Shape fluctuations are different in different directions

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Abstract

We consider the first passage percolation model on $\mathbf{Z}^2$. In this model, we assign independently to each edge e a passage time $t(e)$ with a common distribution F. Let $T(u,v)$ be the passage time from u to v. In this paper, we show that, whenever $F(0) < p_c$, $\sigma^2\left(T((0,0),(n,0))\right) \geq C \log n$ for all $n \geq 1$. Note that if F satisfies an additional special condition, infsupp$(F) = r > 0$ and $F(r) > \vec{p}_c$, it is known that there exists M such that for all n, $\sigma^2\left(T((0,0),(n,n))\right) \leq M$. These results tell us that shape fluctuations not only depend on distribution F, but also on direction. When showing this result, we find the following interesting geometrical property. With the special distribution above, any long piece with r-edges in an optimal path from $(0,0)$ to $(n,0)$ has to be very circuitous.

1 Introduction of the model and results.

The first passage percolation model was introduced in 1965 by Hammersley and Welsh. In this model, we consider the $\mathbf{Z}^2$ lattice as a graph with edges connecting each pair of vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ with $d(u,v) = 1$, where $d(u,v)$ is the Euclidean distance between u and v. We assign independently to each edge a non-negative *passage time* $t(e)$ with a common distribution F. More formally, we consider the following probability space. As the sample space, we take $\Omega = \prod_{e \in \mathbf{Z}^2} [0, \infty)$, whose points are called *configurations*. Let $P = \prod_{e \in \mathbf{Z}^2} \mu_e$ be the corresponding product measure on Ω, where μ_e is the measure on $[0,\infty)$ with distribution F. The expectation and variance with respect to P are denoted by $E(\cdot)$ and $\sigma^2(\cdot)$. For any two vertices u and v, a path γ from u to v is an alternating sequence $(v_0, e_1, v_1, ..., v_i, e_{i+1}, v_{i+1}, ..., v_{n-1}, e_n, v_n)$ of vertices v_i and edges e_i between v_i and v_{i+1} in $\mathbf{Z}^2$ with $v_0 = u$ and $v_n = v$. Given such a path γ, we define its passage time as

$$T(\gamma) = \sum_{i=1}^{n} t(e_i). \tag{1.1}$$

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For any two sets A and B, we define the passage time from A to B as

$$T(A, B) = \inf\{T(\gamma)\},$$

where the infimum is over all possible finite paths from some vertex in A to some vertex in B. A path γ from A to B with $T(\gamma) = T(A, B)$ is called the *optimal path* of $T(A, B)$. The existence of such an optimal path has been proven (see Kesten (1986)). We also want to point out that the optimal path may not be unique. If we focus on a special configuration ω, we may write $T(A, B)(\omega)$ instead of $T(A, B)$. When $A = \{u\}$ and $B = \{v\}$ are single vertex sets, $T(u, v)$ is the passage time from u to v. We may extend the passage time over $\mathbf{R}^2$. If x and y are in $\mathbf{R}^2$, we define $T(x, y) = T(x', y')$, where x' (resp., y') is the nearest neighbor of x (resp., y) in $\mathbf{Z}^2$. Possible indetermination can be eliminated by choosing an order on the vertices of $\mathbf{Z}^2$ and taking the smallest nearest neighbor for this order.

With these definitions, we would like to introduce the basic developments and questions in this field. Hammersley and Welsh (1965) first studied the point-point and the point-line passage times defined as follows:

$$a_{m,n} = \inf\{T(\gamma) : \gamma \text{ is a path from } (m, 0) \text{ to } (n, 0)\},$$

$$b_{m,n} = \inf\{T(\gamma) : \gamma \text{ is a path from } (m, 0) \text{ to } \{x = n\}\}.$$

It is well known (see Smythe and Wierman (1978)) that if $Et(e) < \infty$,

$$\lim_{n\to\infty} \frac{1}{n} a_{0,n} = \lim_{n\to\infty} \frac{1}{n} b_{0,n} = \mu \text{ a.s. and in } L_1, \tag{1.2}$$

where the non-random constant $\mu = \mu(F)$ is called the *time constant*. Later, Kesten showed (see Theorem 6.1 in Kesten (1986)) that

$$\mu = 0 \text{ iff } F(0) \geq p_c, \tag{1.3}$$

where $p_c = 1/2$ is the critical probability for Bernoulli (bond) percolation on $\mathbf{Z}^2$.

