8

PART 2 EXTENDED OBJECTS, MATTER, AND CIRCULAR MOTION

Systems of Particles and Extended Objects

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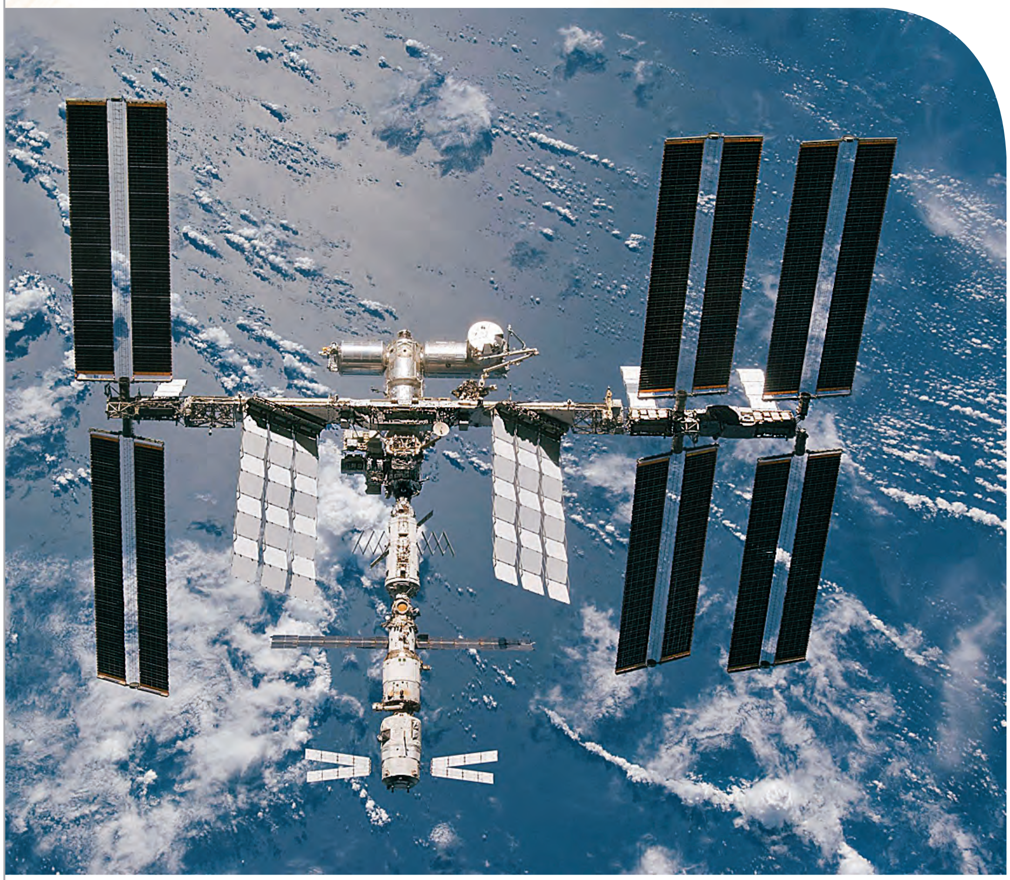


FIGURE 8.1 The International Space Station photographed from the Space Shuttle *Discovery*.

WHAT WE WILL LEARN

- The center of mass is the point at which we can imagine all the mass of an object to be concentrated.
- The position of the combined center of mass of two or more objects is found by taking the sum of their position vectors, weighted by their individual masses.
- The translational motion of the center of mass of an extended object can be described by Newtonian mechanics.
- The center-of-mass momentum is the sum of the linear momentum vectors of the parts of a system. Its time derivative is equal to the total net external force acting on the system, an extended formulation of Newton's Second Law.
- For systems of two particles, working in terms of center-of-mass momentum and relative momentum instead of the individual momentum vectors gives deeper insight into the physics of collisions and recoil phenomena.
- Analyses of rocket motion have to consider systems of varying mass. This variation leads to a logarithmic dependence of the velocity of the rocket on the ratio of initial to final mass.
- It is possible to calculate the location of the center of mass of an extended object by integrating its mass density over its entire volume, weighted by the coordinate vector, and then dividing by the total mass.
- If an object has a plane of symmetry, the center of mass lies in that plane. If the object has more than one symmetry plane, the center of mass lies on the line or point of intersection of the planes.

The International Space Station (ISS), shown in Figure 8.1, is a remarkable engineering achievement. It is scheduled to be completed in 2011, though it has been continuously inhabited since 2000. It orbits Earth at a speed of over 7.5 km/s, in an orbit ranging from 320 to 350 km above Earth's surface. When engineers track the ISS, they treat it as a point particle, even though it measures roughly 109 m by 73 m by 25 m. Presumably this point represents the center of the ISS, but how exactly do engineers determine where the center is?

Every object has a point where all the mass of the object can be considered to be concentrated. Sometimes this point, called the *center of mass,* is not even within the object. This chapter explains how to calculate the location of the center of mass and shows how to use it to simplify calculations involving collisions and conservation of momentum. We have been assuming in earlier chapters that objects could be treated as particles. This chapter shows why that assumption works.

This chapter also discusses changes in momentum for the situation where an object's mass varies as well as its velocity. This occurs with rocket propulsion, where the mass of fuel is often much greater than the mass of the rocket itself.

8.1 Center of Mass and Center of Gravity

So far, we have represented the location of an object by coordinates of a single point. However, a statement such as "a car is located at $x = 3.2$ m" surely does not mean that the entire car is located at that point. So, what does it mean to give the coordinate of one particular point to represent an extended object? Answers to this question depend on the particular application. In auto racing, for example, a car's location is represented by the coordinate of the frontmost part of the car. When this point crosses the finish line, the race is decided. On the other hand, in soccer, a goal is only counted if the entire ball has crossed the goal line; in this case, it makes sense to represent the soccer ball's location by the coordinates of the rearmost part of the ball. However, these examples are exceptions. In almost all situations, there is a natural choice of a point to represent the location of an extended object. This point is called the *center of mass*.

Definition

The **center of mass** is the point at which we can imagine all the mass of an object to be concentrated.

Thus, the center of mass is also the point at which we can imagine the force of gravity acting on the entire object to be concentrated. If we can imagine all of the mass to be concentrated at this point when calculating the force due to gravity, it is legitimate to call this point the *center of gravity,* a term that can often be used interchangeably with *center of mass*. (To be precise, we should note that these two terms are only equivalent in situations where the gravitational force is constant everywhere throughout the object. In Chapter 12, we will see that this is not the case for very large objects.)

It is appropriate to mention here that if an object's mass density is constant, the center of mass (center of gravity) is located in the geometrical center of the object. Thus, for most objects in everyday experience, it is a reasonable first guess that the center of gravity is the middle of the object. The derivations in this chapter will bear out this conjecture.

Combined Center of Mass for Two Objects

If we have two identical objects of equal mass and want to find the center of mass for the combination of the two, it is reasonable to assume from considerations of symmetry that the combined center of mass of this system lies exactly midway between the individual centers of mass of the two objects. If one of the two objects is more massive, then it is equally reasonable to assume that the center of mass for the combination is closer to that of the more massive one. Thus, we have a general formula for calculating the location of the center of mass, $\vec{R}$, for two masses m_1 and m_2 located at positions $\vec{r}_1$ and $\vec{r}_2$ to an arbitrary coordinate system (Figure 8.2):

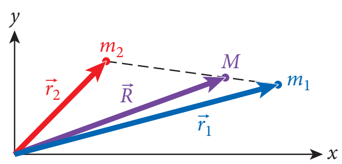


FIGURE 8.2 Location of the center of mass for a system of two masses m_1 and m_2, where $M = m_1 + m_2$.

$$\vec{R} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}. \qquad (8.1)$$

This equation says that the center-of-mass position vector is an average of the position vectors of the individual objects, weighted by their mass. Such a definition is consistent with the empirical evidence we have just cited. For now, we will use this equation as an operating definition and gradually work out its consequences. Later in this chapter and in the following chapters, we will see additional reasons why this definition makes sense.

Note that we can immediately write vector equation 8.1 in Cartesian coordinates as follows:

$$X = \frac{x_1 m_1 + x_2 m_2}{m_1 + m_2}, \quad Y = \frac{y_1 m_1 + y_2 m_2}{m_1 + m_2}, \quad Z = \frac{z_1 m_1 + z_2 m_2}{m_1 + m_2}. \qquad (8.2)$$

8.1 In-Class Exercise

In the case shown in Figure 8.2, what are the relative magnitudes of the two masses m_1 and m_2?

a) $m_1 < m_2$

b) $m_1 > m_2$

c) $m_1 = m_2$

d) Based solely on the information given in the figure, it is not possible to decide which of the two masses is larger.

In Figure 8.2, the location of the center of mass lies exactly on the straight (dashed black) line that connects the two masses. Is this a general result—does the center of mass always lie on this line? If yes, why? If no, what is the special condition that is needed for this to be the case? The answer is that this is a general result for all two-body systems: The center of mass of such a system always lies on the connecting line between the two objects. To see this, we can place the origin of the coordinate system at one of the two masses in Figure 8.2, say m_1. (As we know, we can always shift the origin of a coordinate system without changing the physics results.) Using equation 8.1, we then see that $\vec{R} = \vec{r}_2 m_2 / (m_1 + m_2)$, because with this choice of coordinate system, we define $\vec{r}_1$ as zero. Thus, the two vectors $\vec{R}$ and $\vec{r}_2$ point in the same direction, but $\vec{R}$ is shorter by a factor of $m_2 / (m_1 + m_2) < 1$. This shows that $\vec{R}$ always lies on the straight line that connects the two masses.

SOLVED PROBLEM 8.1 Center of Mass of Earth and Moon

The Earth has a mass of $5.97 \cdot 10^{24}$ kg, and the Moon has a mass of $7.36 \cdot 10^{22}$ kg. The Moon orbits the Earth at a distance of 384,000 km; that is, the center of the Moon is a distance of 384,000 km from the center of Earth, as shown in Figure 8.3a.

PROBLEM

How far from the center of the Earth is the center of mass of the Earth-Moon system?

SOLUTION

THINK

The center of mass of the Earth-Moon system can be calculated by taking the center of the Earth to be located at $x = 0$ and the center of the Moon to be located at $x = 384{,}000$ km. The center of mass of the Earth-Moon system will lie along a line connecting the center of the Earth and the center of the Moon (as in Figure 8.3a).

SKETCH

A sketch showing Earth and Moon to scale is presented in Figure 8.3b.

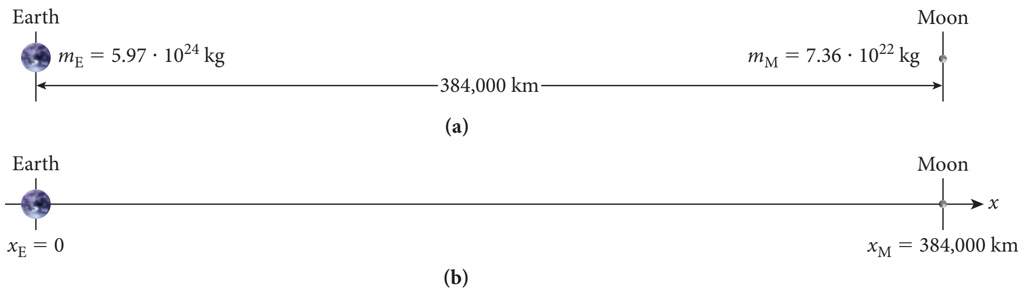


FIGURE 8.3 (a) The Moon orbits the Earth at a distance of 384,000 km (drawing to scale). (b) A sketch showing the Earth at $x_E = 0$ and the Moon at $x_M = 384{,}000$ km.

RESEARCH

We define an x-axis and place the Earth at $x_E = 0$ and the Moon at $x_M = 384{,}000$ km. We can use equation 8.2 to obtain an expression for the x-coordinate of the center of mass of the Earth-Moon system:

$$X = \frac{x_E m_E + x_M m_M}{m_E + m_M}.$$

SIMPLIFY

Since we have put the origin of our coordinate system at the center of Earth, we set $x_E = 0$. This results in

$$X = \frac{x_M m_M}{m_E + m_M}.$$

CALCULATE

Inserting the numerical values, we get the x-coordinate of the center of mass of the Earth-Moon system:

$$X = \frac{x_M m_M}{m_E + m_M} = \frac{(384{,}000 \text{ km})(7.36 \cdot 10^{22} \text{ kg})}{5.97 \cdot 10^{24} \text{ kg} + 7.36 \cdot 10^{22} \text{ kg}} = 4676.418 \text{ km}.$$

ROUND

All of the numerical values were given to three significant figures, so we report our result as

$$X = 4680 \text{ km}.$$

DOUBLE-CHECK

Our result is in kilometers, which is the correct unit for a position. The center of mass of the Earth-Moon system is close to the center of the Earth. This distance is small compared to the distance between the Earth and the Moon, which makes sense because the mass of the Earth is much larger than the mass of the Moon. In fact, this distance is less than the radius of the Earth, $R_E = 6370$ km. The Earth and the Moon actually each orbit the common center of mass. Thus, the Earth seems to wobble as the Moon orbits it.

Combined Center of Mass for Several Objects

The definition of the center of mass in equation 8.1 can be generalized to a total of n objects with different masses, m_i, located at different positions, $\vec{r}_i$. In this general case,

$$\vec{R} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots + \vec{r}_n m_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} \vec{r}_i m_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i, \tag{8.3}$$

where M represents the combined mass of all n objects:

$$M = \sum_{i=1}^{n} m_i. \tag{8.4}$$

Writing equation 8.3 in Cartesian components, we obtain

$$X = \frac{1}{M}\sum_{i=1}^{n} x_i m_i; \quad Y = \frac{1}{M}\sum_{i=1}^{n} y_i m_i; \quad Z = \frac{1}{M}\sum_{i=1}^{n} z_i m_i. \tag{8.5}$$

The location of the center of mass is a fixed point relative to the object or system of objects and does not depend on the location of the coordinate system used to describe it. We can show this by taking the system of equation 8.3 and moving it by $\vec{r}_0$, resulting in a new center-of-mass position, $\vec{R} + \vec{R}_0$. Using equation 8.3, we find

$$\vec{R} + \vec{R}_0 = \frac{\sum_{i=1}^{n} (\vec{r}_0 + \vec{r}_i) m_i}{\sum_{i=1}^{n} m_i} = \vec{r}_0 + \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i.$$

Thus, $\vec{R}_0 = \vec{r}_0$, and the location of the center of mass does not change relative to the system.

Now we can determine the center of mass of a collection of objects in the following example.

EXAMPLE 8.1 Shipping Containers

Large freight containers, which can be transported by truck, railroad, or ship, come in standard sizes. One of the most common sizes is the ISO 20' container, which has a length of 6.1 m, a width of 2.4 m, and a height of 2.6 m. This container is allowed to have a mass (including its contents, of course) of up to 30,400 kg.

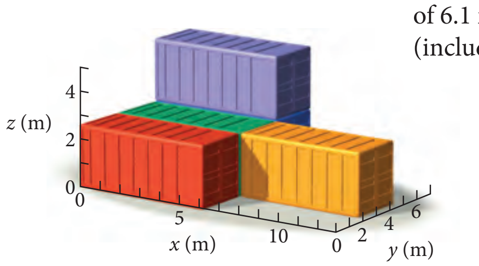


FIGURE 8.4 Freight containers arranged on the deck of a container ship.

PROBLEM

The four freight containers shown in Figure 8.4 sit on the deck of a container ship. Each one has a mass of 9,000 kg, except for the red one, which has a mass of 18,000 kg. Assume that each of the containers has an individual center of mass at its geometric center. What are the x-coordinate and the y-coordinate of the containers' combined center of mass? Use the coordinate system shown in the figure to describe the location of this center of mass.

