

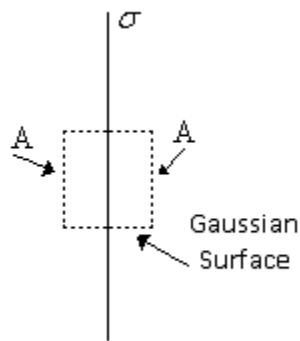
Experiment #4 Gauss's Law Prelab Hints

This lab and prelab will make extensive use of Potentials and Gauss's Law, and using calculus to recast the electric field in terms of potential ... The intent of this is to provide some insight and assistance to help you solve the problems for this prelab.

Prelab Question 1: **In Prelab questions 2 and 3, the electric fields predicted by Gauss' Law for the infinite planar and infinite cylindrical charge symmetries will be given to you. Using these results, you will recast the electric fields and charge densities as potentials. Explain why this recasting is necessary.**

From Lab 3, we plotted out the Electric Field using the carbon paper; it is very easy to measure the Electric Potential using a DMM – but difficult to directly measure the Electric Field. We in turn used the plot of the Electric Potential lines to effectively draw in the Electric Field. Similarly, if we can recast everything in terms of the Electric Potential, it will make the experiment easier to perform and compare to a theoretical value.

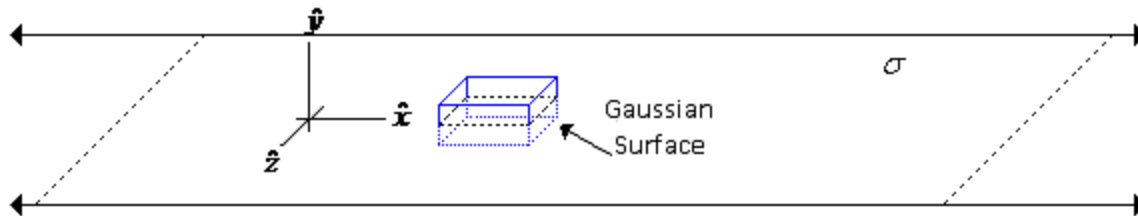
Prelab Question 2: **Consider an infinite plane with a uniform surface charge density of σ .**



From Gauss' Law, the electric field of an infinite plane is:

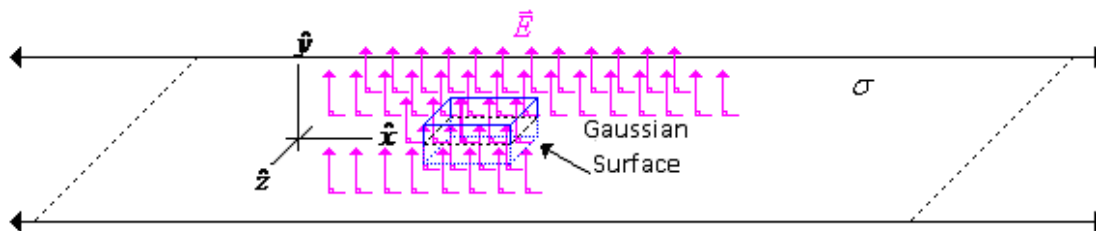
$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$$

First, I had to re-draw the picture, because the one above was a little confusing to me – in the sense of “how is that an infinite plane?”.



Notice that the electric field conveniently points in the $+\hat{y}$ and $-\hat{y}$ direction. Also, notice that we “conveniently” chose the Gaussian Surface to have four of its surfaces with normals perpendicular to the electric field and two of its surfaces with normals in all the same direction as the electric field (i.e., parallel).

If we draw in the Gaussian Surface (in blue), the charged plane (in black) and a coordinate axis frame, we can start to describe the electric field being generated.



(LMAO – and I was just complaining about “bad pictures”...)