Given a vector $x \in \mathbf{R}^2$, by the same arguments as in (1.2) and (1.3), if $Et(e) < \infty$, then

$$\lim_{n\to\infty} \frac{1}{n} T(\mathbf{0}, nx) = \inf_n \frac{1}{n} ET(\mathbf{0}, nx) = \lim_{n\to\infty} \frac{1}{n} ET(\mathbf{0}, nx) = \mu(x) \text{ a.s. and in } L_1, \tag{1.4}$$

and

$$\mu(x) = 0 \text{ iff } F(0) \geq p_c.$$

For convenience, we assume that $t(e)$ is not a constant and satisfies the following:

$$\int e^{\lambda x} dF(x) < \infty \text{ for some } \lambda > 0. \tag{1.5}$$

When $F(0) < p_c$, the map $x \to \mu(x)$ induces a norm on $\mathbf{R}^2$. The unit radius ball for this norm is denoted by $\mathbf{B} := \mathbf{B}(F)$ and is called the *asymptotic shape*. The boundary of $\mathbf{B}$ is

$$\partial\mathbf{B} := \{x \in \mathbf{R}^2 : \mu(x) = 1\}.$$

$\mathbf{B}$ is a compact convex deterministic set and $\partial\mathbf{B}$ is a continuous convex closed curve (Kesten (1986)). Define for all $t > 0$,

$$B(t) := \{v \in \mathbf{R}^2,\ T(\mathbf{0}, v) \leq t\}.$$

The shape theorem (see Theorem 1.7 of Kesten (1986)) is the well-known result stating that for any $\epsilon > 0$,

$$t\mathbf{B}(1-\epsilon) \subset B(t) \subset t\mathbf{B}(1+\epsilon) \text{ eventually w.p.1.}$$

In addition to $t\mathbf{B}$, we can consider the mean of $B(t)$ to be

$$G(t) = \{v \in \mathbf{R}^2 : ET(\mathbf{0}, v) \leq t\}.$$

By (1.4), we also have

$$t\mathbf{B}(1-\epsilon) \subset G(t) \subset t\mathbf{B}(1+\epsilon).$$

The natural and most challenging aspect in this field (see Kesten (1986) and Smythe and Wierman (1978)) is to question the "speed" and "roughness" of the interface $B(t)$ from the deterministic boundaries $t\mathbf{B}$ and $G(t)$. This problem has also received a great amount of attention from statistical physicists because of its equivalence with one version of the Eden growth model. They believe that there is a scaling relation for the shape fluctuations in growth models. For each unit vector x, we may denote by $h_t(x)$ the *height* of the interface (see page 490 in Krug and Spohn (1992)). The initial condition is $h_0(x) = 0$. Being interested in fluctuation, we consider the height fluctuation function

$$\bar{h}_t(x) = h_t(x) - Eh_t(x).$$

Statistical physicists believe that $\bar{h}_t(x)$ should satisfy (see (3.1) in Krug and Spohn (1992)) the following scaling property

$$\bar{h}_t(x) = b^{\zeta}\bar{h}_{b^z t}(bx)$$

with the scaling exponents ζ and z for an arbitrary rescaling factor b. With this scaling equation, we should have (see (7.9) in Krug and Spohn (1992)), for all vectors x,

$$\bar{h}_t(x) \approx t^{\zeta/z} \text{ pointwisely or } \sigma(h_t(x)) \approx t^{\zeta/z}. \tag{1.6}$$

In particular, it is believed that $\zeta = 1/2$ and $z = 2/3$ when $d = 2$.

