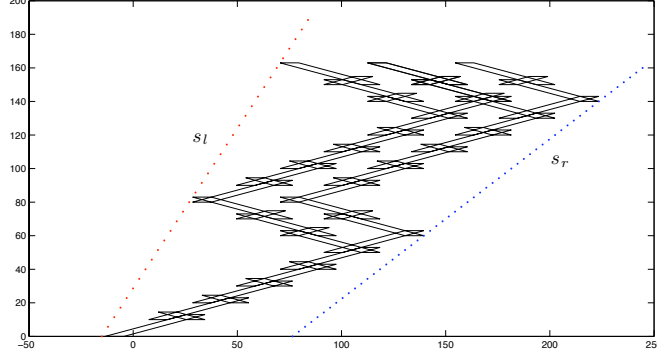


Figure 3: $\mathcal{Y}_{00}, \mathcal{Y}_{1,1}, \mathcal{Y}_{-1,1}$

If we let Ω_∞ be the event that there is an infinite open path in \mathcal{L} starting at $(0, 0)$, then by Lemma 6 above and Theorem VI.3.19 of [8],

$$\lim_{M \rightarrow \infty} P(\Omega_\infty) = 1. \quad (15)$$

Survival of $\xi_t^{\mathcal{W}}$. Let $\mathcal{Y} = \mathcal{Y}(\ell, d, M) = \bigcup_{k=0}^{\infty} \bigcup_{j=-k}^k \mathcal{Y}_{jk}$. On Ω_∞ there must be an infinite active path in the graphical representation starting at some $(x, 0)$, $x \in [-3M\beta/2, -M\beta/2]$, which lies entirely in \mathcal{Y} . Thus if \mathcal{W} is any space-time region such that $\mathcal{Y} \subset \mathcal{W}$, and $\xi_t^{\mathcal{W}}$ is the \mathcal{W} -restricted contact process starting from $\{x : (x, 0) \subset \mathcal{W}\}$, then $\xi_t^{\mathcal{W}} \neq \emptyset \forall t \geq 0$ on Ω_∞ . We will prove the following.

Claim. Assume (7) holds and $\alpha = \alpha(\lambda)$. Then there exists $0 < \beta < \alpha/3$ and integers ℓ', d' such that for all $M > 0$,

$$\mathcal{Y}(\ell', d', M/\alpha(\ell' + 3)) \subset \mathcal{W}(\alpha_l, \alpha_r, M) - (M/(\ell' + 3), 0). \quad (16)$$

Given (16), it follows from translation invariance and (15) that

$$P(\xi_t^{\mathcal{W}(\alpha_l, \alpha_r, M)} \neq \emptyset \forall t \geq 0) \geq P(\Omega_\infty) \rightarrow 1 \text{ as } M \rightarrow \infty,$$

proving (8).

To prove (16) we first suppose that ℓ, d , are positive integers with $d < \ell$ and $M > 0$. For $(j, k) \in \mathcal{L}$, the left upper corner of L_{jk} is $(M(j(\alpha - \beta) - \alpha - \beta/2), M(k + 1 + \beta/\alpha))$, and the right bottom corner of L_{jk} is $(M(j(\alpha - \beta) + 3\beta/2), Mk)$. A little thought shows that \mathcal{Y} must be contained in the space-time region bounded by the following two lines and the x -axis. The first line connects the leftmost point of the top edge of \mathcal{Y}_{00} with the leftmost point of the top edge of $\mathcal{Y}_{-1,1}$, which are the left upper corner of $L_{\ell-d, \ell+d}$ and the left upper corner of $L_{2(\ell+d)-1, 2(\ell+d)+1}$, namely, the points

$$(M((\ell - d)(\alpha - \beta) - \alpha - \beta/2), M(\ell + d + 1 + \beta/\alpha))$$

and

$$(M(2(\ell - d)(\alpha - \beta) - 2\alpha + \beta/2), M(2(\ell + d + 1) + \beta/\alpha)).$$

The slope of this line is

$$s_l = \frac{\ell + d + 1}{\ell - d - 1} \frac{1}{\alpha - \beta} \quad (17)$$

and it contains the point $(x_l, 0)$ where $x_l = -M(3\beta/2 + \beta/\alpha s_l)$. The second line connects the rightmost point of \mathcal{Y}_{00} with the rightmost point of $\mathcal{Y}_{1,1}$, the bottom right corner of $L_{\ell+1, \ell+1}$ and the bottom right $L_{2(\ell+1)-d, 2(\ell+1)+d}$, namely, the points

$$(M((\ell + 1)(\alpha - \beta) + 3\beta/2), M(\ell + 1))$$

and

$$(M((2(\ell + 1) - d)(\alpha - \beta) + 3\beta/2), M(2(\ell + 1) + d)).$$

The slope of this line is

$$s_r = \frac{\ell + d + 1}{\ell - d + 1} \frac{1}{\alpha - \beta} \quad (18)$$

and it contains the point $(x_r, 0)$ where $x_r = M((\ell + 1)(\alpha - \beta - 1/s_r) + 3\beta/2)$.