Using Gauss’ Law:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{IN}}{\epsilon_o}$$

First, let’s consider the Right Hand Side (RHS) of Gauss’ Law:

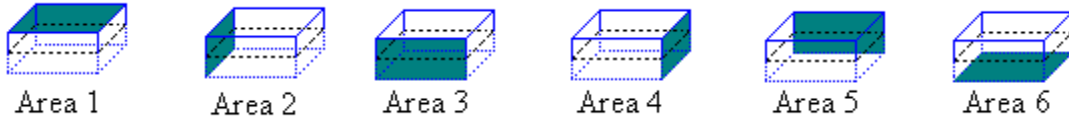
(Notice that since σ is a “charge density” it has units of $[C/m^2]$, so we need to multiply it times the area we are considering to get simply the charge enclosed in that area.)

$$\frac{Q_{IN}}{\epsilon_o} = \frac{\sigma A}{\epsilon_o}$$

Next, let’s consider the Left Hand Side (LHS) of Gauss’ Law:

$$\oint_S \vec{E} \cdot d\vec{A}$$

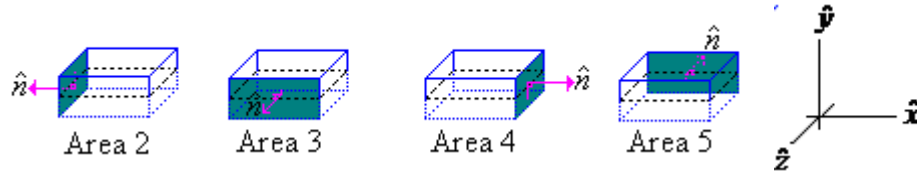
First, if we look simply at the Gaussian Surface, we can see that there are 6 effective areas that we need to consider for the LHS of Gauss’ Law.



This means that the LHS is basically:

$$\oint_S \vec{E} \cdot d\vec{A} = \int_{S1} \vec{E} \cdot d\vec{A}_1 + \int_{S2} \vec{E} \cdot d\vec{A}_2 + \int_{S3} \vec{E} \cdot d\vec{A}_3 + \int_{S4} \vec{E} \cdot d\vec{A}_4 + \int_{S5} \vec{E} \cdot d\vec{A}_5 + \int_{S6} \vec{E} \cdot d\vec{A}_6$$

Let's start with Areas 2-5, since those are "easy":



For Areas 2-5, notice that the area segment normal for each of these areas is pointing perpendicular to the direction of the electric field. This means that the angle between these area segment normals and the electric field will always be 90 degrees. **[PAY ATTENTION TO DIRECTION:** Based upon our coordinate frame above, $\hat{n}_{Area2}$ is pointing in the $-\hat{x}$ direction, $\hat{n}_{Area3}$ is pointing in the $+\hat{z}$ direction, $\hat{n}_{Area4}$ is pointing in the $+\hat{x}$ direction, and $\hat{n}_{Area5}$ is pointing in the $-\hat{z}$ direction.] This is convenient, because then the dot product will always be zero for all these surfaces. ☺

If we consider an area segment of surface 2, we see that the infinitesimal area is given as:

$$dA_2 = dz * dy ; \forall y : 0 \leq y \leq h; \forall z : 0 \leq z \leq w$$

Likewise, following in suit, for surfaces 3-5, we see that their infinitesimal areas are given as:

$$dA_3 = dx * dy ; \forall x : 0 \leq x \leq l; \forall y : 0 \leq y \leq h$$

$$dA_4 = dz * dy ; \forall y : 0 \leq y \leq h; \forall z : 0 \leq z \leq w$$

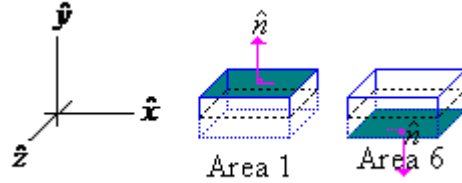
$$dA_5 = dx * dy ; \forall x : 0 \leq x \leq l; \forall y : 0 \leq y \leq h$$