Mathematicians have also made significant efforts in this direction. When $F(0) > p_c$, it is known (see Zhang (1995)) that

$$\sigma^2(a_{0,n}) < \infty. \tag{1.7}$$

When $F(0) = p_c$ and $t(e)$ only takes two values zero and one, it is also known (see Kesten and Zhang (1997)) that

$$\sigma^2(a_{0,n}) = O(\log n). \tag{1.8}$$

In fact, Kesten and Zhang (1997) showed a CLT for the process $a_{0,n}$, a much stronger result than (1.8). For a more general distribution F with $F(0) = p_c$, $\sigma^2(a_{0,n})$ can be either convergent or divergent, depending on the behavior of the derivative of $F(x)$ at $x = 0$ (see Zhang (1999)).

Now we focus on the most interesting situation: when $F(0) < p_c$. It is widely conjectured (see (1.6) above and Kesten (1993)) that if $F(0) < p_c$, then

$$\sigma^2(a_{0,n}) \approx n^{2/3}.$$

The mathematical estimates for the upper bound of $\sigma^2(a_{0,n})$ are quite promising. Kesten (1993) showed that if $F(0) < p_c$, there is a constant C_1 such that

$$\sigma^2(a_{0,n}) \leq C_1 n. \tag{1.9}$$

In this paper, C and C_i are always positive constants that may depend on F, but not on t, m, or n. Their values are not significant and change from appearance to appearance. Benjamini, Kalai and Schramm (2003) also showed that when $t(e)$ only takes two values $0 < a < b$ with a half probability for each one,

$$\sigma^2(a_{0,n}) \leq C_1 n / \log n,$$

where log denotes the natural logarithm.

On the other hand, the lower bound of the variance for $\sigma^2(a_{0,n})$ seems to be much more difficult to estimate. For a high-dimensional lattice, there are some discussions for a lower bound of the fluctuations from $B(t)$ to $t\mathbf{B}$ (see Zhang (2006)). In this paper, we would like to focus on the square lattice. To understand the complexity of the lower bound, we have to deal with the following special distributions investigated by Durrett and Liggett (1981). They defined

$$r = \text{infsupp}(F) = \inf\{x : F(x) = P(t(e) \leq x) > 0\}$$

with $r > 0$. Clearly, if $r > 0$, $F(0) = 0 < p_c$, so shape $\mathbf{B}$ is compact. Without loss of generality, we can suppose that $r = 1$ if we replace $F(x)$ by $F(rx)$. In the following, we always assume that

$$\text{infsupp}(F) = 1 \text{ and } F(1) = P(t(e) = 1) \geq \vec{p}_c, \tag{1.10}$$

where $\vec{p}_c$ is the critical value for the oriented percolation model. Durrett and Liggett (1981) found that shape $\mathbf{B}$ contains a flat segment on the diagonal direction. Later, Marchand (2002) presented the precise locations of the flat segment in the shape when distribution F