SOLUTION

We need to calculate the individual Cartesian components of the center of mass, so we'll use equation 8.5. There does not seem to be a shortcut we can utilize.

Let's call the length of each container ℓ (6.1 m), the width of each container w (2.4 m), and the mass of the green container m_0 (9,000 kg). The mass of the red container is then $2m_0$, and all the others also have a mass of m_0.

First, we need to calculate the combined mass, M. According to equation 8.4, it is

$$\begin{aligned} M &= m_{\text{red}} + m_{\text{green}} + m_{\text{orange}} + m_{\text{blue}} + m_{\text{purple}} \\ &= 2m_0 + m_0 + m_0 + m_0 + m_0 \\ &= 6m_0. \end{aligned}$$

For the x-coordinate of the combined center of mass, we find

$$\begin{aligned} X &= \frac{x_{\text{red}}m_{\text{red}} + x_{\text{green}}m_{\text{green}} + x_{\text{orange}}m_{\text{orange}} + x_{\text{blue}}m_{\text{blue}} + x_{\text{purple}}m_{\text{purple}}}{M} \\ &= \frac{\frac{1}{2}\ell 2m_0 + \frac{1}{2}\ell m_0 + \frac{3}{2}\ell m_0 + \frac{1}{2}\ell m_0 + \frac{1}{2}\ell m_0}{6m_0} \\ &= \frac{\ell(1 + \frac{1}{2} + \frac{3}{2} + \frac{1}{2} + \frac{1}{2})}{6} \\ &= \tfrac{2}{3}\ell = 4.1 \text{ m}. \end{aligned}$$

In the last step, we substituted the value of 6.1 m for ℓ.

In the same way, we can calculate the y-coordinate:

$$\begin{aligned} Y &= \frac{y_{\text{red}}m_{\text{red}} + y_{\text{green}}m_{\text{green}} + y_{\text{orange}}m_{\text{orange}} + y_{\text{blue}}m_{\text{blue}} + y_{\text{purple}}m_{\text{purple}}}{M} \\ &= \frac{\frac{1}{2}w 2m_0 + \frac{3}{2}wm_0 + \frac{3}{2}wm_0 + \frac{5}{2}wm_0 + \frac{5}{2}wm_0}{6m_0} \\ &= \frac{w(1 + \frac{3}{2} + \frac{3}{2} + \frac{5}{2} + \frac{5}{2})}{6} \\ &= \tfrac{3}{2}w = 3.6 \text{ m}. \end{aligned}$$

Here again we substituted the numerical value of 2.4 m in the last step. (Note that we rounded both center-of-mass coordinates to two significant figures to be consistent with the given values.)

8.1 Self-Test Opportunity

Determine the z-coordinate of the center of mass of the container arrangement in Figure 8.4.

8.2 Center-of-Mass Momentum

Now we can take the time derivative of the position vector of the center of mass to get $\vec{V}$, the velocity vector of the center of mass. We take the time derivative of equation 8.3:

$$\vec{V} \equiv \frac{d}{dt}\vec{R} = \frac{d}{dt}\left(\frac{1}{M}\sum_{i=1}^{n}\vec{r}_i m_i\right) = \frac{1}{M}\sum_{i=1}^{n} m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{v}_i = \frac{1}{M}\sum_{i=1}^{n}\vec{p}_i. \tag{8.6}$$

For now, we have assumed that the total mass, M, and the masses, m_i, of the individual objects remain constant. (Later in this chapter, we will give up this assumption and study the consequences for rocket motion.) Equation 8.6 is an expression for the velocity vector of the center of mass, $\vec{V}$. Multiplication of both sides of equation 8.6 by M yields

$$\vec{P} = M\vec{V} = \sum_{i=1}^{n}\vec{p}_i. \tag{8.7}$$

We thus find that the center-of-mass momentum, $\vec{P}$, is the product of the total mass, M, and the center-of-mass velocity, $\vec{V}$, and is the sum of all the individual momentum vectors.

Taking the time derivative of both sides of equation 8.7 yields Newton's Second Law for the center of mass:

$$\frac{d}{dt}\vec{P} = \frac{d}{dt}(M\vec{V}) = \frac{d}{dt}\left(\sum_{i=1}^{n}\vec{p}_i\right) = \sum_{i=1}^{n}\frac{d}{dt}\vec{p}_i = \sum_{i=1}^{n}\vec{F}_i. \tag{8.8}$$

In the last step, we used the result from Chapter 7 that the time derivative of the momentum of particle i is equal to the net force, $\vec{F}_i$, acting on it. Note that if the particles (objects) in a system exert forces on one another, those forces do not make a net contribution to the sum of these forces in equation 8.8. Why? According to Newton's Third Law, the forces that two objects exert on each other are equal in magnitude and opposite in direction. Therefore, adding them yields zero. Thus, we obtain Newton's Second Law for the center of mass:

$$\frac{d}{dt}\vec{P} = \vec{F}_{\text{net}}, \tag{8.9}$$

where $\vec{F}_{\text{net}}$ is the sum of all *external* forces acting on the system of particles.

The center of mass has the same relationships among position, velocity, momentum, force, and mass that have been established for point particles. It is thus possible to consider the center of mass of an extended object or a group of objects as a point particle. This conclusion justifies the approximation we used in earlier chapters that objects can be represented as points.

Two-Body Collisions

One of the most interesting applications of center of mass arises with a frame of reference whose origin is placed at the center of mass of a system of interacting objects. Let's investigate the simplest example of this situation. Consider a system consisting of only two objects. In this case, the total momentum—the sum of individual momenta according to equation 8.7—is

$$\vec{P} = \vec{p}_1 + \vec{p}_2. \tag{8.10}$$

In Chapter 7, we saw that the relative velocity between two colliding objects plays a big role in two-body collisions. Thus, it is natural to define the relative momentum as half of the momentum difference:

$$\vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2). \tag{8.11}$$

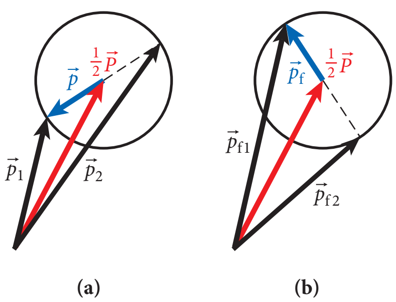


FIGURE 8.5 Relationship between momentum vectors 1 and 2 (black), center-of-mass momentum (red), and relative momentum (blue) in some reference frame: (a) before an elastic collision; (b) after the elastic collision.

Why is the factor $\frac{1}{2}$ appropriate in this definition? The answer is that in a center-of-momentum reference frame—a frame in which the center of mass has zero momentum—the momentum of object 1 is $\vec{p}$ and that of object 2 is $-\vec{p}$. Let's see how this comes about.

Figure 8.5a illustrates the relationship between the center-of-mass momentum $\vec{P}$ (red arrow), the relative momentum $\vec{p}$ (blue arrow), and the momenta of objects 1 and 2 (black arrows). We can express the individual momenta in terms of the center-of-mass momentum and the relative momentum:

$$\begin{aligned} \vec{p}_1 &= \tfrac{1}{2}\vec{P} + \vec{p} \\ \vec{p}_2 &= \tfrac{1}{2}\vec{P} - \vec{p}. \end{aligned} \tag{8.12}$$

The biggest advantage of thinking in terms of center-of-mass momentum and relative momentum becomes clearer when we consider a collision between the two objects. During a collision, the dominant forces that act on the objects are the forces they exert on each other. These internal forces do not enter into the sum of forces in equation 8.8, so we obtain, for the collision of two objects:

$$\frac{d}{dt}\vec{P} = 0.$$

In other words, the center-of-mass momentum does not change; it remains the same during a two-body collision. This is true for elastic or totally inelastic or partially inelastic collisions.

For an inelastic collision, where the two objects stick together after colliding, we found in Chapter 7 that the velocity with which the combined mass moves is

$$\vec{v}_{\text{f}} = \frac{m_1\vec{v}_{\text{i}1} + m_2\vec{v}_{\text{i}2}}{m_1 + m_2}.$$

If we compare this equation with equation 8.6, we see that this velocity is just the center-of-mass velocity. In other words, in the case of a totally inelastic collision, the relative momentum after the collision is zero.

For elastic collisions, the total kinetic energy has to be conserved. If we compute the total kinetic energy in terms of the total momentum, $\vec{P}$, and the relative momentum, $\vec{p}$, the contribution from the total momentum has to remain constant, because $\vec{P}$ is constant. This finding implies that the kinetic energy contained in the relative motion also has to remain constant. Because this kinetic energy, in turn, is proportional to the square of the relative momentum vector, the length of the relative momentum vector has to remain unchanged during an elastic collision. Only the direction of this vector can change. As shown in Figure 8.5b, the new relative momentum vector after the elastic collision lies on the circumference of a circle, whose radius is equal to the length of the initial relative momentum vector and whose center is at the end point of the $\frac{1}{2}\vec{p}$. The situation depicted in Figure 8.5 implies that the motion is restricted to two spatial dimensions. For two-body collisions in three dimensions, the final relative momentum vector is located on the surface of a sphere instead of the perimeter of a circle.

In Figure 8.5, the momentum vectors for particle 1 and particle 2 are plotted in some arbitrary reference frame before and after the collision between the two particles. We can plot the same vectors in a frame that moves with the center of mass. (We have just shown that the center-of-mass momentum does not change in the collision!) The center-of-mass velocity in such a moving frame is zero, and, consequently, $\vec{P}=0$ in that frame. From equation 8.12, we can then see that in this case the initial momentum vectors of the two particles are $\vec{p}_1=+\vec{p}$ and $\vec{p}_2=-\vec{p}$. In the moving center-of-mass frame, the two-particle collision simply results in a rotation of the relative momentum vector about the origin, as shown in Figure 8.6, which automatically ensures that the conservation laws of momentum and kinetic energy (because this is an elastic collision!) are obeyed.

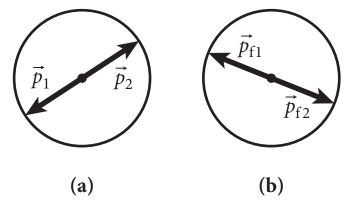


FIGURE 8.6 Same collision as in Figure 8.5 but displayed in the center-of-mass frame.

Recoil

When a bullet is fired from a gun, the gun **recoils;** that is, it moves in the direction opposite to that in which the bullet is fired. Another demonstration of the same physical principle occurs if you are sitting in a boat that is at rest and you throw an object off the boat: The boat moves in the direction opposite from that of the object. You also experience the same effect if you stand on a skateboard and toss a (reasonably heavy) ball. This well-known recoil effect can be understood using the framework we have just developed for two-body collisions. It is also a consequence of Newton's Third Law.

SOLVED PROBLEM 8.2 Cannon Recoil

Suppose a cannonball of mass 13.7 kg is fired at a target that is 2.30 km away from the cannon, which has a mass of 249.0 kg. The distance 2.30 km is also the maximum range of the cannon. The target and cannon are at the same elevation, and the cannon is resting on a horizontal surface.

PROBLEM

What is the velocity with which the cannon will recoil?

SOLUTION

THINK

First, we realize that the cannon can recoil only in the horizontal direction, because the normal force exerted by the ground will prevent it from acquiring a downward velocity component. We use the fact that the x-component of the center-of-mass momentum of the system (cannon and cannonball) remains unchanged in the process of firing the cannon, because the explosion of the gunpowder inside the cannon, which sets the cannonball in motion, creates only forces internal to the system. No net external force component occurs in the horizontal direction because the two external forces (normal force and gravity) are both vertical. The y-component of the center-of-mass velocity changes because a net external force component does occur in the y-direction when the normal force increases to prevent the cannon from penetrating the ground. Because the cannonball and cannon are both initially at rest, the center-of-mass momentum of this system is initially zero, and its x-component remains zero after the firing of the cannon.

Continued—

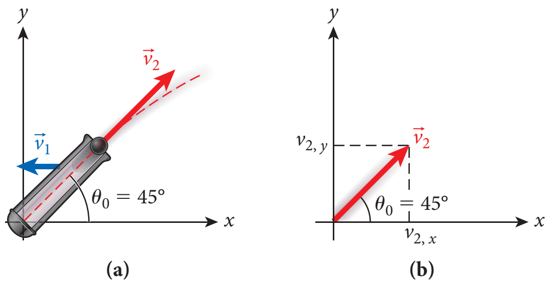


FIGURE 8.7 (a) Cannonball being fired from a cannon. (b) The initial velocity vector of the cannonball.

SKETCH

Figure 8.7a is a sketch of the cannon just as the cannonball is fired. Figure 8.7b shows the velocity vector of the cannonball, $\vec{v}_2$, including the x- and y-components.

RESEARCH

Using equation 8.10 with index 1 for the cannon and index 2 for the cannonball, we obtain

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = m_1\vec{v}_1 + m_2\vec{v}_2 = 0 \Rightarrow \vec{v}_1 = -\frac{m_2}{m_1}\vec{v}_2.$$

For the horizontal component of the velocity, we then have

$$v_{1,x} = -\frac{m_2}{m_1}v_{2,x}. \quad \text{(i)}$$

We can obtain the horizontal component of the cannonball's initial velocity (at firing) from the fact that the range of the cannon is 2.30 km. In Chapter 3, we saw that the range of the cannon is related to the initial velocity via $R = (v_0^2/g)(\sin 2\theta_0)$. The maximum range is reached for $\theta_0 = 45°$ and is $R = v_0^2/g \Rightarrow v_0 = \sqrt{gR}$. For $\theta_0 = 45°$, the initial speed and horizontal velocity component are related via $v_{2,x} = v_0 \cos 45° = v_0/\sqrt{2}$. Combining these two results, we can relate the maximum range to the horizontal component of the initial velocity of the cannonball:

$$v_{2,x} = \frac{v_0}{\sqrt{2}} = \sqrt{\frac{gR}{2}}. \quad \text{(ii)}$$

SIMPLIFY

Substituting from equation (ii) into equation (i) gives us the result we are looking for:

$$v_{1,x} = -\frac{m_2}{m_1}v_{2,x} = -\frac{m_2}{m_1}\sqrt{\frac{gR}{2}}.$$

CALCULATE

Inserting the numbers given in the problem statement, we obtain

$$v_{1,x} = -\frac{m_2}{m_1}\sqrt{\frac{gR}{2}} = -\frac{13.7 \text{ kg}}{249 \text{ kg}}\sqrt{\frac{(9.81 \text{ m/s}^2)(2.30 \cdot 10^3 \text{ m})}{2}} = -5.84392 \text{ m/s}.$$

ROUND

Expressing our answer to three significant figures gives

$$v_{1,x} = -5.84 \text{ m/s}.$$

DOUBLE-CHECK

The minus sign means that the cannon moves in the opposite direction to that of the cannonball, which is reasonable. The cannonball should have a much larger initial velocity than the cannon because the cannon is much more massive. The initial velocity of the cannonball was

$$v_0 = \sqrt{gR} = \sqrt{\left(9.81 \text{ m/s}^2\right)\left(2.3 \cdot 10^3 \text{ m}\right)} = 150 \text{ m/s}.$$

The fact that our answer for the velocity of the cannon is much less than the initial velocity of the cannonball also seems reasonable.

Mass can be ejected continuously from a system, producing a continuous recoil. As an example, let's consider the spraying of water from a fire hose.

EXAMPLE 8.2 Fire Hose

PROBLEM

What is the magnitude of the force, F, that acts on a firefighter holding a fire hose that ejects 360 L of water per minute with a muzzle speed of $v = 39.0$ m/s, as shown in Figure 8.8?