This analysis shows that $\mathcal{Y}(\ell, d, M)$ is contained in the wedge $\mathcal{W}(1/s_l, 1/s_r, M')$ and $(x_l, 0)$, where $M' = x_r - x_l$. A little algebra shows that $-M\alpha < x_l < x_r < M\alpha(\ell + 2)$, and thus

$$\mathcal{Y}(\ell, d, M) \subset \mathcal{W}(1/s_l, 1/s_r, M\alpha(\ell + 3)) - (M\alpha, 0). \quad (19)$$

We now set $s_\ell = 1/\alpha_\ell, s_r = 1/\alpha_r$ and solve (17) and (18) for d and ℓ , obtaining

$$\ell = \frac{s_r(s_l(\alpha - \beta) + 1)}{s_l - s_r}, \quad d = \frac{s_l(s_r(\alpha - \beta) - 1)}{s_l - s_r}. \quad (20)$$

Unfortunately, ℓ, d need not be integers. To deal with this problem we first note that if $s_l \geq s'_l > s_r$ then for any M , the wedge $\mathcal{W}(\alpha_l, \alpha_r, M)$ contains the narrower wedge $\mathcal{W}(1/s'_l, 1/s_r, M)$. If we can find s'_l and $0 < \beta < \alpha/3$ such that

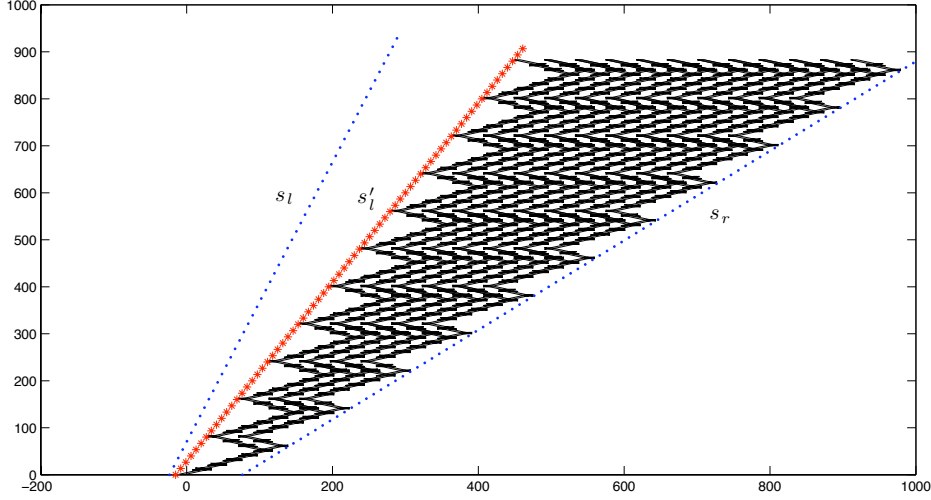
$$\ell' = \frac{s_r(s'_l(\alpha - \beta) + 1)}{s'_l - s_r} \quad \text{and} \quad d' = \frac{s'_l(s_r(\alpha - \beta) - 1)}{s'_l - s_r} \quad (21)$$

are both integers, then (16) follows from (19).

We can find s'_l, β as follows. Let $m_0 = 3/\alpha s_r$ and take any integer $m > m_0$ such that $s_r \frac{m}{m-1} < s_l$. Put $s'_l = s_r \frac{m}{m-1}$, so that $s_l > s'_l > s_r$. Since $m > 3/\alpha s_r, 1/3\alpha m s_r > 1$

and the interval $(\frac{2}{3} \alpha m s_r, \alpha m s_r)$ must contain at least one integer. Since $\alpha s_r > 1$, the right endpoint of this interval is greater than m . Choose any integer $c \geq m$ from the interval and put $\beta = \alpha - \frac{c}{m s_r}$. Then $0 < \beta < \alpha/3$ and $s_r(\alpha - \beta) = c/m$. A little algebra shows that ℓ', d' given in (21) are the integers $\ell' = c + m - 1, d' = c - m$, and we are done.