This means we can now evaluate the integrals for surface 2 (S2) through surface 5 (S5):

$$\begin{aligned} \int_{S2} \vec{E} \cdot d\vec{A}_2 &= \int_0^h \int_0^w E \, dy \, dz (\pm \hat{y} \cdot -\hat{x}) \cos(\phi) = E(0) \cos(90^\circ) \int_0^h dy \int_0^w dz = 0 \\ \int_{S3} \vec{E} \cdot d\vec{A}_3 &= \int_0^l \int_0^w E \, dx \, dy (\pm \hat{y} \cdot \hat{z}) \cos(\phi) = E(0) \cos(90^\circ) \int_0^l dx \int_0^w dz = 0 \\ \int_{S4} \vec{E} \cdot d\vec{A}_4 &= \int_0^h \int_0^w E \, dy \, dz (\pm \hat{y} \cdot +\hat{x}) \cos(\phi) = E(0) \cos(90^\circ) \int_0^h dy \int_0^w dz = 0 \end{aligned}$$

$$\int_{S5} \vec{E} \cdot d\vec{A}_5 = \int_0^l \int_0^w E dx dy (\pm \hat{y} \cdot -\hat{z}) \cos(\phi) = E(0) \cos(90^\circ) \int_0^l ds \int_0^w dz = 0$$

Next, let's continue with Areas 1 and 6:



For Areas 1 and 6, notice that the area segment normal for each of these areas is pointing parallel (and in the same direction) to the direction of the electric field. This means that the angle between these area segment normals and the electric field will always be 0 degrees. **[PAY ATTENTION TO DIRECTION:** Based upon our coordinate frame above, $\hat{n}_{Area1}$ is pointing in the $+\hat{y}$ direction, and $\hat{n}_{Area6}$ is pointing in the $-\hat{y}$ direction.]

This is convenient, because then the dot product will always be one for all these surfaces. ☺

If we consider an area segment of surface 1, we see that the infinitesimal area is given as:

$$dA_1 = dz * dx ; \forall x : 0 \leq x \leq l ; \forall z : 0 \leq z \leq w$$

Likewise, following in suit, for surface 6, we see that its infinitesimal area is also given as:

$$dA_6 = dz * dx ; \forall x : 0 \leq x \leq l ; \forall y : 0 \leq y \leq w$$

This means the integrals are practically identical, and we can now evaluate the integrals for surface 1 (S1) and surface 6 (S6):

$$\begin{aligned} \int_{S1} \vec{E} \cdot d\vec{A}_1 &= \int_0^l \int_0^w E dx dz (+\hat{y} \cdot +\hat{y}) \cos(\phi) = E(1) \cos(0^\circ) \int_0^l dy \int_0^w dz = E(lw) \\ \int_{S6} \vec{E} \cdot d\vec{A}_6 &= \int_0^l \int_0^w E dx dz (-\hat{y} \cdot -\hat{y}) \cos(\phi) = E(1) \cos(0^\circ) \int_0^l dy \int_0^w dz = E(lw) \end{aligned}$$

This means that the LHS of Gauss' Law is simply:

$$\begin{aligned} \oint_S \vec{E} \cdot d\vec{A} &= \int_{S1} \vec{E} \cdot d\vec{A}_1 + \int_{S2} \vec{E} \cdot d\vec{A}_2 + \int_{S3} \vec{E} \cdot d\vec{A}_3 + \int_{S4} \vec{E} \cdot d\vec{A}_4 + \int_{S5} \vec{E} \cdot d\vec{A}_5 + \int_{S6} \vec{E} \cdot d\vec{A}_6 \\ &= E A_1 + 0 + 0 + 0 + 0 + E A_6 = 2EA \end{aligned}$$

Now, setting the LHS equal to the RHS:

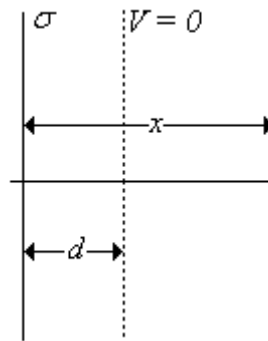
$$2EA = \frac{\sigma A}{\epsilon_0}$$

Thus putting everything together and solving for the Electric Field:

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \hat{n}$$

OK, this looks good ... let's continue with the rest of the problem.