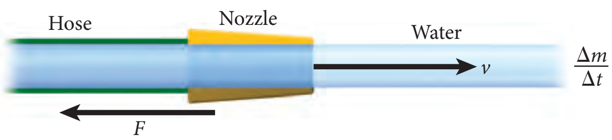


FIGURE 8.8 A fire hose with water leaving at speed v.

SOLUTION

Let's first find the total mass of the water that is being ejected per minute. The mass density of water is $\rho = 1000\ \text{kg/m}^3 = 1.0$ kg/L. Because $\Delta V = 360$ L, we get for the total mass of water ejected in a minute:

$$\Delta m = \Delta V \rho = (360\ \text{L})(1.0\ \text{kg/L}) = 360\ \text{kg}.$$

The momentum of the water is then $\Delta p = v\Delta m$, and, from the definition of the average force, $F = \Delta p/\Delta t$, we have:

$$F = \frac{v\Delta m}{\Delta t} = \frac{(39.0\ \text{m/s})(360\ \text{kg})}{60\ \text{s}} = 234\ \text{N}.$$

This force is sizable, which is why it is so dangerous for firefighters to let go of operating fire hoses: They would whip around, potentially causing injury.

8.2 In-Class Exercise

A garden hose is used to fill a 20-L bucket in 1 min. The velocity of the water leaving the hose is 1.05 m/s. What force is required to hold the hose in place?

a) 0.35 N

b) 2.1 N

c) 9.8 N

d) 12 N

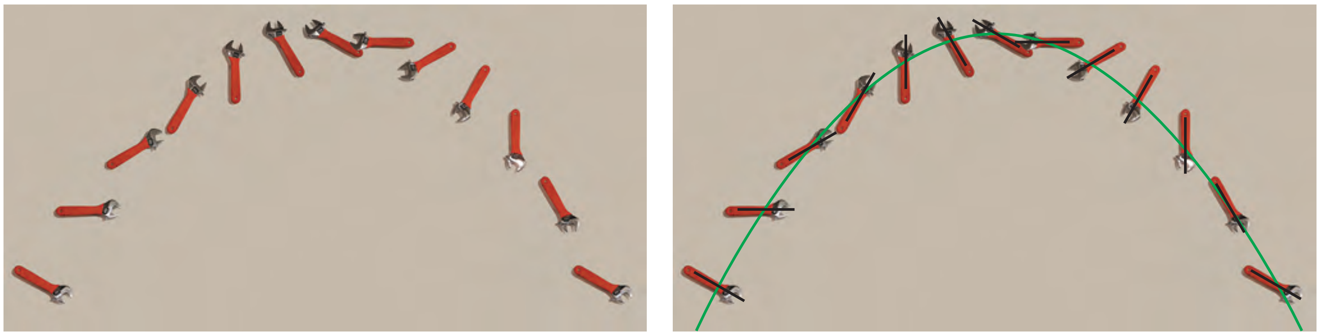
e) 21 N

General Motion of the Center of Mass

Extended solid objects can have motions that appear, at first sight, rather complicated. One example of such motion is high jumping. During the 1968 Olympic Games in Mexico City, the American track-and-field star Dick Fosbury won a gold medal using a new high-jump technique, which became known as the Fosbury flop (see Figure 8.9). Properly executed, the technique allows the athlete to cross over the bar while his or her center of mass remains below it, thus adding effective height to the jump.

Figure 8.10a shows a wrench twirling through the air, in a multiple-exposure series of images with equal time intervals between sequential frames. While this motion looks complicated, we can use what we know about the center of mass to perform a straightforward analysis of this motion. If we assume that all the mass of the wrench is concentrated at a point, then this point will move on a parabola through the air under the influence of gravity, as discussed in Chapter 3. Superimposed on this motion is a rotation of the wrench

FIGURE 8.9 Dick Fosbury clears the high-jump bar during the finals of the Olympic Games in Mexico City on October 20, 1968.



(a) (b)

FIGURE 8.10 (a) Digitally processed multiple-exposure series of images of a wrench tossed through the air. (b) Same series as in part (a), but with a parabola for the center-of-mass motion superimposed.

about its center of mass. You can see this parabolic trajectory clearly in Figure 8.10b, where a superimposed parabola (green) passes through the location of the center of mass of the wrench in each exposure. In addition, a superimposed black line rotates with a constant rate about the center of mass of the wrench. You can clearly see that the handle of the wrench is always aligned with the black line, indicating that the wrench rotates with constant rate about its center of mass (we will analyze such rotational motion in Chapter 10).

The techniques introduced here allow us to analyze many kinds of complicated problems involving moving solid objects in terms of superposition of the motion of the center of mass and a rotation of the object about the center of mass.

8.3 Rocket Motion

FIGURE 8.11 A Delta II rocket lifting a GPS satellite into orbit.

Example 8.2 about the fire hose is the first situation we have examined that involves a change in momentum due to a change in mass rather than in velocity. Another important situation in which changing momentum is due to changing mass is rocket motion, where part of the mass of the rocket is ejected through a nozzle or nozzles at the rear (Figure 8.11). Rocket motion is an important case of the recoil effect discussed in Section 8.2. A rocket does not "push against" anything. Instead, its forward thrust is gained from ejecting its propellant from its rear, according to the law of conservation of total momentum.

In order to obtain an expression for the acceleration of a rocket, we'll first consider ejecting discrete amounts of mass out of the rocket. Then we can approach the continuum limit. Let's use a toy model of a rocket that moves in interstellar space, propelling itself forward by shooting cannonballs out its back end (Figure 8.12). (We specify that the rocket is in interstellar space so we can treat it and its components as an isolated system, for which we can neglect outside forces.) Initially, the rocket is at rest. All motion is in the x-direction, so we can use notation for one-dimensional motion, with the signs of the x-components of the velocities (which, for simplicity, we will refer to as velocities) indicating their direction. Each cannonball has a mass of Δm, and the initial mass of the rocket, including all cannonballs, is m_0. Each cannonball is fired with a velocity of v_c relative to the rocket, resulting in a cannonball momentum of $v_c\Delta m$.

After the first cannonball is fired, the mass of the rocket is reduced to $m_0 - \Delta m$. Firing the cannonball does not change the center-of-mass momentum of the system (rocket plus cannonball). (Remember, this is an isolated system, on which no net external forces act.) Thus, the rocket receives a recoil momentum opposite to that of the cannonball. The momentum of the cannonball is

$$p_c = v_c\Delta m,$$

and the momentum of the rocket is

$$p_r = (m_0 - \Delta m)v_1,$$

where v_1 is the velocity of the rocket after the cannonball is fired. Because momentum is conserved, we can write $p_r + p_c = 0$, and then substitute p_r and p_c from the preceding two expressions:

$$(m_0 - \Delta m)v_1 + v_c\Delta m = 0.$$

We define the change in the velocity, Δv_1, of the rocket after firing one cannonball as

$$v_1 = v_0 + \Delta v = 0 + \Delta v = \Delta v_1,$$

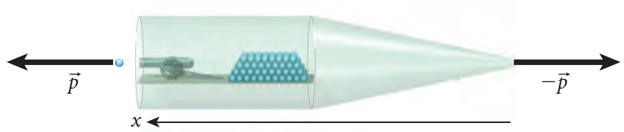


FIGURE 8.12 Toy model for rocket propulsion: firing cannonballs.

where the assumption that the rocket was initially at rest means $v_0 = 0$. This gives us the recoil velocity of the rocket due to the firing of one cannonball:

$$\Delta v_1 = -\frac{v_c \Delta m}{m_0 - \Delta m}.$$

In the moving system of the rocket, we can then fire the second cannonball. Firing the second cannonball reduces the mass of the rocket from $m_0 - \Delta m$ to $m_0 - 2\Delta m$, which results in an additional recoil velocity of

$$\Delta v_2 = -\frac{v_c \Delta m}{m_0 - 2\Delta m}.$$

The total velocity of the rocket then increases to $v_2 = v_1 + \Delta v_2$. After firing the nth cannonball, the velocity change is:

$$\Delta v_n = -\frac{v_c \Delta m}{m_0 - n\Delta m}. \tag{8.13}$$

Thus, the velocity of the rocket after firing the nth cannonball is:

$$v_n = v_{n-1} + \Delta v_n.$$

This kind of equation, which defined the nth term of a sequence where each term is expressed as a function of the preceding terms, is called a *recursion relation.* It can be solved in a straightforward manner by using a computer. However, we can use a very helpful approximation for the case where the mass emitted per unit time is constant and small compared to m, the overall (time-dependent) mass of the rocket. In this limit, we obtain from equation 8.13

$$\Delta v = -\frac{v_c \Delta m}{m} \Rightarrow \frac{\Delta v}{\Delta m} = -\frac{v_c}{m}. \tag{8.14}$$

Here v_c is the velocity with which the cannonball is ejected. In the limit $\Delta m \rightarrow 0$, we then obtain the derivative

$$\frac{dv}{dm} = -\frac{v_c}{m}. \tag{8.15}$$

The solution of this differential equation is

$$v(m) = -v_c \int_{m_0}^{m} \frac{1}{m'} dm' = -v_c \ln m \Big|_{m_0}^{m} = v_c \ln\left(\frac{m_0}{m}\right). \tag{8.16}$$

(You can verify that equation 8.16 is indeed the solution of equation 8.15 by taking the derivative of equation 8.16 with respect to m.)

If m_i is the initial value for the total mass at some time t_i and m_f is the final mass at a later time, we can use equation 8.16 to obtain $v_i = v_c \ln(m_0/m_i)$ and $v_f = v_c \ln(m_0/m_f)$ for the initial and final velocities of the rocket. Then, using the property of logarithms, $\ln(a/b) = \ln a - \ln b$, we find the difference in those two velocities:

$$v_f - v_i = v_c \ln\left(\frac{m_0}{m_f}\right) - v_c \ln\left(\frac{m_0}{m_i}\right) = v_c \ln\left(\frac{m_i}{m_f}\right). \tag{8.17}$$

EXAMPLE 8.3 Rocket Launch to Mars

One proposed scheme for sending astronauts to Mars involves assembling a spaceship in orbit around Earth, thus avoiding the need for the spaceship to overcome most of Earth's gravity at the start. Suppose such a spaceship has a payload of 50,000 kg, carries 2,000,000 kg of fuel, and is able to eject the propellant with a velocity of 23.5 km/s. (Current chemical rocket propellants yield a maximum velocity of approximately 5 km/s, but electromagnetic rocket propulsion is predicted to yield a velocity of perhaps 40 km/s.)

Continued—

PROBLEM

What is the final velocity that this spaceship can reach, relative to the velocity it initially had in its orbit around Earth?

SOLUTION

Using equation 8.17 and substituting the numbers given in this problem, we find

$$v_f - v_i = v_c \ln\left(\frac{m_i}{m_f}\right) = (23.5 \text{ km/s})\ln\left(\frac{2{,}050{,}000 \text{ kg}}{50{,}000 \text{ kg}}\right) = (23.5 \text{ km/s})(\ln 41) = 87.3 \text{ km/s}.$$

For comparison, the Saturn V multistage rocket that carried astronauts to the Moon in the late 1960s and early 1970s was able to reach a speed of only about 12 km/s.

However, even with advanced technology such as electromagnetic propulsion, it would still take several months for astronauts to reach Mars, even under the most favorable conditions. The *Mars Rover,* for example, took 207 days to travel from Earth to Mars. NASA estimates that astronauts on such a mission would receive approximately 10 to 20 times more radiation than the maximally allowable annual dose for radiation workers, leading to high probabilities of developing cancer and brain damage. No shielding mechanism has yet been proposed that could protect the astronauts from this danger.

Another and perhaps easier way to think of rocket motion is to go back to the definition of momentum as the product of mass and velocity and take the time derivative to obtain the force. However, now the mass of the object can change as well:

$$\vec{F}_{\text{net}} = \frac{d}{dt}\vec{p} = \frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} + \vec{v}\frac{dm}{dt}.$$

(The last step in this equation represents the application of the product rule of differentiation from calculus.) If no external force is acting on an object ($\vec{F}_{\text{net}} = 0$), then we obtain

$$m\frac{d\vec{v}}{dt} = -\vec{v}\frac{dm}{dt}.$$

In the case of rocket motion (as illustrated in Figure 8.13), the outflow of propellant, *dm/dt,* is constant and creates the change in mass of the rocket. The propellant moves with a constant velocity, $\vec{v}_c$, relative to the rocket, so we obtain

$$m\frac{d\vec{v}}{dt} = m\vec{a} = -\vec{v}_c\frac{dm}{dt}.$$

The combination $v_c(dm/dt)$ is called the **thrust** of the rocket. It is a force and thus is measured in newtons:

$$\vec{F}_{\text{thrust}} = -\vec{v}_c\frac{dm}{dt}. \qquad \textbf{(8.18)}$$

The thrust generated by space shuttle rocket engines and boosters is approximately 31.3 MN (31.3 meganewtons, or approximately 7.8 million pounds). The initial total mass of the Space Shuttle, including payload, fuel tanks, and rocket fuel, is slightly greater than 2.0 million kg; thus, the shuttle's rocket engines and boosters can produce an initial acceleration of

$$a = \frac{\vec{F}_{\text{net}}}{m} = \frac{3.13 \cdot 10^7 \text{ N}}{2.0 \cdot 10^6 \text{ kg}} = 16 \text{ m/s}^2.$$

FIGURE 8.13 Rocket motion.

This acceleration is sufficient to lift the shuttle off the launch pad against the acceleration of gravity (-9.81 m/s^2). Once the shuttle rises and its mass decreases, it can generate a larger acceleration. As the fuel is expended, the main engines are throttled back to make sure that the acceleration does not exceed $3g$ (three times gravitational acceleration) in order to avoid damaging the cargo or injuring the astronauts.

8.4 Calculating the Center of Mass

So far, we have not addressed a key question: How do we calculate the location of the center of mass for an arbitrarily shaped object? To answer this question, let's find the location of the center of mass of the hammer shown in Figure 8.14. To do this, we can represent the hammer by small identical-sized cubes, as shown in the lower part of the figure. The centers of the cubes are their individual centers of mass, marked with red dots. The red arrows are the position vectors of the cubes. If we accept the collection of cubes as a good approximation for the hammer, we can use equation 8.3 to find the center of mass of the collection of cubes and thus that of the hammer.

FIGURE 8.14 Calculating the center of mass for a hammer.

Note that not all the cubes have the same mass, because the densities of the wooden handle and the iron head are very different. The relationship between mass density (ρ), mass, and volume is given by

$$\rho = \frac{dm}{dV}. \tag{8.19}$$

If the mass density is uniform throughout an object, we simply have

$$\rho = \frac{M}{V} \quad \text{(for constant } \rho\text{)}. \tag{8.20}$$

We can then use the mass density and rewrite equation 8.3:

$$\vec{R} = \frac{1}{M}\sum_{i=1}^{n}\vec{r}_i m_i = \frac{1}{M}\sum_{i=1}^{n}\vec{r}_i \rho(\vec{r}_i)V.$$

Here we have assumed that the mass density of each small cube is uniform (but still possibly different from one cube to another) and that each cube has the same (small) volume, V.

We can obtain a better and better approximation by shrinking the volume of each cube and using a larger and larger number of cubes. This procedure should look very familiar to you, because it is exactly what is done in calculus to arrive at the limit for an integral. In this limit, we obtain for the location of the center of mass for an arbitrarily shaped object:

$$\vec{R} = \frac{1}{M}\int_V \vec{r}\rho(\vec{r})\,dV. \tag{8.21}$$

Here the three-dimensional volume integral extends over the entire volume of the object under consideration.