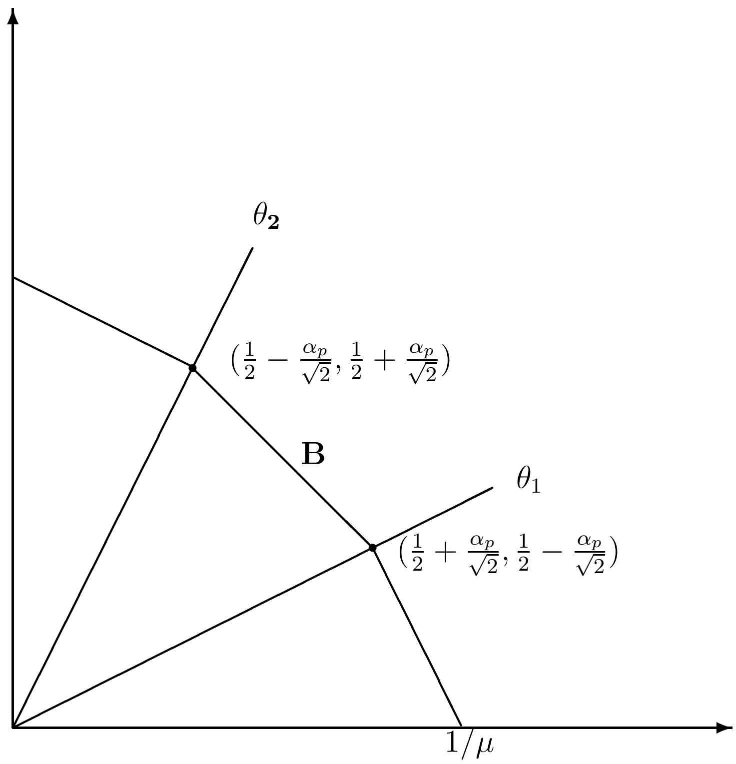

Figure 1: *The graph shows that shape* **B** *contains a flat segment when F satisfies (1.10).*

satisfies (1.10). More precisely, two polar coordinates in the first quadrant are denoted by $(\sqrt{1/2+\alpha_p^2}, \theta_i)$ for $i = 1, 2$ (see Fig. 1), where

$$\theta_i = \arctan\left(\frac{1/2 \mp \alpha_p/\sqrt{2}}{1/2 \pm \alpha_p/\sqrt{2}}\right),$$

and $\alpha_p \geq 0$ is a constant defined in (2.4) below. Note that $\theta_1 < \theta_2$ if $F(1) > \vec{p}_c$ and $\theta_1 = \theta_2$ if $F(1) = \vec{p}_c$. Marchand (see Theorem 1.3 of Marchand (2002)) showed that under (1.10),

$$\mathbf{B} \cap \{(x,y) \in \mathbf{R}^2, |x|+|y| = 1\} = \text{ the segment from } (\sqrt{1/2+\alpha_p^2}, \theta_1) \text{ to } (\sqrt{1/2+\alpha_p^2}, \theta_2),$$

where the segment will shrink as a point $(1/\sqrt{2}, \pi/4)$ when $F(1) = \vec{p}_c$. This segment is called the *flat edge* of shape **B**. The cone between θ_1 and θ_2 is called the *oriented percolation cone*. To understand why this is called the oriented percolation cone, we introduce the following oriented paths. Let us define the northeast- and the southeast-oriented paths. A path (not necessary to be a 1-path) is said to be a northeast path if each vertex u of the path has only one exiting edge, either from u to $u+(1,0)$ or to $u+(0,1)$. Similarly, a path is said to be a southeast path if each vertex u of the path has only one existing edge, either from u to $u+(1,0)$ or to $u+(0,-1)$.

For any vector (r,θ) with $\theta_1 \leq \theta \leq \theta_2$, under (1.10), with a positive probability, there is a northeast path γ from $(0,0)$ to (nr, θ) with only 1-edges (see (3.2) in Yukich and Zhang (2006)). Thus, we call the cone between θ_1 and θ_2 the oriented percolation cone.

With this observation, for $\theta_1 < \theta < \theta_2$ (see special case 1 in Newman and Piza (1995)), there exists a constant $C = C(F, \theta)$ such that

$$\sigma^2(T((0,0),(n,\theta))) \leq C \text{ for all } n, \tag{1.11}$$

where both $(0,0)$ and (n,θ) are polar coordinates. On the other hand, it has been proven (see Newman and Piza (1995)) that if $F(0) < p_c$ and $\text{infsupp}(F) = 0$, or $\text{infsupp}(F) = 1$ and $F(1) < \vec{p}_c$, then

$$\sigma^2(T((0,0),(n,\theta))) \geq C \log n \text{ for all } \theta. \tag{1.12}$$

Even though (1.12) is far from the correct order $n^{2/3}$, it at least tells us that $\sigma^2(a_{0,n})$ diverges as $n \to \infty$. As we mentioned earlier, both the convergence and divergence in (1.11) and (1.12) indicate the complexity of an estimate for the lower bound of the variance.