The uniformly charged infinite plane has equipotential surfaces that are parallel to it. Since we can pick anywhere we want to be zero potential, we will call the equipotential at $x = d$ the $V = 0$ equipotential, as shown in the figure below:



a.) Using the relationship between potential and electric field:

$$\Delta V = - \int_d^x \vec{E} \cdot d\vec{s}$$

find the potential difference between the points $x = d$ and $x = x$.

First, let's plug in the electric field we found above and the infinitesimal displacement (pointing in the same direction as the electric field) **[IMPORTANT: Note that if we went with the bounds of $x \rightarrow d$ for the integral, instead of $d \rightarrow x$ (as we have) ... then the displacement vector wouldn't have the same sign and would be pointing in the opposite direction as the electric field! This would result in a -1 for the dot product; but still gives us the same answer. ☺]**:

$$\Delta V = - \int_d^x \frac{\sigma}{2\epsilon_0} dn (\hat{n} \bullet \hat{n}) = - \frac{\sigma}{2\epsilon_0} \int_d^x dn (1) = - \frac{\sigma}{2\epsilon_0} \int_d^x dn$$

Next, the trick is to use the negative sign to “flip the ratio”:

$$\Delta V = - \frac{\sigma}{2\epsilon_0} (n |_d^x = - \frac{\sigma}{2\epsilon_0} (x - d) = \frac{\sigma}{2\epsilon_0} (d - x)$$

This means we can determine the potential at any arbitrary distance x from the surface of the charged plane.

$$V(x) = - \frac{\sigma}{2\epsilon_0} (x - d)$$

b.) The potential you found in part (a) has the form:

$$V = mx + b$$

where m and b depend on σ . Use the fact that at $x = 0$, $V = V_0$ to find m and b in terms of V_0 instead of σ . In this lab, you will compare the theoretical values of m and b to the experimentally measured values of m and b .

It's easy to see that the form mentioned here and the form we found do match, where:

$$V = mx + b = \left(- \frac{\sigma}{2\epsilon_0} \right) x + \left(\frac{\sigma d}{2\epsilon_0} \right)$$

$$m = - \frac{\sigma}{2\epsilon_0} \text{ and } b = \frac{\sigma d}{2\epsilon_0}$$

If we apply the boundary condition at $x = 0$:

$$V(0) = \frac{\sigma}{2\epsilon_0} (d - 0) = \frac{\sigma d}{2\epsilon_0} = V_0$$

Now, solve for $\frac{\sigma}{2\epsilon_0}$ and plug this back into our generic $V(x)$ equation:

$$\frac{\sigma}{2\epsilon_o} = \frac{V_o}{d}$$

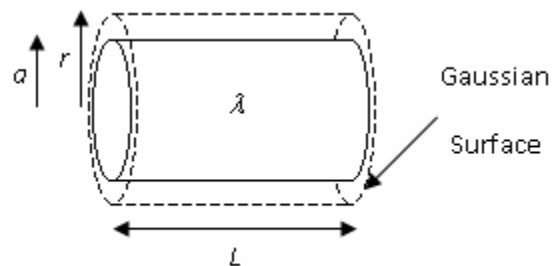
$$V(x) = \left(-\frac{V_o}{d}\right)x + (V_o)$$

This means, now:

$$m = -\frac{V_o}{d} \text{ and } b = V_o$$

We have now successfully recast the electric field of an infinite charged plane into potential (something we can measure and compare in the lab).

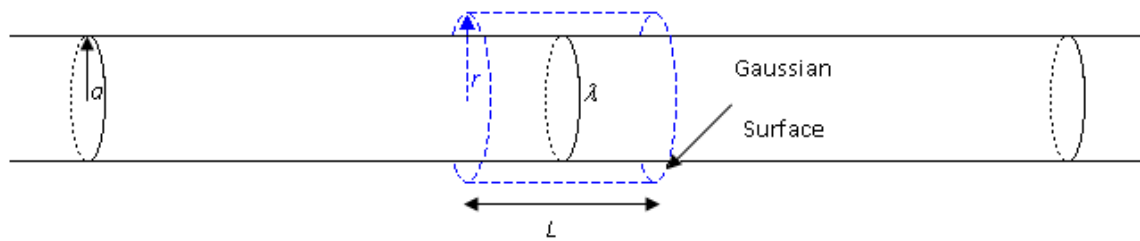
Prelab Question 3: **Consider an infinite cylinder of radius a with a uniform charge per length L of λ .**