The next question that arises is what coordinate system to choose in order to evaluate this integral. You may have never seen a three-dimensional integral before and may have worked only with one-dimensional integrals of the form $\int f(x)dx$. However, all three-dimensional integrals that we will use in this chapter can be reduced to (at most) three successive one-dimensional integrals, most of which are very straightforward to evaluate, provided one selects an appropriate system of coordinates.

Three-Dimensional Non-Cartesian Coordinate Systems

Chapter 1 introduced a three-dimensional orthogonal coordinate system, the Cartesian coordinate system, with coordinates x, y, and z. However, for some applications, it is mathematically simpler to represent the position vector in another coordinate system. This section briefly introduces two commonly used three-dimensional coordinate systems that can be used to specify a vector in three-dimensional space: spherical coordinates and cylindrical coordinates.

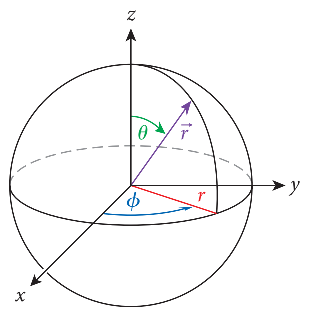


FIGURE 8.15 Three-dimensional spherical coordinate system.

Spherical Coordinates

In **spherical coordinates,** the position vector $\vec{r}$ is represented by giving its length, r; its polar angle relative to the positive z-axis, θ; and the azimuthal angle of the vector's projection onto the xy-plane relative to the positive x-axis, ϕ (Figure 8.15).

We can obtain the Cartesian coordinates of the vector $\vec{r}$ from its spherical coordinates via the transformation

$$\begin{aligned} x &= r\cos\phi\sin\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\theta. \end{aligned} \tag{8.22}$$

The inverse transformation from Cartesian to spherical coordinates is

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right). \end{aligned} \tag{8.23}$$

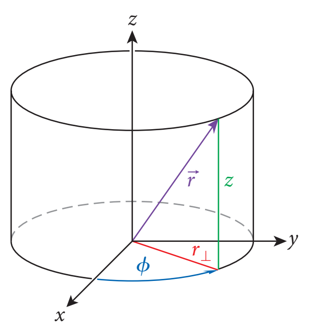


FIGURE 8.16 Cylindrical coordinate system in three dimensions.

Cylindrical Coordinates

Cylindrical coordinates can be thought of as an intermediate between Cartesian and spherical coordinate systems, in the sense that the Cartesian z-coordinate is retained, but the Cartesian coordinates x and y are replaced by the coordinates $r_\perp$ and ϕ (Figure 8.16). Here $r_\perp$ specifies the length of the projection of the position vector $\vec{r}$ onto the xy-plane, so it measures the perpendicular distance to the z-axis. Just as in spherical coordinates, ϕ is the angle of the vector's projection into the xy-plane relative to the positive x-axis.

We obtain the Cartesian coordinates from the cylindrical coordinates via

$$\begin{aligned} x &= r_\perp\cos\phi \\ y &= r_\perp\sin\phi \\ z &= z. \end{aligned} \tag{8.24}$$

The inverse transformation from Cartesian to cylindrical coordinates is

$$\begin{aligned} r_\perp &= \sqrt{x^2+y^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z. \end{aligned} \tag{8.25}$$

As a rule of thumb, you should use a Cartesian coordinate system in your first attempt to describe any physical situation. However, cylindrical and spherical coordinate systems are often preferable when working with objects that have symmetry about a point or a line. Later in this chapter, we will make use of a cylindrical coordinate system to perform a three-dimensional volume integral. Chapter 9 will discuss polar coordinates, which can be thought of as the two-dimensional equivalent of either cylindrical or spherical coordinates. Finally, in Chapter 10, we will again use spherical and cylindrical coordinates to solve slightly more complicated problems requiring integration.

Mathematical Insert: Volume Integrals

Even though calculus is a prerequisite for physics, many universities allow students to take introductory physics and calculus courses concurrently. In general, this approach works well, but when students encounter multidimensional integrals in physics, it is often the first time they have seen this notation. Therefore, let's review the basic procedure for performing these integrations.

If we want to integrate any function over a three-dimensional volume, we need to find an expression for the volume element dV in an appropriate set of coordinates. Unless there is an extremely important reason not to, you should always use orthogonal coordinate systems. The three commonly used three-dimensional orthogonal coordinate systems are the Cartesian, cylindrical, and spherical systems.

It is easiest by far to express the volume element dV in Cartesian coordinates; it is simply the product of the three individual coordinate elements (Figure 8.17). The three-dimensional volume integral written in Cartesian coordinates is

$$\int_V f(\vec{r})dV = \int_{z_{\min}}^{z_{\max}} \left(\int_{y_{\min}}^{y_{\max}} \left(\int_{x_{\min}}^{x_{\max}} f(\vec{r})dx \right) dy \right) dz. \tag{8.26}$$

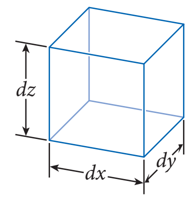


FIGURE 8.17 Volume element in Cartesian coordinates.

In this equation, $f(\vec{r})$ can be an arbitrary function of the position. The lower and upper boundaries for the individual coordinates are denoted by $x_{\min}, x_{\max}, \ldots$. The convention is to solve the innermost integral first and then work outward. For equation 8.26, this means that we first execute the integration over x, then the integration over y, and finally the integration over z. However, any other order is possible. An equally valid way of writing the integral in equation 8.26 is

$$\int_V f(\vec{r})dV = \int_{x_{\min}}^{x_{\max}} \left(\int_{y_{\min}}^{y_{\max}} \left(\int_{z_{\min}}^{z_{\max}} f(\vec{r})dz \right) dy \right) dx, \tag{8.27}$$

which implies that the order of integration is now z, y, x. Why might the order of integration make a difference? The only time the order of the integration matters is when the integration boundaries in a particular coordinate depend on one or both of the other coordinates. Example 8.4 will consider such a situation.

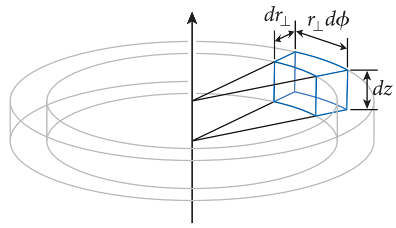


FIGURE 8.18 Volume element in cylindrical coordinates.

Because the angle ϕ is one of the coordinates in the cylindrical coordinate system, the volume element is not cube-shaped. For a given differential angle, $d\phi$, the size of the volume element depends on how far away from the z-axis the volume element is located. This size increases linearly with the distance $r_\perp$ from the z-axis (Figure 8.18) and is given by

$$dV = r_\perp dr_\perp d\phi dz. \tag{8.28}$$

The volume integral is then

$$\int_V f(\vec{r})dV = \int_{z_{\min}}^{z_{\max}} \left(\int_{\phi_{\min}}^{\phi_{\max}} \left(\int_{r_{\perp\min}}^{r_{\perp\max}} f(\vec{r})r_\perp dr_\perp \right) d\phi \right) dz. \tag{8.29}$$

Again the order of integration can be chosen to make the task as simple as possible.

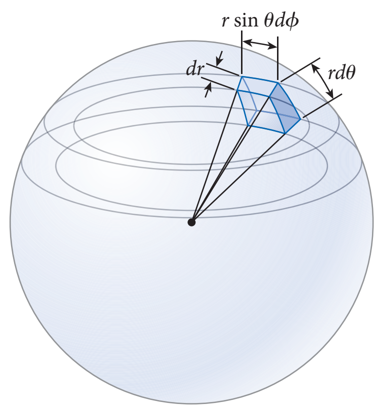


FIGURE 8.19 Volume element in spherical coordinates.

Finally, in spherical coordinates, we use two angular variables, θ and ϕ (Figure 8.19). Here the size of the volume element for a given value of the differential coordinates depends on the distance r to the origin as well as the angle relative to the $\theta = 0$ axis (equivalent to the z-axis in Cartesian or cylindrical coordinates). The differential volume element in spherical coordinates is

$$dV = r^2 dr \sin\theta d\theta d\phi. \tag{8.30}$$

The volume integral in spherical coordinates is given by

$$\int_V f(\vec{r})dV = \int_{r_{\min}}^{r_{\max}} \left(\int_{\phi_{\min}}^{\phi_{\max}} \left(\int_{\theta_{\min}}^{\theta_{\max}} f(\vec{r})\sin\theta d\theta \right) d\phi \right) r^2 dr. \tag{8.31}$$

EXAMPLE 8.4 Volume of a Cylinder

To illustrate why it may be simpler to use non-Cartesian coordinates in certain circumstances, let's use volume integrals to find the volume of a cylinder with radius R and height H. We have to integrate the function $f(\vec{r}) = 1$ over the entire cylinder to obtain the volume.

Continued—

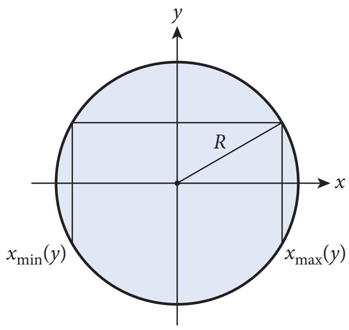


FIGURE 8.20 Bottom surface of a right cylinder of radius R.

PROBLEM

Use a volume integral to find the volume of a right cylinder of height H and radius R.

SOLUTION

In Cartesian coordinates, we place the origin of our coordinate system at the center of the cylinder's circular base (bottom surface), so the shape in the xy-plane that we have to integrate over is a circle with radius R (Figure 8.20). The volume integral in Cartesian coordinates is then

$$\int_V dV = \int_0^H \left(\int_{y_{\min}}^{y_{\max}} \left(\int_{x_{\min}(y)}^{x_{\max}(y)} dx \right) dy \right) dz. \tag{i}$$

The innermost integral has to be done first and is straightforward:

$$\int_{x_{\min}(y)}^{x_{\max}(y)} dx = x_{\max}(y) - x_{\min}(y). \tag{ii}$$

The integration boundaries depend on y: $x_{\max} = \sqrt{R^2 - y^2}$ and $x_{\min} = -\sqrt{R^2 - y^2}$. Thus, the solution of equation (ii) is $x_{\max}(y) - x_{\min}(y) = 2\sqrt{R^2 - y^2}$. We insert this into equation (i) and obtain

$$\int_V dV = \int_0^H \left(\int_{-R}^{R} 2\sqrt{R^2 - y^2}\, dy \right) dz. \tag{iii}$$

The inner of these two remaining integrals evaluates to

$$\int_{-R}^{R} 2\sqrt{R^2 - y^2}\, dy = \left(y\sqrt{R^2 - y^2} + R^2 \tan^{-1}\left(\frac{y}{\sqrt{R^2 - y^2}} \right) \right)\Bigg|_{-R}^{R} = \pi R^2.$$

You can check this result by looking up the definite integral in an integral table. Inserting this result into equation (iii) finally yields our answer:

$$\int_V dV = \int_0^H \pi R^2 dz = \pi R^2 \int_0^H dz = \pi R^2 H.$$

As you can see, obtaining the volume of the cylinder was rather cumbersome in Cartesian coordinates. What about using cylindrical coordinates? According to equation 8.29, the volume integral is then

$$\int_V f(\vec{r})\, dV = \int_0^H \left(\int_0^{2\pi} \left(\int_0^R r_\perp dr_\perp \right) d\phi \right) dz = \int_0^H \left(\int_0^{2\pi} \left(\tfrac{1}{2} R^2 \right) d\phi \right) dz$$

$$= \tfrac{1}{2} R^2 \int_0^H \left(\int_0^{2\pi} d\phi \right) dz = \tfrac{1}{2} R^2 \int_0^H 2\pi\, dz = \pi R^2 \int_0^H dz = \pi R^2 H.$$

In this case, it was much easier to use cylindrical coordinates, a consequence of the geometry of the object over which we had to integrate.

8.2 Self-Test Opportunity

Using spherical coordinates, show that the volume V of a sphere with radius R is $V = \frac{4}{3}\pi R^3$.

Now we can return to the problem of calculating the location of an object's center of mass. For the Cartesian components of the position vector, we find, from equation 8.21:

$$X = \frac{1}{M}\int_V x\rho(\vec{r})dV, \quad Y = \frac{1}{M}\int_V y\rho(\vec{r})dV, \quad Z = \frac{1}{M}\int_V z\rho(\vec{r})dV. \tag{8.32}$$

If the mass density for the entire object is constant, $\rho(\vec{r}) \equiv \rho$, we can remove this constant factor from the integral and obtain a special case of equation 8.21 for constant mass density:

$$\vec{R} = \frac{\rho}{M}\int_V \vec{r}\,dV = \frac{1}{V}\int_V \vec{r}\,dV \quad \text{(for constant } \rho\text{)}, \tag{8.33}$$

where we have used equation 8.20 in the last step. Expressed in Cartesian components, we obtain for this case:

$$X = \frac{1}{V}\int_V x\,dV, \quad Y = \frac{1}{V}\int_V y\,dV, \quad Z = \frac{1}{V}\int_V z\,dV. \tag{8.34}$$

Equations 8.33 and 8.34 indicate that any object that has a symmetry plane has its center of mass located in that plane. An object having three mutually perpendicular symmetry planes (such as a cylinder, a rectangular solid, or a sphere) has its center of mass where these three planes intersect, which is the geometric center. Example 8.5 develops this idea further.

EXAMPLE 8.5 Center of Mass for a Half-Sphere

PROBLEM

Consider a solid half-sphere of constant mass density with radius R_0 (Figure 8.21a). Where is its center of mass?

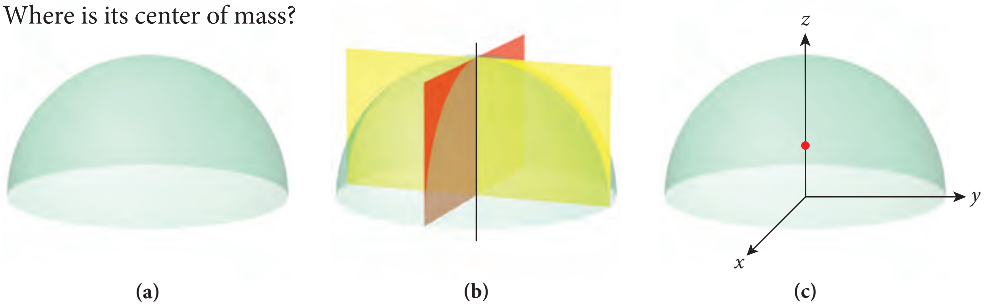


FIGURE 8.21 Determination of the center of mass: (a) half-sphere; (b) symmetry planes and symmetry axis; (c) coordinate system, with location of center of mass marked by red dot.

SOLUTION

As shown in Figure 8.21b, symmetry planes can divide this object into equal, mirror-image halves. Shown are two perpendicular planes in red and yellow, but any plane through the vertical symmetry axis (indicated by the thin black line) is a symmetry plane.

We now position the coordinate system so that one axis (the z-axis, in this case) coincides with this symmetry axis. We are then assured that the center of mass is located exactly on this axis. Because the mass distribution is symmetric and the integrands of equations 8.33 or 8.34 are odd powers of $\vec{r}$ the integral for X or Y has to have the value zero. Specifically,

$$\int_{-a}^{a} x\,dx = 0 \text{ for all values of the constant } a.$$

Positioning the coordinate system so that the z-axis is the symmetry axis ensures that $X = Y = 0$. This is shown in Figure 8.21c, where the origin of the coordinate system is positioned at the center of the half-sphere's circular bottom surface.