From (1.11) and (1.12), we may ask the behaviors of the variance for the passage time on a non-oriented percolation cone or simply ask whether with (1.10), for the most popular first passage time $a_{0,n}$,

$$\sigma^2(a_{0,n}) \text{ diverges as } n \to \infty. \tag{1.13}$$

Indeed, if there were a proof for (1.13), the proof would be tricky because one has to show that two different behaviors exist in the oriented percolation and the non-oriented percolation cones. As Newman and Piza (1995) described, "either the new techniques, or additional hypotheses seem to need to investigate conjecture (1.13) when (1.10) holds." This is the same flavor as the extension of the strict inequality on the time constant (see van den Berg and Kesten (1993)) to the non-oriented percolation cone (see Marchand (2002)). In this paper, one of the main works is to investigate the different behaviors in the oriented and non-oriented percolation cones. We discovered that, unlike the oriented percolation cone, any long piece with 1-edges in an optimal path from $(0,0)$ to $(n,0)$ contains proportional circuitous pieces. With this geometric property, we will show (1.13).

Before we mention our result, we would like to introduce Newman and Piza's martingale method that was used to show (1.12). To describe their method simply, we assume that $t(e)$ can only take two values a and b with $0 \leq a < b$. The key to these martingale arguments is to show that there are proportionally many b-edges in an optimal path. If we change b-edges from b to a, then the passage time will shrink at least $b - a$ from the original passage time. This tells us why the variance of passage time should be large if the number of b-edges is large. The remaining task is to estimate the number of b-edges. However, if we assume (1.10) holds, it is possible that all edges in any optimal path have value a. Therefore, the Newman and Piza method will not be applied.

To develop new techniques for case (1.10), we need to investigate the geometric properties of an optimal path. For an optimal path from $(0,0)$ to $(n,0)$, we may guess that an optimal path should not be northeast or southeast since it is not in the oriented percolation cone. Let us give a more precise definition of what "non-northeast" or "non-southeast" means. Let γ_n be an optimal path from $(0,0)$ to $(n,0)$. Note that the existence of such a γ_n has been mentioned before. With this existence, there might be many such optimal paths for

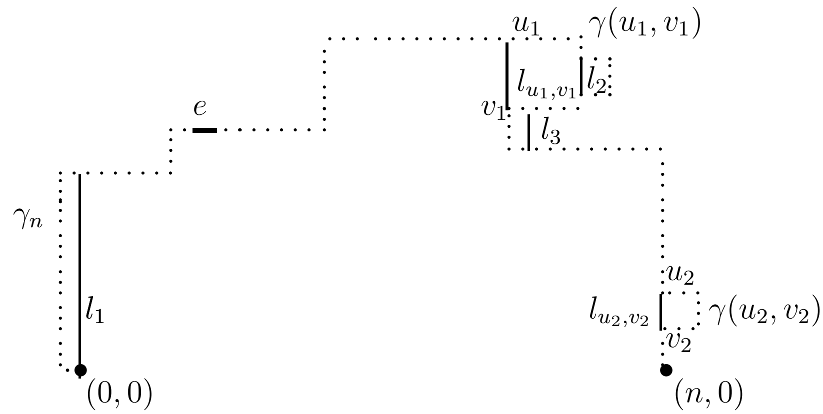

Figure 2: *The dotted line in the graph is the optimal path γ_n. e is a 1^+-edge. l_1, l_2, l_3, l_{u_1,v_2}, and l_{u_2,v_2} are broken bridges of γ_n. In fact, there are many vertical M-broken bridges parallel to l_{u_1,v_1} and l_3, but we will not list all of them in this graph. l_{u_2,v_2} is the only broken bridge for the remaining path from v_1 to $(n, 0)$. l_1 is not an M-bridge because its length is more than $2M$. After ordering all M-broken bridges of γ_n, l_2 and l_3 are no longer M-broken bridges for γ_n, so l_{u_1,v_1} and l_{u_2,v_2} are M-broken bridges for γ_n.*