From Gauss's Law, the electric field of an infinite cylinder at a radial distance r from its center is:

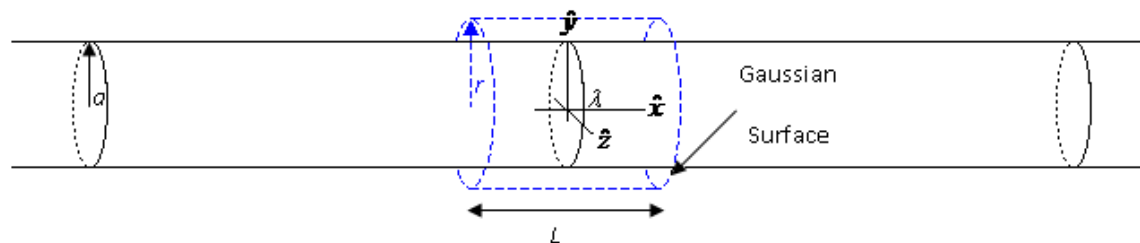
$$\vec{E} = \frac{2k\lambda}{r} \hat{r}$$

First, I had to re-draw the picture, again, because the one above was a little confusing to me – in the sense of “how is that an infinite cylinder?”. **WHO DRAWS THESE PICTURES?!?**

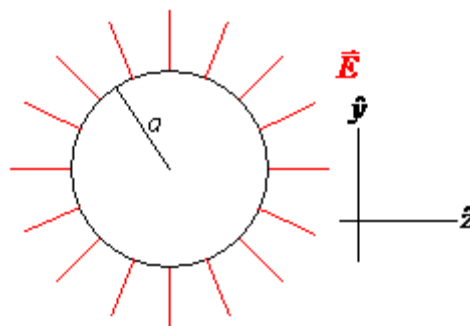


Notice that the electric field direction is given as radiating off of the infinite cylinder. Also, notice that we “conveniently” chose the Gaussian Surface to have two of its surfaces with normals perpendicular to the radiating electric field and one of its surfaces with normals in all the same direction as the electric field (i.e., parallel).

If we draw in the Gaussian Surface (in blue), the charged cylinder (in black) and a coordinate axis frame, we can start to describe the electric field being generated.



If we take an “edge on” view of this, it is very easy to see that the electric field sits in the Y-Z plane and radiates away from the cylinder (this is given as the direction $+\hat{r}$).



Using Gauss' Law:

$$\oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{IN}}{\epsilon_o}$$

First, let's consider the Right Hand Side (RHS) of Gauss' Law:

$$\frac{Q_{IN}}{\epsilon_o} = \frac{\lambda L}{\epsilon_o}$$

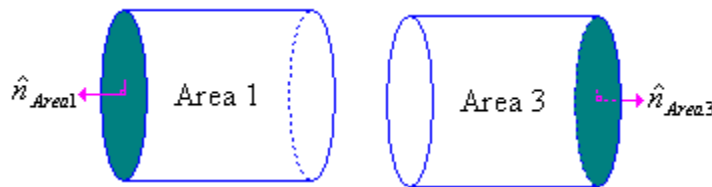
Next, let's consider the Left Hand Side (LHS) of Gauss' Law:

$$\oint_S \vec{E} \cdot d\vec{A}$$

First, if we look simply at the Gaussian Surface, we can see that there are 3 effective areas that we need to consider for the LHS of Gauss' Law. This means that the LHS is basically:

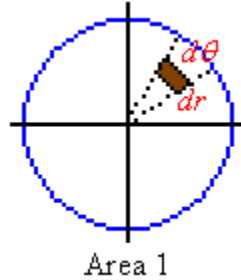
$$\oint_S \vec{E} \cdot d\vec{A} = \int_{S1} \vec{E} \cdot d\vec{A}_1 + \int_{S2} \vec{E} \cdot d\vec{A}_2 + \int_{S3} \vec{E} \cdot d\vec{A}_3$$