Now we have to find the value of the third integral in equation 8.34:

$$Z = \frac{1}{V}\int_V z\,dV.$$

The volume of a half-sphere is half the volume of a sphere, or

$$V = \frac{2\pi}{3}R_0^3. \tag{i}$$

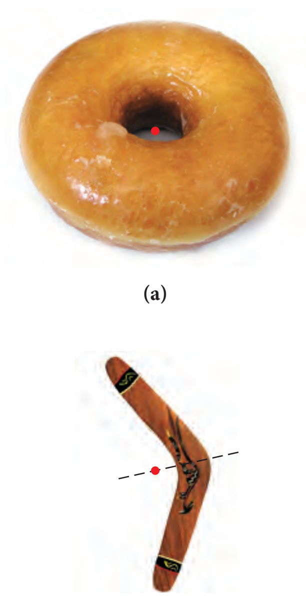
Continued—

To evaluate the integral for Z, we use cylindrical coordinates, in which the differential volume element is given (see equation 8.28) as $dV = r_\perp dr_\perp d\phi\, dz$. The integral is then evaluated as follows:

$$\begin{aligned}\int_V z\,dV &= \int_0^{R_0}\left(\int_0^{\sqrt{R_0^2-z^2}}\left(\int_0^{2\pi} zr_\perp d\phi\right)dr_\perp\right)dz\\ &= \int_0^{R_0} z\left(\int_0^{\sqrt{R_0^2-z^2}} r_\perp\left(\int_0^{2\pi} d\phi\right)dr_\perp\right)dz\\ &= 2\pi\int_0^{R_0} z\left(\int_0^{\sqrt{R_0^2-z^2}} r_\perp dr_\perp\right)dz\\ &= \pi\int_0^{R_0} z(R_0^2-z^2)dz\\ &= \frac{\pi}{4}R_0^4.\end{aligned}$$

Combining this result and the expression for the volume of a half-sphere from equation (i), we obtain the z-coordinate of the center of mass:

$$Z = \frac{1}{V}\int_V z\,dV = \frac{3}{2\pi R_0^3}\frac{\pi R_0^4}{4} = \frac{3}{8}R_0.$$



(a)

(b)

FIGURE 8.22 Objects with a center of mass (indicated by the red dot) outside their mass distribution: (a) donut; (b) boomerang. The symmetry axis of the boomerang is shown by a dashed line.

Note that the center of mass of an object does not always have to be located inside the object. Two obvious examples are shown in Figure 8.22. From symmetry considerations, it follows that the center of mass of the donut (Figure 8.22a) is exactly in the center of its hole, at a point outside the donut. The center of mass of the boomerang (Figure 8.22b) lies on the dashed symmetry axis but, again, outside the object.

Center of Mass for One- and Two-Dimensional Objects

Not all problems involving calculation of the center of mass focus on three-dimensional objects. For example, you may want to calculate the center of mass of a two-dimensional object, such as a flat metal plate. We can write the equations for the center-of-mass coordinates of a two-dimensional object whose area mass density (or mass per unit area) is $\sigma(\vec{r})$ by modifying the expressions for X and Y given in equation 8.32:

$$X = \frac{1}{M}\int_A x\sigma(\vec{r})dA,\quad Y = \frac{1}{M}\int_A y\sigma(\vec{r})dA, \tag{8.35}$$

where the mass is

$$M = \int_A \sigma(\vec{r})dA. \tag{8.36}$$

If the area mass density of the object is constant, then $\sigma = M/A$, and we can rewrite equation 8.35 to give the coordinates of the center of mass of a two-dimensional object in terms of the area, A, and the coordinates x and y:

$$X = \frac{1}{A}\int_A x\,dA,\quad Y = \frac{1}{A}\int_A y\,dA, \tag{8.37}$$

where the total area is obtained from

$$A = \int_A dA. \tag{8.38}$$

If the object is effectively one-dimensional, such as a long, thin rod with length L and linear mass density (or mass per unit length) of $\lambda(x)$, the coordinate of the center of mass is given by

$$X = \frac{1}{M}\int_L x\lambda(x)dx, \tag{8.39}$$

where the mass is

$$M = \int_L \lambda(x)dx. \tag{8.40}$$

If the linear mass density of the rod is constant, then clearly the center of mass is located at the geometric center—the middle of the rod—and no further calculation is required.

SOLVED PROBLEM 8.3 Center of Mass of a Long, Thin Rod

PROBLEM

A long, thin rod lies along the x-axis. One end of the rod is located at $x = 1.00$ m, and the other end of the rod is located at $x = 3.00$ m. The linear mass density of the rod is given by $\lambda(x) = ax^2 + b$, where $a = 0.300$ kg/m^3 and $b = 0.600$ kg/m. What are the mass of the rod and the x-coordinate of its center of mass?

SOLUTION

THINK

The linear mass density of the rod is not uniform but depends on the x-coordinate. Therefore, to get the mass, we must integrate the linear mass density over the length of the rod. To get the center of mass, we need to integrate the linear mass density, weighted by the distance in the x-direction, and then divide by the mass of the rod.

SKETCH

The long, thin rod oriented along the x-axis is shown in Figure 8.23.

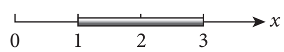


FIGURE 8.23 A long, thin rod oriented along the x-axis.

RESEARCH

We obtain the mass of the rod by integrating the linear mass density, λ, over the rod from $x_1 = 1.00$ m to $x_2 = 3.00$ m (see equation 8.40):

$$M = \int_{x_1}^{x_2} \lambda(x)dx = \int_{x_1}^{x_2} \left(ax^2 + b\right)dx = \left[a\frac{x^3}{3} + bx\right]_{x_1}^{x_2}.$$

To find the x-coordinate of the center of mass of the rod, X, we evaluate the integral of the differential mass times x and then divide by the mass, which we have just calculated (see equation 8.39):

$$X = \frac{1}{M}\int_{x_1}^{x_2} \lambda(x)x\,dx = \frac{1}{M}\int_{x_1}^{x_2} \left(ax^2 + b\right)x\,dx = \frac{1}{M}\int_{x_1}^{x_2} \left(ax^3 + bx\right)dx = \frac{1}{M}\left[a\frac{x^4}{4} + b\frac{x^2}{2}\right]_{x_1}^{x_2}.$$

SIMPLIFY

Inserting the upper and lower limits, x_2 and x_1, we get the mass of the rod:

$$M = \left[a\frac{x^3}{3} + bx\right]_{x_1}^{x_2} = \left(a\frac{x_2^3}{3} + bx_2\right) - \left(a\frac{x_1^3}{3} + bx_1\right) = \frac{a}{3}\left(x_2^3 - x_1^3\right) + b\left(x_2 - x_1\right).$$

Continued—

And, in the same way, we find the x-coordinate of the center of mass of the rod:

$$X = \frac{1}{M}\left[a\frac{x^4}{4} + b\frac{x^2}{2}\right]_{x_1}^{x_2} = \frac{1}{M}\left\{\left(a\frac{x_2^4}{4} + b\frac{x_2^2}{2}\right) - \left(a\frac{x_1^4}{4} + b\frac{x_1^2}{2}\right)\right\},$$

which we can further simplify to

$$X = \frac{1}{M}\left\{\frac{a}{4}\left(x_2^4 - x_1^4\right) + \frac{b}{2}\left(x_2^2 - x_1^2\right)\right\}.$$

CALCULATE

Substituting the given numerical values, we compute the mass of the rod:

$$M = \frac{0.300\ \text{kg/m}^3}{3}\left(\left(3.00\ \text{m}\right)^3 - \left(1.00\ \text{m}\right)^3\right) + \left(0.600\ \text{kg/m}\right)\left(3.00\ \text{m} - 1.00\ \text{m}\right) = 3.8\ \text{kg}.$$

With the numerical values, the x-coordinate of the rod is

$$X = \frac{1}{3.8\ \text{kg}}\left\{\frac{0.300\ \text{kg/m}^3}{4}\left(\left(3.00\ \text{m}\right)^4 - \left(1.00\ \text{m}\right)^4\right) + \frac{0.600\ \text{kg/m}}{2}\left(\left(3.00\ \text{m}\right)^2 - \left(1.00\ \text{m}\right)^2\right)\right\}$$

$$= 2.210526316\ \text{m}.$$

ROUND

All of the numerical values in the problem statement were specified to three significant figures, so we report our results as

$$M = 3.80\ \text{kg}$$

and

$$X = 2.21\ \text{m}.$$

DOUBLE-CHECK

To double-check our answer for the mass of the rod, let's assume that the rod has a constant linear mass density equal to the linear mass density obtained by setting $x = 2$m (the middle of the rod) in the expression for λ in the problem, that is,

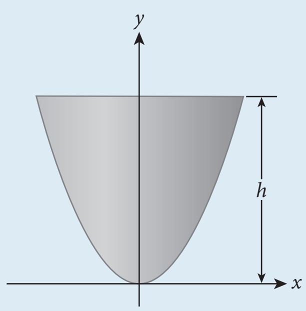
$$\lambda = \left(0.3 \cdot 4 + 0.6\right)\ \text{kg/m} = 1.8\ \text{kg/m}.$$

The mass of the rod is then $m \approx 2\text{m} \cdot 1.8\ \text{kg/m} = 3.6$ kg, which is reasonably close to our exact calculation of $M = 3.80$ kg.

To double-check the x-coordinate of the center of mass of the rod, we again assume that the linear mass density is constant. Then the center of mass will be located at the middle of the rod, or $X \approx 2$m. Our calculated answer is $X = 2.21$ m, which is slightly to the right of the middle of the rod. Looking at the function for the linear mass density, we see that the linear mass of the rod increases toward the right, which means that the center of mass of the rod must be to the right of the rod's geometric center. Our result is therefore reasonable.

8.3 Self-Test Opportunity

A plate with height h is cut from a thin metal sheet with uniform mass density, as shown in the figure. The lower boundary of the plate is defined by $y = 2x^2$. Show that the center of mass of this plate is located at $x = 0$ and $y = \frac{3}{5}h$.



WHAT WE HAVE LEARNED | EXAM STUDY GUIDE

- The center of mass is the point at which we can imagine all the mass of an object to be concentrated.
- The location of the center of mass for an arbitrarily shaped object is given by $\vec{R} = \frac{1}{M}\int_V \vec{r}\rho(\vec{r})dV$, where the mass density of the object is $\rho = \frac{dm}{dV}$, the integration extends over the entire volume V of the object, and M is its total mass.
- When the mass density is uniform throughout the object, that is, $\rho = \frac{M}{V}$, the center of mass is $\vec{R} = \frac{1}{V}\int_V \vec{r}\,dV$.
- If an object has a plane of symmetry, the location of the center of mass must be in that plane.
- The location of the center of mass for a combination of several objects can be found by taking the

mass-weighted average of the locations of the centers of mass of the individual objects: $\vec{R} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots + \vec{r}_n m_n}{m_1 + m_2 + \cdots + m_n} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i.$

- The motion of an extended rigid object can be described by the motion of its center of mass.
- The velocity of the center of mass is given by the derivative of its position vector: $\vec{V} \equiv \frac{d}{dt}\vec{R}.$
- The center-of-mass momentum for a combination of several objects is $\vec{P} = M\vec{V} = \sum_{i=1}^{n} \vec{p}_i$. This momentum obeys Newton's Second Law: $\frac{d}{dt}\vec{P} = \frac{d}{dt}(M\vec{V}) = \sum_{i=1}^{n} \vec{F}_i = \vec{F}_{\text{net}}$. Internal forces between the objects do not contribute to the sum that yields the net force (because they always come in action-reaction pairs adding up to zero) and thus do not change the center-of-mass momentum.
- For a system of two objects, the total momentum is $\vec{P} = \vec{p}_1 + \vec{p}_2$, and the relative momentum is $\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2)$. In collisions between two objects, the total momentum remains unchanged.
- Rocket motion is an example of motion during which the mass of the moving object is not constant. The equation of motion for a rocket in interstellar space is given by $\vec{F}_{\text{thrust}} = m\vec{a} = -\vec{v}_{\text{c}}\frac{dm}{dt}$, where $\vec{v}_{\text{c}}$ is the velocity of the propellant relative to the rocket and $\frac{dm}{dt}$ is the rate of change in mass due to outflow of propellant.
- The velocity of a rocket as a function of its mass is given by $v_{\text{f}} - v_{\text{i}} = v_{\text{c}} \ln (m_{\text{i}}/m_{\text{f}})$, where the indices i and f indicate initial and final masses and velocities.

KEY TERMS

center of mass, p. 247
recoil, p. 253
thrust, p. 258
spherical coordinates, p. 260
cylindrical coordinates, p. 260

NEW SYMBOLS AND EQUATIONS

$\vec{R} = \frac{1}{M}\sum_{i=1}^{n} \vec{r}_i m_i$, combined center-of-mass position vector

$\vec{R} = \frac{1}{M}\int_V \vec{r}\rho(\vec{r})dV$, center of mass for an extended object

$dV = r_\perp dr_\perp d\phi\, dz$, volume element in cylindrical coordinates

$dV = r^2 dr \sin\theta\, d\theta\, d\phi$, volume element in spherical coordinates

$\vec{F}_{\text{thrust}}$, rocket thrust

ANSWERS TO SELF-TEST OPPORTUNITIES

8.1

$$Z = \frac{z_{\text{red}} m_{\text{red}} + z_{\text{green}} m_{\text{green}} + z_{\text{orange}} m_{\text{orange}} + z_{\text{blue}} m_{\text{blue}} + z_{\text{purple}} m_{\text{purple}}}{M}$$

$$= \frac{\frac{1}{2}h2m_0 + \frac{1}{2}hm_0 + \frac{1}{2}hm_0 + \frac{1}{2}hm_0 + \frac{3}{2}hm_0}{6m_0}$$

$$= \frac{w1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{3}{2}}{6} = \frac{2}{3}h = 1.7 \text{ m}$$

8.2 We use spherical coordinates and integrate the angle θ from 0 to π, the angle ϕ from 0 to 2π, and the radial coordinate r from 0 to R.

$$V = \int_0^R \left(\int_0^{2\pi} \left(\int_0^{\pi} \sin\theta\, d\theta \right) d\phi \right) r^2 dr$$

First, evaluate the integral over the azimuthal angle:

$$\int_0^{\pi} \sin\theta\, d\theta = \left[-\cos\theta\right]_0^{\pi} = -\left[\cos(\pi) - \cos(0)\right] = 2$$

$$V = 2\int_0^R \left(\int_0^{2\pi} d\phi \right) r^2 dr$$

Now evaluate the polar angle integral:

$$\int_0^{2\pi} d\phi = \left[\phi\right]_0^{2\pi} = 2\pi$$

Continued—

Finally:

$$V = 4\pi \int_0^R r^2 dr = 4\pi \left[\frac{r^3}{3}\right]_0^R = \frac{4}{3}\pi R^3$$

8.3 $dA = x(y)dy;\ y = 2x^2 \Rightarrow\ x = \sqrt{y/2}$

$$x(y) = 2\sqrt{y/2} = \sqrt{2y}dA = \sqrt{2y}dy$$

$$Y = \frac{\int_0^h y\sqrt{2y}dy}{\int_0^h \sqrt{2y}dy} = \frac{\sqrt{2}\int_0^h y^{3/2}\,dy}{\sqrt{2}\int_0^h y^{1/2}\,dy} = \frac{\left[\frac{y^{5/2}}{5/2}\right]_0^h}{\left[\frac{y^{3/2}}{3/2}\right]_0^h}$$

$$Y = \tfrac{3}{5}h$$

PROBLEM-SOLVING PRACTICE

Problem-Solving Guidelines: Center of Mass

1. The first step in locating the center of mass of an object or a system of particles is to look for planes of symmetry. The center of mass must be located on the plane of symmetry, on the line of intersection of two planes of symmetry, or at the point of intersection of more than two planes.