Figure 4: Wedge containing \mathcal{Y}



4 Proof of Theorem 2

We begin by analyzing the rightmost particle. Let $\mathcal{W}(\alpha_r, M) = \{(x, t) : t \geq 0, x \in (-\infty, M + \alpha_r t] \cap \mathbb{Z}\}$ and consider the restricted contact process $\xi_t^{\mathcal{W}(\alpha_r, M)}$ with initial state $\xi_0^{\mathcal{W}(\alpha_r, M)} = (-\infty, M] \cap \mathbb{Z}$. Let \bar{r}_t^M be the corresponding right-edge process, $\bar{r}_t^M = \max\{x : \xi_t^{\mathcal{W}(\alpha_r, M)}(x) = 1\}$. We claim that for every M ,

$$\lim_{t \rightarrow \infty} \frac{\bar{r}_t^M}{t} = \alpha_r \quad a.s. \quad (22)$$

By stationarity of the basic Poisson processes, the law of \bar{r}^{M+1} is the same as the law of $\bar{r}^M + 1$, so it suffices to prove (22) for $M = 0$. By construction, $\limsup_{t \rightarrow \infty} \bar{r}_t^0/t \leq \alpha_r$ a.s..

For the lower bound, fix $0 < \varepsilon < \alpha_r$ and consider the region $\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M)$ and the restricted contact process $\xi_t^{\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M)}$ with initial state $[0, M] \cap \mathbb{Z}$. For fixed $\delta > 0$, Theorem 1 implies there exists M_0 such that the event $\{\xi_t^{\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M_0)} \neq \emptyset \forall t \geq 0\}$ has probability at least $1 - \delta$. On this event, since $\xi_t^{\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M_0)} \subset \xi_t^{\mathcal{W}(\alpha_r, M_0)}$, $\liminf_{t \rightarrow \infty} \bar{r}_t^{M_0}/t \geq \alpha_r - \varepsilon$. Consequently,

$$P(\liminf_{t \rightarrow \infty} \bar{r}_t^0/t \geq \alpha_r - \varepsilon) = P(\liminf_{t \rightarrow \infty} \bar{r}_t^{M_0}/t \geq \alpha_r - \varepsilon) \geq 1 - \delta.$$

This shows $P(\liminf_{t \rightarrow \infty} \bar{r}_t^0/t \geq \alpha_r - \varepsilon) = 1$ for every $\varepsilon > 0$, and together with the previous limsup bound shows $\bar{r}_t^0/t \rightarrow \alpha_r$ a.s., establishing (22).

It is a consequence of the nearest-neighbor interaction mechanism that for any $\alpha_l < \alpha_r$ and M , with $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$,

$$\xi_t^{\mathcal{W}}(x) = \xi_t^{\mathcal{W}(\alpha_r, M)}(x) \forall x \in [l_t^{\mathcal{W}}, r_t^{\mathcal{W}}] \text{ on } \{\xi_t^{\mathcal{W}} \neq \emptyset\}.$$

This implies $r_t^{\mathcal{W}} = \bar{r}_t^M$ on $\{\xi_t^{\mathcal{W}} \neq \emptyset\}$, and so by (22), $\lim_{t \rightarrow \infty} r_t^{\mathcal{W}}/t = \alpha_r$ on the event $\{\xi_t^{\mathcal{W}} \neq \emptyset \forall t \geq 0\}$. We omit the similar argument for $\lim_{t \rightarrow \infty} l_t^{\mathcal{W}}/t = \alpha_l$.

For (11), let $\xi_t^{\mathbb{Z}}$ denote the unrestricted process with initial state $\xi_0^{\mathbb{Z}} = \mathbb{Z}$, and let $\hat{\xi}_t$ be the unrestricted process constructed in Section 2 started in the invariant measure ν . (That is, $\hat{\xi}_0$ is random with law ν , independent of the Poisson processes, and given $\hat{\xi}_0$ the construction of Section 2 is used.) We observe again that the nearest-neighbor interaction implies

$$\xi_t^{\mathbb{Z}}(x) = \xi_t^{\mathcal{W}}(x) \forall x \in [l_t^{\mathcal{W}}, r_t^{\mathcal{W}}] \text{ on } \{\xi_t^{\mathcal{W}} \neq \emptyset \forall t \geq 0\}.$$

Standard exponential estimates for $P(\xi_t^{\mathbb{Z}}(x) \neq \hat{\xi}_t(x)) = P(\xi_t^{\mathbb{Z}}(x) = 1) - P(\hat{\xi}_t(x) = 1)$, a ‘‘filling in’’ argument and Borel-Cantelli (see Theorem I.2.30 of [9]) imply that for any $A > 0$,

$$P(\xi_t^{\mathbb{Z}} = \hat{\xi}_t \text{ on } [-At, At] \text{ for all large } t) = 1$$

Combining the above with (10) gives (11).