$a_{0,n}$. We now select a unique optimal path. For example, we may start at the origin and select vertices among all optimal paths in each step such that the selected vertex is closer to the X-axis. We still write this unique optimal path as γ_n, without loss of generality. Later, we always use γ_n as the unique selected optimal path. For vertices u, v, we say $l_{u,v}$ is an *M-bridge* if $l_{u,v}$ is a horizontal or vertical segment, including both vertices and edges, from u to v whose number of vertices is less than $2M$. Furthermore, we say $l_{u,v}$ is an *M-broken bridge* of γ_n if $l_{u,v}$ is an M-bridge and

$$l_{u,v} \cap \gamma_n = \{u, v\}. \tag{1.14}$$

In other words, the bridge from u to v is broken and γ_n has to go around from u to v to avoid using any vertex in $l_{u,v}$ except u and v.

Now we choose a list of peculiar broken bridges of γ_n (see Fig. 2), as follows. We first list all the possible M-broken bridges of γ_n. We then go along γ_n from the origin to meet u_1, the first vertex of γ_n, such that there exists an M-broken bridge for γ_n at u_1. Note that there may be up to three such M-broken bridges at u_1, since u_1 may be the origin. If there are two or three M-broken bridges at u_1, for example, $l_{u_1,v_1'}$, $l_{u_1,v_1''}$, and $l_{u_1,v_1'''}$, we use the following way to select one. Going along γ_n from the origin, we first meet either v_1', v_1'' or v_1'''. We then select the first vertex that we encounter and denote it by v_1. We can go along γ_n from u_1 to $v_1 \neq u_1$, denoted by $\gamma(u_1, v_1)$. Note that $\gamma(u_1, v_1) \cup l_{u_1,v_1}$ is a loop (see Fig. 2). After selecting l_{u_1,v_1}, we list all the possible M-broken bridges of the path from v_1 to $(n, 0)$ along the remaining part of γ_n. As we go from v_1 along the remaining part of γ_n, we meet u_2, the first vertex in the remaining part, such that there exists an M-broken bridge at u_2 for the remaining part. Note that u_2 may equal v_1. Similarly, if there are two M-broken bridges at u_2, denoted by $l_{u_2,v_2'}$ and $l_{u_2,v_2''}$, we select v_2, from v_2' and v_2'', as the first vertex encountered on the remaining part of γ_n from v_1. We now go along the remaining part of γ_n from u_2 to v_2, denoted by $\gamma(u_2, v_2)$. Thus, $\gamma(u_2, v_2) \cup l_{u_2,v_2}$ is the second loop. Since γ_n is finite, we continue this process until the last M-broken bridge, l_{u_τ,v_τ}. The corresponding piece of γ_n from u_τ to v_τ is $\gamma(u_\tau, v_\tau)$, and the loop is $\gamma(u_\tau, v_\tau) \cup l_{u_\tau,v_\tau}$. In the following discussion, for γ_n, we always consider these M-broken bridges $\{l_{u_i,v_i}\}$ $(i = 1, \cdots, \tau)$ for γ_n by this arrangement and by ignoring the other listed M-broken bridges.

Furthermore, by the definition (see Fig. 2),

$$\text{the subpath of } \gamma_n \text{ from } v_i \text{ to } u_{i+1} \text{ has none of its own } M\text{-broken bridges.} \tag{1.15}$$

Note that γ_n is self-avoiding, so

$$\gamma(u_i, v_i) \text{ and } \gamma(u_j, v_j) \text{ have no common edge for } i \neq j. \tag{1.16}$$

On the other hand, the interior of the loop $\gamma(u_i, v_i) \cup l_{u_i,v_i}$ cannot contain a vertex of γ_n by (1.16) and the definition of the M-broken bridge. With this observation,

$$l_{u_i,v_i} \text{ and } l_{u_j,v_j} \text{ have no common edge for } i \neq j. \tag{1.17}$$

Since l_{u_i,v_i} is shorter than $\gamma(u_i, v_i)$ by at least two edges, and each edge costs at least time one,

$$\exists\ e \in l_{u_i,v_i} \text{ such that } t(e) > 1. \tag{1.18}$$

Clearly, for a northeast or southeast path, there is no broken bridge. From this point of view, we may guess that there are many broken bridges for the optimal path γ_n from the origin to $(n, 0)$. We shall show the following theorem to describe this fact.