2. For complicated shapes, break the object down into simpler geometric forms and locate the center of mass for each individual form. Then combine the separate centers of mass into one overall center of mass using the weighted average of distances and masses. Treat holes as objects of negative mass.

3. Any motion of an object can be treated as a superposition of motion of its center of mass (according to Newton's Second Law) and rotation of the object about the center of mass. Collisions can often be conveniently analyzed by considering a reference frame with the origin located at the center of mass.

4. Often, integration is unavoidable when you need to locate the center of mass. In such a case, it is always best to think carefully about the dimensionality of the situation and the choice of the coordinate system (Cartesian, cylindrical, or spherical).

SOLVED PROBLEM 8.4 Thruster Firing

PROBLEM

Suppose a spacecraft has an initial mass of 1,850,000 kg. Without its propellant, the spacecraft has a mass of 50,000 kg. The rocket that powers the spacecraft is designed to eject the propellant with a speed of 25 km/s with respect to the rocket at a constant rate of 15,000 kg/s. The spacecraft is initially at rest in space and travels in a straight line. How far will the spacecraft travel before its rocket uses all the propellant and shuts down?

SOLUTION

THINK

The total mass of propellant is the total mass of the spacecraft minus the mass of the spacecraft after all the propellant is ejected. The rocket ejects the propellant at a fixed rate, so we can calculate the amount of time during which the rocket operates. As the propellant is used up, the mass of the spacecraft decreases and the speed of the spacecraft increases. If the spacecraft starts from rest, the speed $v(t)$ at any time while the rocket is operating can be obtained from equation 8.17, with the final mass of the spacecraft replaced by the mass of the spacecraft at that time. The distance traveled before all the propellant is used is given by the integral of the speed as a function of time.

SKETCH

The flight of the spacecraft is sketched in Figure 8.24.

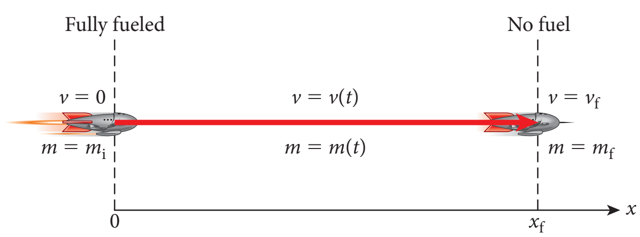


FIGURE 8.24 The various parameters for the spacecraft as the rocket operates.

RESEARCH

We symbolize the rate at which the propellant is ejected by r_p. The time t_{max} during which the rocket will operate is then given by

$$t_{max} = \frac{(m_i - m_f)}{r_p},$$

where m_i is the initial mass of the spacecraft and m_f is the mass of the spacecraft after all the propellant is ejected. The total distance the spacecraft travels in this time interval is the integral of the speed over time:

$$x_f = \int_0^{t_{max}} v(t)\,dt. \tag{i}$$

While the rocket is operating, the mass of the spacecraft at a time t is given by

$$m(t) = m_i - r_p t.$$

The speed of the spacecraft at any given time after the rocket starts to operate and before all the propellant is used up is given by (compare to equation 8.17)

$$v(t) = v_c \ln\left(\frac{m_i}{m(t)}\right) = v_c \ln\left(\frac{m_i}{m_i - r_p t}\right) = v_c \ln\left(\frac{1}{1 - r_p t / m_i}\right), \tag{ii}$$

where v_c is the speed of the ejected propellant with respect to the rocket.

SIMPLIFY

Now we substitute from equation (ii) for the time dependence of the speed of the spacecraft into equation (i) and obtain

$$x_f = \int_0^{t_{max}} v(t)\,dt = \int_0^{t_{max}} v_c \ln\left(\frac{1}{1 - r_p t / m_i}\right) dt = -v_c \int_0^{t_{max}} \ln\left(\frac{1 - r_p t}{m_i}\right) dt. \tag{iii}$$

Because $\int \ln(1 - ax)dx = \frac{ax - 1}{a}\ln(1 - ax) - x$ (you can look up this result in an integral table), the integral evaluates to

$$\begin{aligned}\int_0^{t_{max}} \ln(1 - r_p t / m_i)dt &= \left[\left(\frac{r_p t / m_i - 1}{r_p / m_i}\right)\ln(1 - r_p t / m_i) - t\right]_0^{t_{max}} \\ &= \left(\frac{r_p t_{max} / m_i - 1}{r_p / m_i}\right)\ln(1 - r_p t_{max} / m_i) - t_{max} \\ &= (t_{max} - m_i / r_p)\ln(1 - r_p t_{max} / m_i) - t_{max}.\end{aligned}$$

The distance traveled is then

$$x_f = -v_c\left[(t_{max} - m_i / r_p)\ln(1 - r_p t_{max} / m)_i - t_{max}\right].$$

CALCULATE

The time during which the rocket is operating is

$$t_{max} = \frac{m_i - m_f}{r_p} = \frac{1{,}850{,}000 \text{ kg} - 50{,}000 \text{ kg}}{15{,}000 \text{ kg/s}} = 120 \text{ s}.$$

Putting numerical values into the factor $1 - r_p t_{max} / m_i$ gives

$$1 - \frac{r_p t_{max}}{m_i} = 1 - \frac{15{,}000 \text{ kg/s} \cdot 120 \text{ s}}{1{,}850{,}000 \text{ kg}} = 0.027027.$$

Continued—

Thus, we find for the distance traveled

$$x_f = -\left(25\cdot 10^3\text{ m/s}\right)\left[-(120\text{ s}) + \{(120\text{ s}) - \left(1.85\cdot 10^6\text{ kg}\right)/\left(15\cdot 10^3\text{ kg/s}\right)\}\ln(0.027027)\right]$$
$$= 2.69909\cdot 10^6\text{ m.}$$

ROUND

Because the propellant speed was given to only two significant figures, we need to round to that accuracy:

$$x_f = 2.7\cdot 10^6\text{ m.}$$

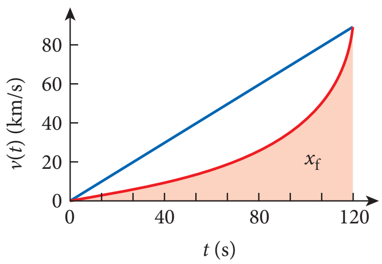


FIGURE 8.25 Comparison of the exact solution for $v(t)$ (red curve) to one for constant acceleration (blue line).

DOUBLE-CHECK

To double-check our answer for the distance traveled, we use equation 8.17 to calculate the final velocity of the spacecraft:

$$v_f = v_c \ln\left(\frac{m_i}{m_f}\right) = \left(25\text{ km/s}\right)\ln\left(\frac{1.85\cdot 10^6\text{ kg}}{5\cdot 10^4\text{ kg}}\right) = 90.3\text{ km/s.}$$

If the spacecraft accelerated at a constant rate, the speed would increase linearly with time, as shown in Figure 8.25, and the average speed during the time the propellant was being ejected would be $\bar{v} = v_f/2$. Taking this average speed and multiplying by that time gives

$$x_{a\text{-const}} \approx \bar{v}t_{max} = \left(v_f/2\right)t_{max} = \left(90.3/2\text{ km/s}\cdot 120\text{ s}\right)/2 = 5.4\cdot 10^6\text{ m.}$$

This approximate distance is bigger than our calculated answer, because in the calculation the velocity increases logarithmically in time until it reaches the value of 90.3 km/s. The approximation is about twice the calculated distance, giving us confidence that our answer at least has the right order of magnitude.

Figure 8.25 shows the exact solution for $v(t)$ (red curve). The distance traveled, x_f, is the area under the red curve. The blue line shows the case where constant acceleration leads to the same final velocity. As you can see, the area under the blue line is approximately twice that under the red curve. Since we just calculated the area under the blue line, $x_{a\text{-const}}$, and found it to be about twice as big as our calculated result, we gain confidence that we interpreted correctly.

SOLVED PROBLEM 8.5 Center of Mass of a Disk with a Hole in It

PROBLEM

Where is the center of mass of a disk with a rectangular hole in it (Figure 8.26)? The height of the disk is h =11.0 cm, and its radius is R =11.5 cm. The rectangular hole has a width w = 7.0 cm and a depth d = 8.0 cm. The right side of the hole is located so that its midpoint coincides with the central axis of the disk.

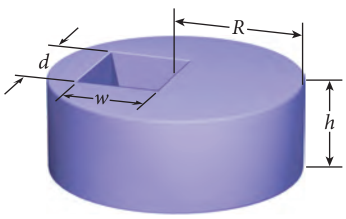


FIGURE 8.26 Three-dimensional view of a disk with a rectangular hole in it.

SOLUTION

THINK

One way to approach this problem is to write mathematical formulas that describe the three-dimensional geometry of the disk with a hole in it and then integrate over that volume to obtain the coordinates of the center of mass. If we did that, we would be faced with several difficult integrals. A simpler way to approach this problem is to think of the disk with a hole in it as a solid disk *minus* a rectangular hole. That is, we treat the hole as a solid object with a negative mass. Using the symmetry of the solid disk and of the hole, we can specify the coordinates of the center of mass of the solid disk and of the center of mass of the hole. We can then combine these coordinates, using equation 8.1, to find the center of mass of the disk with a hole in it.

SKETCH

Figure 8.27a shows a top view of the disk with a hole in it, with x- and y-axes assigned.

Figure 8.27b shows the two symmetry planes of the disk with a hole in it. One plane corresponds to the x-y plane, and the second plane is a plane along the x-axis and perpendicular to the x-y plane. The line where the two planes intersect is marked A.

RESEARCH

The center of mass must lie along the intersection of the two planes of symmetry. Therefore, we know that the center of mass can only be located along the x-axis. The center of mass for the disk without the hole is at the origin of the coordinate system, at $x_d = 0$, and the volume of the solid disk is $V_d = \pi R^2 h$. If the hole were a solid object with the same dimensions ($h = 11.0$ cm, $w = 7.0$ cm, and $d = 8.0$ cm), that object would have a volume of $V_h = hwd$. If this imagined solid object were located where the hole is, its center of mass would be in the middle of the hole, at $x_h = -3.5$ cm. We now multiply each of the volumes by ρ, the mass density of the material of the disk, to get the corresponding masses, and assign a negative mass to the hole. Then we use equation 8.1 to get the x-coordinate of the center of mass:

$$X = \frac{x_d V_d \rho - x_h V_h \rho}{V_d \rho - V_h \rho}. \quad \text{(i)}$$

This method of treating a hole as an object of the same shape and then using its volume in calculations, but with negative mass (or charge), is very common in atomic and subatomic physics. We will encounter it again when we explore atomic physics (Chapter 37) and nuclear and particle physics (Chapters 39 and 40).

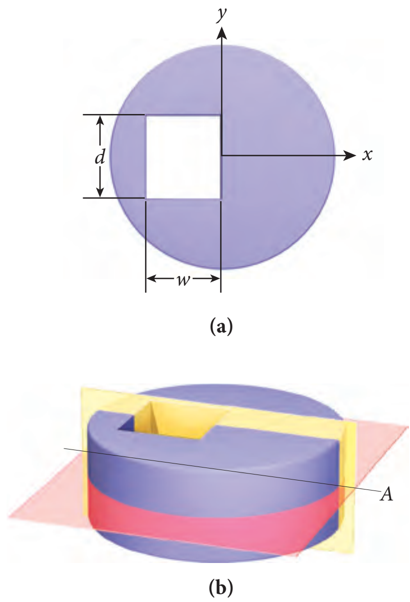


FIGURE 8.27 (a) Top view of the disk with a hole in it with a coordinate system assigned. (b) Symmetry planes of the disk with a hole in it.

SIMPLIFY

We can simplify equation (i) by realizing that $x_d = 0$ and that ρ is a common factor:

$$X = \frac{-x_h V_h}{V_d - V_h}.$$

Substituting the expressions we obtained above for V_d and V_h, we get

$$X = \frac{-x_h V_h}{V_d - V_h} = \frac{-x_h (hwd)}{\pi R^2 h - hwd} = \frac{-x_h wd}{\pi R^2 - wd}.$$

Defining the area of the disk in the x-y plane to be $A_d = \pi R^2$ and the area of the hole in the x-y plane to be $A_h = wd$, we can write

$$X = \frac{-x_h wd}{\pi R^2 - wd} = \frac{-x_h A_h}{A_d - A_h}.$$

CALCULATE

Inserting the given numbers, we find that the area of the disk is

$$A_d = \pi R^2 = \pi (11.5 \text{ cm})^2 = 415.475 \text{ cm}^2,$$

and the area of the hole is

$$A_h = wd = (7.0 \text{ cm})(8.0 \text{ cm}) = 56 \text{ cm}^2.$$

Therefore, the location of the center of mass of the disk with the hole in it (remember that $x_h = -3.5$ cm) is

$$X = \frac{-x_h A_h}{A_d - A_h} = \frac{-(-3.5 \text{ cm})(56 \text{ cm}^2)}{(415.475 \text{ cm}^2) - (56 \text{ cm}^2)} = 0.545239 \text{ cm}.$$

ROUND

Expressing our answer with two significant figures, we report the x-coordinate of the center of mass of the disk with a hole in it as

$$X = 0.55 \text{ cm}.$$

Continued—

DOUBLE-CHECK

This point is slightly to the right of the center of the solid disk, by a distance that is a small fraction of the radius of the disk. This result seems reasonable because taking material out of the disk to the left of $x = 0$ should shift the center of gravity to the right, just as we calculated.

MULTIPLE-CHOICE QUESTIONS

8.1 A man standing on frictionless ice throws a boomerang, which returns to him. Choose the correct statement:

a) Since the momentum of the man-boomerang system is conserved, the man will come to rest holding the boomerang at the same location from which he threw it.

b) It is impossible for the man to throw a boomerang in this situation.

c) It is possible for the man to throw a boomerang, but because he is standing on frictionless ice when he throws it, the boomerang cannot return.

d) The total momentum of the man-boomerang system is not conserved, so the man will be sliding backward holding the boomerang after he catches it.

8.2 When a bismuth-208 nucleus at rest decays, thallium-204 is produced, along with an alpha particle (helium-4 nucleus). The mass numbers of bismuth-208, thallium-204, and helium-4 are 208, 204, and 4, respectively. (The mass number represents the total number of protons and neutrons in the nucleus.) The kinetic energy of the thallium nucleus is

a) equal to that of the alpha particle.

b) less than that of the alpha particle.

c) greater than that of the alpha particle.

8.3 Two objects with masses m_1 and m_2 are moving along the x-axis in the positive direction with speeds v_1 and v_2, respectively, where v_1 is less than v_2. The speed of the center of mass of this system of two bodies is

a) less than v_1.

b) equal to v_1.

c) equal to the average of v_1 and v_2.

d) greater than v_1 and less than v_2.

e) greater than v_2.

8.4 An artillery shell is moving on a parabolic trajectory when it explodes in midair. The shell shatters into a very large number of fragments. Which of the following statements is true (select all that apply)?

a) The force of the explosion will increase the momentum of the system of fragments, and so the momentum of the shell is *not* conserved during the explosion.

b) The force of the explosion is an internal force and thus cannot alter the total momentum of the system.

c) The center of mass of the system of fragments will continue to move on the initial parabolic trajectory until the last fragment touches the ground.

d) The center of mass of the system of fragments will continue to move on the initial parabolic trajectory until the first fragment touches the ground.

e) The center of mass of the system of fragments will have a trajectory that depends on the number of fragments and their velocities right after the explosion.