5 Proof of Corollary 4

We will make use of the graphical construction in Section 2 and define independent events $\Omega_1, \Omega_2, \Omega_3$, each with positive probability, and such that $\|\zeta_t\|_1 \rightarrow \infty$ as $t \rightarrow \infty$ on their intersection.

First, since $\alpha(\lambda)$ is strictly increasing we may choose $\alpha(\lambda_2) < \alpha_l < \alpha_r < \alpha(\lambda_1)$. Fix $M > 2$ and write \mathcal{W} for $\mathcal{W}(\alpha_l, \alpha_r, M)$. The first event is

$$\Omega_1 = \{\text{there is no active 2-path from any } (x, 0), x < 0, \text{ to any point of } \mathcal{W}(\alpha_l, \alpha_r, M)\}.$$

Since the process of 2’s is a contact process with parameter λ_2 , and $\alpha(\lambda_2) < \alpha_l$, it follows from (4) that Ω_1 has positive probability.

For the second event, choose $x_0 \in \mathbb{Z}$ and $t_0 > 0$ such that $x_0 = \alpha_l t_0$ and $(x, t_0) \subset \mathcal{W}$ for all $x \in [x_0, x_0 + M] \cap \mathbb{Z}$. Since $M > 2$ the event,

$$\Omega_2 = \{\text{there is an active path in } \mathcal{W} \text{ from } (0, 0) \text{ to each of } (x, t_0), x \in [x_0, x_0 + M] \cap \mathbb{Z}\}$$

has positive probability.

For the third event, define, for $t \geq t_0$,

$$A_t = \{y : \text{there is an infinite active path in } \mathcal{W} \text{ from } (x, t_0) \text{ to } (y, t) \\ \text{for some } x \in [x_0, x_0 + M] \cap \mathbb{Z}\}$$

and put $\Omega_3 = \{|A_t| \rightarrow \infty \text{ as } t \rightarrow \infty\}$. It follows from Theorems 1 and 2 that Ω_3 has positive probability.

The events Ω_i are independent since they are defined in terms of our Poisson processes over disjoint space-time regions. Furthermore, it is easy to see from Remark 3 that $\|\zeta_t\|_1 \rightarrow \infty$ on their intersection, so we are done.

References

- [1] Alarcon, T., Byrne, H.M. and Maini, P.K. (2003) A cellular automaton for tumour growth in inhomogeneous environment, *J.Theor. Biol.* **225**, 257-274.
- [2] Campanino, M. and Klein, A. (1991) Decay of two-point functions for $(d + 1)$ -dimensional percolation, Ising and Potts models with d -dimensional disorder, *Comm. Math. Phys.* **135**, 489-497.
- [3] Chayes, J.T. and Chayes, L. (1986) Critical points and intermediate phases on wedges of \mathbb{Z}^d , *Journal of Physics A: Mathematical and General* **19**, 3033-3048.
- [4] Durrett, R. (1980) On the growth of one-dimensional contact process, *Ann. Probab.* **8**, 890-907.
- [5] Durrett, R. and Swindle, G. (1991) Are there bushes in a forest? *Stoch. Proc. Appl.* **37**, 19-31.
- [6] Grimmett, G. (1983) Bond percolation on subsets of the square lattice, and the transition between one-dimensional and two-dimensional behavior, *Journal of Physics A: Mathematical and General* **16**, 599-604.
- [7] Harris, T.E. (1974) Contact interactions on a lattice, *Ann. Probab.* **2**, 969-988.
- [8] Liggett, T.M. (1985) *Interacting Particle Systems*, Springer-Verlag, Berlin.
- [9] Liggett, T.M. (1999) *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Springer-Verlag, Berlin.
- [10] Madras, N., Schinazi, R. and Schonmann, R. (1994) On the critical behavior of the contact process in deterministic inhomogeneous environments, *Ann. Prob.* **22**, 1140-1159.
- [11] McCoy, B. and Wu, T.T. (1968) Theory of the two-dimensional Ising model with random impurities. I. Thermodynamics, *Phys. Rev. B* **76**, 631-643.

J. Theodore Cox
Department of Mathematics, Syracuse University Syracuse, NY 13244
e-mail: jtcox@syr.edu

Nevena Marić
Department of Mathematics and CS, University of Missouri- St. Louis St. Louis, MO
63121
e-mail: maricn@umsl.edu

Rinaldo Schinazi
Department of Mathematics, University of Colorado-Colorado Springs Colorado Springs,
CO 80933
e-mail: rschinaz@uccs.edu