Let's start with Areas 1 and 3, since those are "easy":



For Areas 1 and 3, notice that the area segment normal for each of these areas is pointing perpendicular to the direction of the electric field. This means that the angle between these area segment normals and the electric field will always be 90 degrees (for area 3 or 270 degrees – for area 1). **[PAY ATTENTION TO DIRECTION:** Based upon our coordinate frame above, $\hat{n}_{Area1}$ is pointing in the $-\hat{x}$ direction and $\hat{n}_{Area3}$ is pointing in

the $+\hat{x}$ direction.] This is convenient, because then the dot product will always be zero for both surfaces. ☺ If we consider an area segment of surface 1 (which is identical to surface 3 – just pointing in the opposite direction), we see that the infinitesimal area is given as:



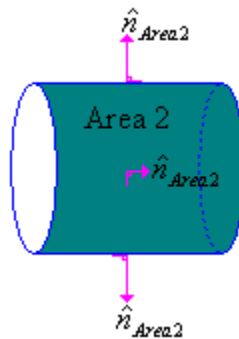
$$dA_{1,3} = dr * d\theta ; \forall r : 0 \leq r \leq R; \forall \theta : 0 \leq \theta \leq 2\pi;$$

This means we can now evaluate the integrals for surface 1 (S1) and surface 3 (S3):

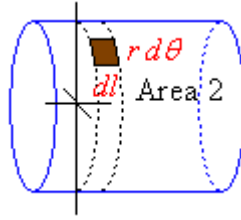
$$\int_{S1} \vec{E} \cdot d\vec{A}_1 = \int_0^R \int_0^{2\pi} E dr d\theta (\hat{r} \cdot -\hat{x}) \cos(\phi) = E(0) \cos(270^\circ) \int_0^R dr \int_0^{2\pi} d\theta = 0$$

$$\int_{S3} \vec{E} \cdot d\vec{A}_3 = \int_0^R \int_0^{2\pi} E dr d\theta (\hat{r} \cdot \hat{x}) \cos(\phi) = E(0) \cos(90^\circ) \int_0^R dr \int_0^{2\pi} d\theta = 0$$

Next, let's continue with Area 2:



For Area 2, notice that for the Gaussian Surface we selected, the area segment normal (on the surface around the cylinder) is always pointing in the same direction as the electric field. [AGAIN, PAY ATTENTION TO DIRECTION: Based upon our coordinate frame above, $\hat{n}_{Area2}$ is pointing in the $+\hat{r}$ direction.] This means that the angle between the area segment normal and the electric field will always be 0. This is convenient, because then the dot product will always be one for the surface. ☺ If we consider an area segment of surface 2, we see that the infinitesimal area is given as:



$$dA_2 = dl * r d\theta ; \forall l : 0 \leq l \leq L; \forall \theta : 0 \leq \theta \leq 2\pi;$$

This means we can now evaluate the integrals for surface 2 (S2):

$$\oint_{S2} \vec{E} \cdot d\vec{A}_2 = \int_0^L \int_0^{2\pi} E dl r d\theta \cos(\phi) (\hat{r} \cdot \hat{r}) = E r \cos(0) (1) \int_0^L dl \int_0^{2\pi} d\theta$$

$$\oint_{S2} \vec{E} \cdot d\vec{A}_2 = E r l \Big|_0^L \theta \Big|_0^{2\pi} = E r (L - 0) (2\pi - 0) = 2\pi r E L$$

Phew, now we can see that the LHS is simply:

$$\oint_S \vec{E} \cdot d\vec{A} = \int_{S1} \vec{E} \cdot d\vec{A}_1 + \int_{S2} \vec{E} \cdot d\vec{A}_2 + \int_{S3} \vec{E} \cdot d\vec{A}_3 = 0 + 2\pi r E L + 0 = 2\pi r E L$$