8.5 An 80-kg astronaut becomes separated from his spaceship. He is 15.0 m away from it and at rest relative to it. In an effort to get back, he throws a 500-g object with a speed of 8.0 m/s in a direction away from the ship. How long does it take him to get back to the ship?

a) 1 s
b) 10 s
c) 20 s
d) 200 s
e) 300 s

8.6 You find yourself in the (realistic?) situation of being stuck on a 300-kg raft (including yourself) in the middle of a pond with nothing but a pile of 7-kg bowling balls and 55-g tennis balls. Using your knowledge of rocket propulsion, you decide to start throwing balls from the raft to move toward shore. Which of the following will allow you to reach the shore faster?

a) throwing the tennis balls at 35 m/s at a rate of 1 tennis ball per second

b) throwing the bowling balls at 0.5 m/s at a rate of 1 bowling ball every 3 s

c) throwing a tennis ball and a bowling ball simultaneously, with the tennis ball moving at 15 m/s and the bowling ball moving at 0.3 m/s, at a rate of 1 tennis ball and 1 bowling ball every 4 s

d) not enough information to decide

8.7 The figures show a high jumper using different techniques to get over the crossbar. Which technique would allow the jumper to clear the highest setting of the bar?

(a) (b) (c) (d)

8.8 The center of mass of an irregular rigid object is *always* located

a) at the geometrical center of the object.

b) somewhere within the object.

c) both of the above

d) none of the above

8.9 A catapult on a level field tosses a 3-kg stone a horizontal distance of 100 m. A second 3-kg stone tossed in an identical fashion breaks apart in the air into 2 pieces, one with a mass of 1 kg and one with a mass of 2 kg. Both of the pieces hit the ground at the same time. If the 1-kg piece lands a distance of 180 m away from the catapult, how far away from the catapult does the 2-kg piece land? Ignore air resistance.

a) 20 m
b) 60 m
c) 100 m
d) 120 m
e) 180 m

8.10 Two point masses are located in the same plane. The distance from mass 1 to the center of mass is 3.0 m. The distance from mass 2 to the center of mass is 1.0 m. What is m_1/m_2, the ratio of mass 1 to mass 2?

a) 3/4
b) 4/3
c) 4/7
d) 7/4
e) 1/3
f) 3/1

8.11 A cylindrical bottle of oil-and-vinegar salad dressing whose volume is 1/3 vinegar (ρ = 1.01 g/cm^3) and 2/3 oil (ρ = 0.910 g/cm^3) is at rest on a table. Initially, the oil and the vinegar are separated, with the oil floating on top of the vinegar. The bottle is shaken so that the oil and vinegar mix uniformly, and the bottle is returned to the table. How has the height of the center of mass of the salad dressing changed as a result of the mixing?

a) It is higher.
b) It is lower.
c) It is the same.
d) There is not enough information to answer this question.

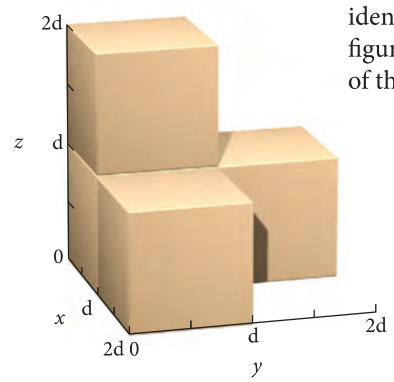
8.12 A one-dimensional rod has a linear density that varies with position according to the relationship $\lambda(x) = cx$, where c is a constant and $x = 0$ is the left end of the rod. Where do you expect the center of mass to be located?

a) the middle of the rod
b) to the left of the middle of the rod
c) to the right of the middle of the rod
d) at the right end of the rod
e) at the left end of the rod

QUESTIONS

8.13 A projectile is launched into the air. Part way through its flight, it explodes. How does the explosion affect the motion of the center of mass of the projectile?

8.14 Find the center of mass of the arrangement of uniform identical cubes shown in the figure. The length of the sides of the each cube is d.



8.15 A model rocket that has a horizontal range of 100 m is fired. A small explosion splits the rocket into two equal parts. What can you say about the points where the fragments land on the ground?

8.16 Can the center of mass of an object be located at a point outside the object, that is, at a point in space where no part of the object is located? Explain.

8.17 Is it possible for two masses to undergo a collision such that the system of two masses has more kinetic energy than the two separate masses had? Explain.

8.18 Prove that the center of mass of a thin metal plate in the shape of an equilateral triangle is located at the intersection of the triangle's altitudes by direct calculation and by physical reasoning.

8.19 A soda can of mass m and height L is filled with soda of mass M. A hole is punched in the bottom of the can to drain out the soda.

a) What is the center of mass of the system consisting of the can and the soda remaining in it when the level of soda in the can is h, where $0 < h < L$?

b) What is the minimum value of the center of mass as the soda drains out?

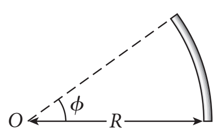
8.20 An astronaut of mass M is floating in space at a constant distance D from his spaceship when his safety line breaks. He is carrying a toolbox of mass $M/2$ that contains a big sledgehammer of mass $M/4$, for a total mass of $3M/4$. He can throw the items with a speed v relative to his final speed after each item is thrown. He wants to return to the spaceship as soon as possible.

a) To attain the maximum final speed, should the astronaut throw the two items together, or should he throw them one at a time? Explain.

b) To attain the maximum speed, is it best to throw the hammer first or the toolbox first, or does the order make no difference? Explain.

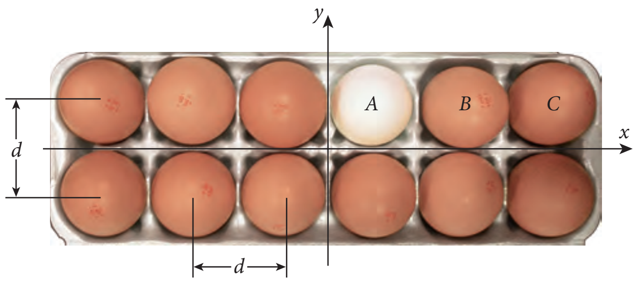
c) Find the maximum speed at which the astronaut can start moving toward the spaceship.

8.21 A metal rod with a length density (mass per unit length) λ is bent into a circular arc of radius R and subtending a total angle of ϕ, as shown in the figure. What is the distance of the center of mass of this arc from O as a function of the angle ϕ? Plot this center-of-mass coordinate as a function of ϕ.



8.22 The carton shown in the figure is filled with a dozen eggs, each of mass m. Initially, the center of mass of the eggs is at the center of the carton, which is the same point as the origin of the Cartesian coordinate system shown. Where is the center of mass of the remaining eggs, in terms of the egg-to-egg distance d, in each of the following situations? Neglect the mass of the carton.

Continued—



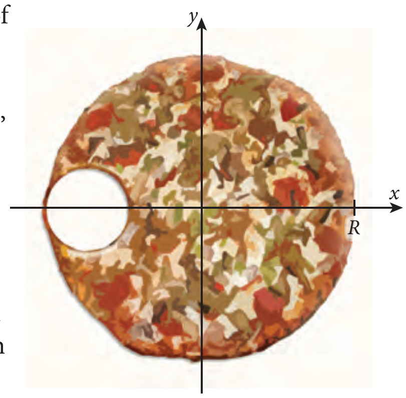
a) Only egg A is removed.

b) Only egg B is removed.

c) Only egg C is removed.

d) Eggs A, B and C are removed.

8.23 A circular pizza of radius R has a circular piece of radius $R/4$ removed from one side, as shown in the figure. Where is the center of mass of the pizza with the hole in it?



8.24 Suppose you place an old-fashioned hourglass, with sand in the bottom, on a very sensitive analytical balance to determine its mass. You then turn it over (handling it with very clean gloves) and place it back on the balance. You want to predict whether the reading on the balance will be less than, greater than, or the same as before. What do you need to calculate to answer this question? Explain carefully what should be calculated and what the results would imply. You do not need to attempt the calculation.

PROBLEMS

A blue problem number indicates a worked-out solution is available in the Student Solutions Manual. One • and two •• indicate increasing level of problem difficulty.

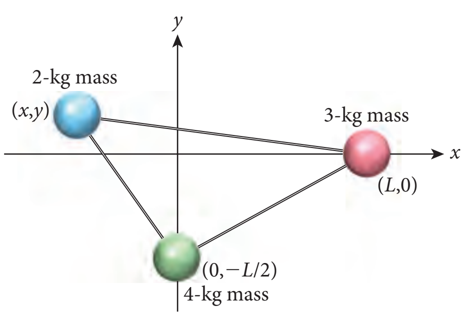
Section 8.1

8.25 Find the following center-of-mass information about objects in the Solar System. You can look up the necessary data on the Internet or in the tables in Chapter 12 of this book. Assume spherically symmetrical mass distributions for all objects under consideration.

a) Determine the distance from the center of mass of the Earth-Moon system to the geometric center of Earth.

b) Determine the distance from the center of mass of the Sun-Jupiter system to the geometric center of the Sun.

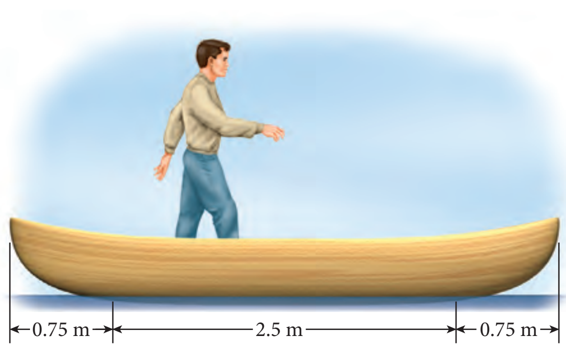
•8.26 The coordinates of the center of mass for the extended object shown in the figure are $(L/4, -L/5)$. What are the coordinates of the 2-kg mass?



•8.27 Young acrobats are standing still on a circular horizontal platform suspended at the center. The origin of the two-dimensional Cartesian coordinate system is assumed to be at the center of the platform. A 30-kg acrobat is located at (3m, 4m), and a 40-kg acrobat is located at (–2m, –2m). Assuming that the acrobats stand still in their positions, where must a 20-kg acrobat be located so that the center of mass of the system consisting of the three acrobats is at the origin and the platform is balanced?

Section 8.2

8.28 A man with a mass of 55 kg stands up in a 65-kg canoe of length 4.0 m floating on water. He walks from a point 0.75 m from the back of the canoe to a point 0.75 m from the front of the canoe. Assume negligible friction between the canoe and the water. How far does the canoe move?

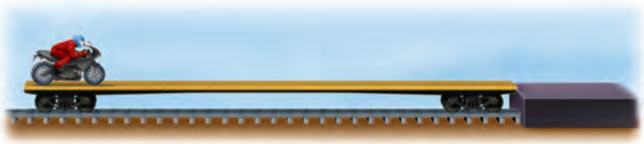


8.29 A toy car of mass 2.0 kg is stationary, and a child rolls a toy truck of mass 3.5 kg straight toward it with a speed of 4.0 m/s.

a) What is the velocity of the center of mass of the system consisting of the two toys?

b) What are the velocities of the truck and the car with respect to the center of mass of the system consisting of the two toys?

8.30 A motorcycle stunt rider plans to start from one end of a railroad flatcar, accelerate toward the other end of the car, and jump from the flatcar to a platform. The motorcycle and rider have a mass of 350. kg and a length of 2.00 m. The flatcar has a mass of 1500. kg and a length of 20.0 m. Assume that there is negligible friction between the flatcar's wheels and the rails and that the motorcycle and rider can move through the air with negligible resistance. The flatcar is initially touching the platform. The promoters of the event have asked you how far the flatcar will be from the platform when the stunt rider reaches the end of the flatcar. What is your answer?



•**8.31** Starting at rest, two students stand on 10-kg sleds, which point away from each other on ice, and they pass a 5-kg medicine ball back and forth. The student on the left has a mass of 50 kg and can throw the ball with a relative speed of 10 m/s. The student on the right has a mass of 45 kg and can throw the ball with a relative speed of 12 m/s. (Assume there is no friction between the ice and the sleds and no air resistance.)

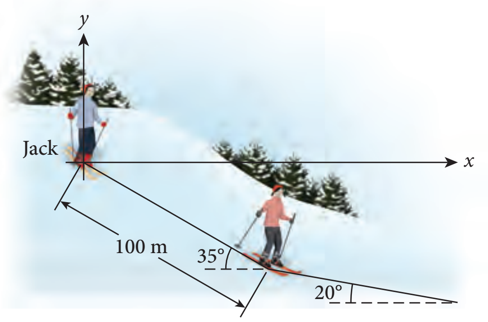
a) If the student on the left throws the ball horizontally to the student on the right, how fast is the student on the left moving right after the throw?

b) How fast is the student on the right moving right after catching the ball?

c) If the student on the right passes the ball back, how fast will the student on the left be moving after catching the pass from the student on the right?

(d) How fast is the student on the right moving after the pass?

•**8.32** Two skiers, Annie and Jack, start skiing from rest at different points on a hill at the same time. Jack, with mass 88 kg, skis from the top of the hill down a steeper section with an angle of inclination of 35°. Annie, with mass 64 kg, starts from a lower point and skis a less steep section, with an angle of inclination of 20°. The length of the steeper section is 100 m. Determine the acceleration, velocity, and position vectors of the combined center of mass for Annie and Jack as a function of time before Jack reaches the less steep section.



•**8.33** Many nuclear collisions studied in laboratories are analyzed in a frame of reference relative to the laboratory. A proton, with a mass of $1.6605 \cdot 10^{-27}$ kg and traveling at a speed of 70% of the speed of light, *c,* collides with a tin-116 (^{116}Sn) nucleus with a mass of $1.9096 \cdot 10^{-25}$ kg. What is the speed of the center of mass with respect to the laboratory frame? Answer in terms of *c,* the speed of light.

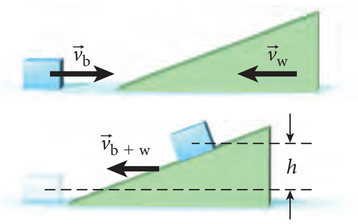
•**8.34** A system consists of two particles. Particle 1 with mass 2.0 kg is located at (2.0 m, 6.0 m) and has a velocity of (4.0 m/s, 2.0 m/s). Particle 2 with mass 3.0 kg is located at (4.0 m, 1.0 m) and has a velocity of (0, 4.0 m/s).

a) Determine the position and the velocity of the center of mass of the system.

b) Sketch the position and velocity vectors for the individual particles and for the center of mass.

•**8.35** A fire hose 4.0 cm in diameter is capable of spraying water at a velocity of 10 m/s. For a continuous horizontal flow of water, what horizontal force should a fireman exert on the hose to keep it stationary?

••**8.36** A block of mass $m_b = 1.2$ kg slides to the right at a speed of 2.5 m/s on a frictionless horizontal surface, as shown in the figure. It "collides" with a wedge of mass m_w, which moves to the left at a speed of 1.1 m/s. The wedge is shaped so that the block slides seamlessly up the Teflon (frictionless!) surface, as the two come together. Relative to the horizontal surface, block and wedge are moving with a common velocity v_{b+w} at the instant the block stops sliding up the wedge.



a) If the block's center of mass rises by a distance $h = 0.37$ m, what is the mass of the wedge?

b) What is v_{b+w}?

Section 8.3

8.37 One important characteristic of rocket engines is the specific impulse, which is defined as the total impulse (time integral of the thrust) per unit ground weight of fuel/oxidizer expended. (The use of weight, instead of mass, in this definition is due to purely historical reasons.)

a) Consider a rocket engine operating in free space with an exhaust nozzle speed of v. Calculate the specific impulse of this engine.

b) A model rocket engine has a typical exhaust speed of $v_{toy} = 800$ m/s. The best chemical rocket engines have exhaust speeds of approximately $v_{chem} = 4.00$ km/s. Evaluate and compare the specific impulse values for these engines.

•**8.38** An astronaut is performing a space walk outside the International Space Station. The total mass of the astronaut with her space suit and all her gear is 115 kg. A small leak develops in her propulsion system and 7 g of gas are ejected each second into space with a speed of 800 m/s. She notices the leak 6 s after it starts. How much will the gas leak have caused her to move from her original location in space by that time?

•**8.39** A rocket in outer space has a payload of 5190.0 kg and $1.551 \cdot 10^5$ kg of fuel. The rocket can expel propellant at a speed of 5.600 km/s. Assume that the rocket starts from rest, accelerates to its final velocity, and then begins its trip. How long will it take the rocket to travel a distance of $3.82 \cdot 10^5$ km (approximately the distance between Earth and Moon)?

••**8.40** A uniform chain with a mass of 1.32 kg per meter of length is coiled on a table. One end is pulled upward at a constant rate of 0.47 m/s.

a) Calculate the net force acting on the chain.

b) At the instant when 0.15 m of the chain has been lifted off the table, how much force must be applied to the end being raised?

••**8.41** A spacecraft engine creates 53.2 MN of thrust with a propellant velocity of 4.78 km/s.

a) Find the rate (dm/dt) at which the propellant is expelled.

b) If the initial mass is $2.12 \cdot 10^6$ kg and the final mass is $7.04 \cdot 10^4$ kg, find the final speed of the spacecraft (assume the initial speed is zero and any gravitational fields are small enough to be ignored).

c) Find the average acceleration till burnout (the time at which the propellant is used up; assume the mass flow rate is constant until that time).

••**8.42** A cart running on frictionless air tracks is propelled by a stream of water expelled by a gas-powered pressure washer stationed on the cart. There is a 1.0-m^3 water tank on the cart to provide the water for the pressure washer. The mass of the cart, including the operator riding it, the pressure washer with its fuel, and the empty water tank, is 400. kg. The water can be directed, by switching a valve, either backward or forward. In both directions, the pressure washer ejects 200 L of water per min with a muzzle velocity of 25.0 m/s.

a) If the cart starts from rest, after what time should the valve be switched from backward (forward thrust) to forward (backward thrust) for the cart to end up at rest?

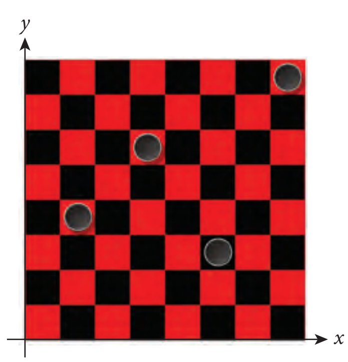
b) What is the mass of the cart at that time, and what is its velocity? (*Hint:* It is safe to neglect the decrease in mass due to the gas consumption of the gas-powered pressure washer!)

c) What is the thrust of this "rocket"?

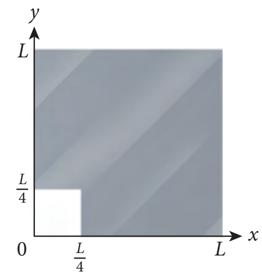
d) What is the acceleration of the cart immediately before the valve is switched?

Section 8.4

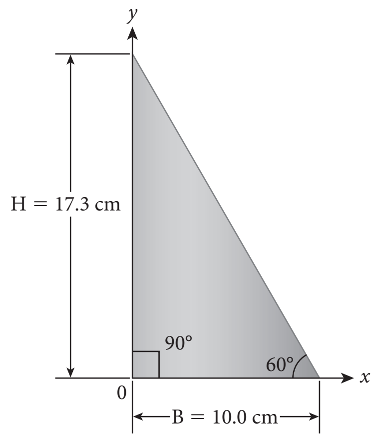
8.43 A 32-cm-by-32-cm checkerboard has a mass of 100 g. There are four 20-g checkers located on the checkerboard, as shown in the figure. Relative to the origin located at the bottom left corner of the checkerboard, where is the center of mass of the checkerboard-checkers system?



•**8.44** A uniform, square metal plate with side $L = 5.70$ cm and mass 0.205 kg is located with its lower left corner at $(x, y) = (0, 0)$, as shown in the figure. A square with side $L/4$ and its lower left edge located at $(x, y) = (0, 0)$ is removed from the plate. What is the distance from the origin of the center of mass of the remaining plate?

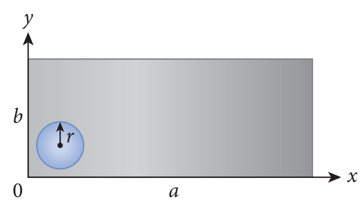


•**8.45** Find the x- and y-coordinates of the center of mass of the flat triangular plate of height $H = 17.3$ cm and base $B = 10.0$ cm shown in the figure.



•**8.46** The density of a 1.0-m long rod can be described by the linear density function $\lambda(x) = 100\text{ g/m} + 10.0x\text{ g/m}^2$. One end of the rod is positioned at $x = 0$ and the other at $x = 1$ m. Determine (a) the total mass of the rod, and (b) the center-of-mass coordinate.

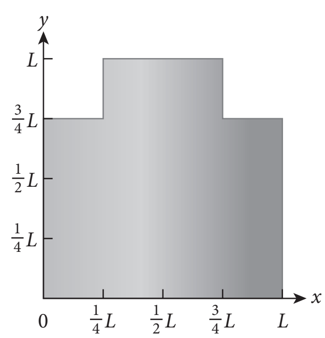
•**8.47** A thin rectangular plate of uniform area density $\sigma_1 = 1.05\text{ kg/m}^2$ has a length $a = 0.600$ m and a width $b = 0.250$ m. The lower left corner is placed at the origin, $(x, y) = (0, 0)$. A circular hole of radius $r = 0.048$ m with center at $(x, y) = (0.068\text{ m}, 0.068\text{ m})$ is cut in the plate. The hole is plugged with a disk of the same radius that is composed of another material of uniform area density $\sigma_2 = 5.32\text{ kg/m}^2$. What is the distance from the origin of the resulting plate's center of mass?



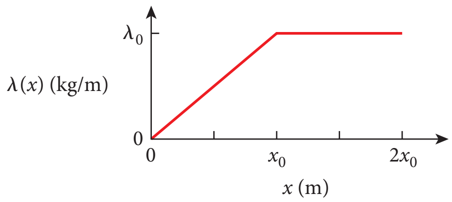
•**8.48** A uniform, square metal plate with side $L = 5.70$ cm and mass 0.205 kg is located with its lower left corner at $(x, y) = (0, 0)$ as shown in the figure. Two squares with side length $L/4$ are removed from the plate.

a) What is the x-coordinate of the center of mass?

b) What is the y-coordinate of the center of mass?



••**8.49** The linear mass density, $\lambda(x)$, for a one-dimensional object is plotted in the graph. What is the location of the center of mass for this object?



Additional Problems

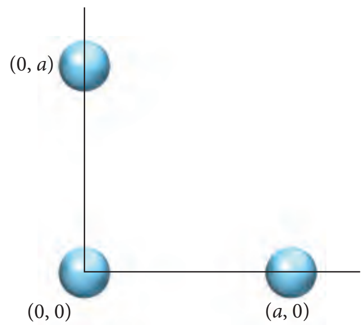
8.50 A 750-kg cannon fires a 15-kg projectile with a speed of 250 m/s with respect to the muzzle. The cannon is on wheels and can recoil with negligible friction. Just after the cannon fires the projectile, what is the speed of the projectile with respect to the ground?

8.51 The distance between a carbon atom (m = 12 u) and an oxygen atom (m = 16 u) in a carbon monoxide (CO) molecule is $1.13 \cdot 10^{-10}$ m. How far from the carbon atom is the center of mass of the molecule? (1 u = 1 atomic mass unit.)

8.52 One method of detecting extrasolar planets involves looking for indirect evidence of a planet in the form of wobbling of its star about the star-planet system's center of mass. Assuming that the Solar System consisted mainly of the Sun and Jupiter, how much would the Sun wobble? That is, what back-and-forth distance would it move due to its rotation about the center of mass of the Sun-Jupiter system? How far from the center of the Sun is that center of mass?

8.53 The USS *Montana* is a massive battleship with a weight of 136,634,000 lb. It has twelve 16-inch guns, which are capable of firing 2700-lb projectiles at a speed of 2300 ft/s. If the battleship fires three of these guns (in the same direction), what is the recoil velocity of the ship?

8.54 Three identical balls of mass m are placed in the configuration shown in the figure. Find the location of the center of mass.



8.55 Sam (61 kg) and Alice (44 kg) stand on an ice rink, providing them with a nearly frictionless surface to slide on. Sam gives Alice a push, causing her to slide away at a speed (with respect to the rink) of 1.20 m/s.

a) With what speed does Sam recoil?

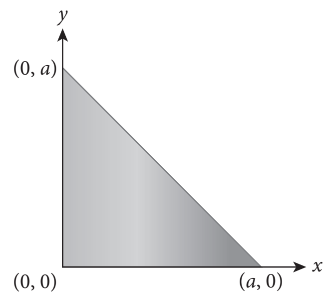
b) Calculate the change in the kinetic energy of the Sam-Alice system.

c) Energy cannot be created or destroyed. What is the source of the final kinetic energy of this system?

8.56 A baseball player uses a bat with mass m_{bat} to hit a ball with mass m_{ball}. Right before he hits the ball, the bat's initial velocity is 35 m/s, and the ball's initial velocity is –30 m/s (the positive direction is along the positive x-axis). The bat and ball undergo a one-dimensional elastic collision. Find the speed of the ball after the collision. Assume that m_{bat} is much greater than m_{ball}, so the center of mass of the two objects is essentially at the bat.

8.57 A student with a mass of 40 kg can throw a 5-kg ball with a relative speed of 10.0 m/s. The student is standing at rest on a cart of mass 10 kg that can move without friction. If the student throws the ball horizontally, what will the velocity of the ball with respect to the ground be?

•**8.58** Find the location of the center of mass of a two-dimensional sheet of constant density σ that has the shape of an isosceles triangle (see the figure).



•**8.59** A rocket consists of a payload of 4390.0 kg and $1.761 \cdot 10^5$ kg of fuel. Assume that the rocket starts from rest in outer space, accelerates to its final velocity, and then begins its trip. What is the speed at which the propellant must be expelled to make the trip from the Earth to the Moon, a distance of $3.82 \cdot 10^5$ km, in 7.0 h?

•**8.60** A 350-kg cannon, sliding freely on a frictionless horizontal plane at a speed of 7.5 m/s, shoots a 15-kg cannonball at an angle of 55° above the horizontal. The velocity of the ball relative to the cannon is such that when the shot occurs, the cannon stops cold. What is the velocity of the ball relative to the cannon?

•**8.61** The Saturn V rocket, which was used to launch the *Apollo* spacecraft on their way to the Moon, has an initial mass $M_0 = 2.8 \cdot 10^6$ kg and a final mass $M_1 = 0.8 \cdot 10^6$ kg and burns fuel at a constant rate for 160. s. The speed of the exhaust relative to the rocket is about v = 2700. m/s.

a) Find the upward acceleration of the rocket, as it lifts off the launch pad (while its mass is the initial mass).

b) Find the upward acceleration of the rocket, just as it finishes burning its fuel (when its mass is the final mass).

c) If the same rocket were fired in deep space, where there is negligible gravitational force, what would be the net change in the speed of the rocket during the time it was burning fuel?

•**8.62** Find the location of the center of mass for a one-dimensional rod of length L and of linear density $\lambda(x) = cx$, where c is a constant. (*Hint:* You will need to calculate the mass in terms of c and L.)

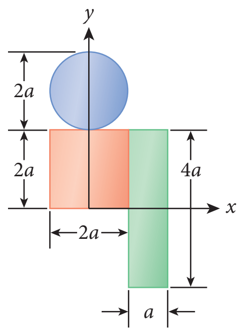
•**8.63** Find the center of mass of a rectangular plate of length 20 cm and width 10 cm. The mass density varies linearly along the length. At one end, it is 5 g/cm^2; at the other end, it is 20 g/cm^2.

•**8.64** A uniform log of length 2.50 m has a mass of 91 kg and is floating in water. Standing on this log is a 72-kg man, located 22 cm from one end. On the other end is his daughter (m = 20 kg), standing 1 m from the end.

a) Find the center of mass of this system.

b) If the father jumps off the log backward away from his daughter (v = 3.14 m/s), what is the initial speed of log and child?

8.65 A sculptor has commissioned you to perform an engineering analysis of one of his works, which consists of regularly shaped metal plates of uniform thickness and density, welded together as shown in the figure. Using the intersection of the two axes shown as the origin of the coordinate system, determine the Cartesian coordinates of the center of mass of this piece.



•**8.66** A jet aircraft is traveling at 223 m/s in horizontal flight. The engine takes in air at a rate of 80.0 kg/s and burns fuel at a rate of 3.00 kg/s. The exhaust gases are ejected at 600. m/s relative to the speed of the aircraft. Find the thrust of the jet engine.

•**8.67** A bucket is mounted on a skateboard, which rolls across a horizontal road with no friction. Rain is falling vertically into the bucket. The bucket is filled with water, and the total mass of the skateboard, bucket, and water is M = 10 kg. The rain enters the top of the bucket and simultaneously leaks out of a hole at the bottom of the bucket at equal rates of λ = 0.10 kg/s. Initially, bucket and skateboard are moving at a speed of v_0. How long will it take before the speed is reduced by half?

•**8.68** A 1000-kg cannon shoots a 30-kg shell at an angle of 25° above the horizontal and a speed of 500 m/s. What is the recoil velocity of the cannon?

•**8.69** Two masses, m_1 = 2.0 kg and m_2 = 3.0 kg, are moving in the xy-plane. The velocity of their center of mass and the velocity of mass 1 relative to mass 2 are given by the vectors v_{cm} = (–1.0, +2.4) m/s and v_{rel} = (+5.0, +1.0) m/s. Determine

a) the total momentum of the system

b) the momentum of mass 1, and

c) the momentum of mass 2.

••**8.70** You are piloting a spacecraft whose total mass is 1000 kg and attempting to dock with a space station in deep space. Assume for simplicity that the station is stationary, that your spacecraft is moving at 1.0 m/s toward the station, and that both are perfectly aligned for docking. Your spacecraft has a small retro-rocket at its front end to slow its approach, which can burn fuel at a rate of 1.0 kg/s and with an exhaust velocity of 100 m/s relative to the rocket. Assume that your spacecraft has only 20 kg of fuel left and sufficient distance for docking.

a) What is the initial thrust exerted on your spacecraft by the retro-rocket? What is the thrust's direction?

b) For safety in docking, NASA allows a maximum docking speed of 0.02 m/s. Assuming you fire the retro-rocket from time t = 0 in one sustained burst, how much fuel (in kilograms) has to be burned to slow your spacecraft to this speed relative to the space station?

c) How long should you sustain the firing of the retro-rocket?

d) If the space station's mass is 500,000 kg (close to the value for the ISS), what is the final velocity of the station after the docking of your spacecraft, which arrives with a speed of 0.02 m/s?

••**8.71** A chain whose mass is 3.0 kg and length is 5.0 m is held at one end so that the bottom end of the chain just touches the floor (see the figure). The top end of the chain is released. What is the force exerted by the chain on the floor just as the last link of the chain lands on the floor?