Now, setting the LHS equal to the RHS:

$$2\pi r E L = \frac{\lambda L}{\epsilon_0}$$

Thus putting everything together and solving for the Electric Field:

$$\vec{E} = \frac{\lambda}{2\pi r \epsilon_o} \hat{r}$$

Using the relationship that:

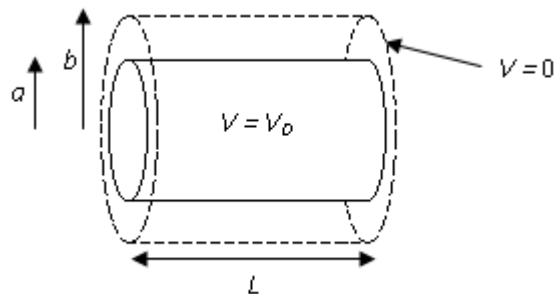
$$\frac{1}{\epsilon_o} = 4\pi k$$

$$\vec{E} = \frac{4\pi k \lambda}{2\pi r} \hat{r} = \frac{2k\lambda}{r} \hat{r}$$

OK, this looks good ... let's continue with the rest of the problem.

The uniformly charged infinite cylinder has equipotential surfaces that are also cylinders and concentric with the charged cylinder. We will call the equipotential at $r = b$ the

$V(b) = 0$ and at $r = a$ the $V(a) = V_o$:



a.) Using the relationship between potential and electric field:

$$\Delta V = - \int_R^b \vec{E} \cdot d\vec{s}$$

find the potential difference between the points $r = R$ and $r = b$.

First, let's plug in the electric field we found above and the infinitesimal displacement (pointing in the same direction as the electric field) **[IMPORTANT: Note that if we went with the bounds of $b \rightarrow R$ for the integral, instead of $R \rightarrow b$ (as we have) ... then the displacement vector wouldn't have the same sign and would be pointing in the opposite direction as the electric field! This would result in a -1 for the dot product; but still gives us the same answer. ☺]**:

$$\Delta V = - \int_R^b \frac{2k\lambda}{r} dr (\hat{r} \bullet \hat{r}) = -2k\lambda \int_R^b \frac{1}{r} dr (1) = -2k\lambda \int_R^b \frac{1}{r} dr$$

Next, the trick is to use the negative sign to “flip the log”:

$$\Delta V = -2k\lambda (\ln(r)) \Big|_R^b = -2k\lambda (\ln(b) - \ln(R)) = 2k\lambda (\ln(R) - \ln(b)) = 2k\lambda \ln\left(\frac{R}{b}\right)$$

This means we can determine the potential at any arbitrary distance R from the surface of the charged rod.

$$V(R) = - \int_R^b \frac{2k\lambda}{r} dr = 2k\lambda \ln\left(\frac{R}{b}\right)$$

b.) The potential you found in part a has the form:

$$V = A \ln(Br)$$

where A depends on λ and B is a geometric parameter. Use the fact that at $r = a$, $V(a) = V_0$ to find A and B in terms of V_0 , a and b instead of λ . In this lab, you will compare the theoretical values of A and B to the experimentally measure values of A and B .

It's easy to see that the form mentioned here and the form we found do match, where:

$$V = A \ln(Br) = 2k\lambda \ln\left(\frac{R}{b}\right)$$

$$A = 2k\lambda \quad \text{and} \quad B = \frac{1}{b}$$

If we apply the boundary condition at $R = a$:

$$V(a) = 2k\lambda \ln\left(\frac{a}{b}\right) = V_o$$

Now, solve for $2k\lambda$ and plug this back into our generic $V(R)$ equation:

$$2k\lambda = \frac{V_o}{\ln\left(\frac{a}{b}\right)}$$

$$V(R) = A \ln(BR) = \left(\frac{V_o}{\ln\left(\frac{a}{b}\right)} \right) \ln\left(\frac{R}{b}\right)$$

This means, now:

$$A = \frac{V_o}{\ln\left(\frac{a}{b}\right)} \quad \text{and} \quad B = \frac{1}{b}$$

We have now successfully recast the electric field of an infinite charged cylinder into potential (something we can measure and compare in the lab).

~~ TOLD YOU THIS WOULD NOT BE EASY!! ~~