An edge e is called a 1-edge if $t(e) = 1$. A path is called a 1-path if all of its edges are 1-edges. Note that we assume that $t(e)$ is not a constant, so

$$0 < P(1 < t(e)). \tag{1.19}$$

We say edge $e \in \gamma_n$ is a 1^+-edge if $t(e) > 1$. We collect all vertices in γ_n that are adjacent to 1^+-edges on γ_n and denote them by $D(\gamma_n)$. If γ_n is not northeast or southeast, there may exist M-broken bridges $\{l_{u_i,v_i}\}_{1\leq i\leq\tau}$ of γ_n. Note that $u_i, v_i \in \gamma_n$, so we collect all vertices u_i and v_i in γ_n for $0 \leq i \leq \tau$ and denote them by $S_M(\gamma_n)$. With these definitions, we will have the following theorem.

Theorem 1. *Let F be a distribution such that infsupp$(F) = 1$, $\vec{p}_c \leq F(1)$ and satisfying the tail assumption in (1.5). Then there exist constants $\delta = \delta(F, M) > 0$ and $C_i = C_i(F, M, \delta)$ $(i = 1, 2)$ such that for all $m, n \geq 1$ with $n/2 \geq m \geq n^{2/3}$,*

$$P(|B(m) \cap [S_M(\gamma_n) \cup D(\gamma_n)]| \leq \delta m) \leq C_1 \exp(-C_2 n^{1/14}),$$

where $B(m) = [-m, m]^2$ and $|A|$ represents the number of vertices in set A.

Remark 1. When $m = O(n)$, we can generalize Theorem 1 for any vector $x = (1, \theta)$ with $0 < \theta < \theta_1$. More precisely, for a polar coordinate $x = (1, \theta)$ with $0 < \theta < \theta_1$, let $\gamma_n(\theta)$ be an optimal path from the origin to (n, θ). Similarly, we choose a list of peculiar M-broken bridges $\{l_{u_i,v_i}(\theta)\}$ for $\gamma_n(\theta)$ as we did for γ_n. We denote by $D(\gamma_n(\theta))$ all vertices in $\gamma_n(\theta)$ that are adjacent to 1^+-edges on $\gamma_n(\theta)$. We also denote by $S_M(\gamma_n(\theta))$ all vertices $\{u_i, v_i\}$ for the M-broken bridges $\{l_{u_i,v_i(\theta)}\}$ of $\gamma_n(\theta)$. If $m = O(n)$, we can show, under the assumptions of Theorem 1,

$$P(|B(m) \cap (S_M(\gamma_n(\theta)) \cup D(\gamma_n(\theta))| \leq \delta n) \leq C_1 \exp(-C_2 n). \tag{1.20}$$

However, because of the lack of symmetry, we cannot show (1.20) for all $\theta < \theta_1$ when $m = o(n)$.

Remark 2. We may consider the same problem as Theorem 1 when $d \geq 3$. It is possible to show a similar result as Theorem 1 when $d \geq 3$ and (1.10) holds. However, we do not know whether a similar result of Theorem 1 holds when $\vec{p}_c(d) \leq F(1) < \vec{p}_c$, where $\vec{p}_c(d)$ is a critical probability for the d dimensional oriented percolation. The main reason is that we need to use Lemma 3, proven by Marchand (2002), in our section 2 to show Theorem 1, but Lemma 3 has not been proven for all $d \geq 3$.

Remark 3. The term $n^{2/3}$ in Theorem 1 can be improved to $Cn^{1/2} \log n$ for large constant C.

With Theorem 1, we can see that an optimal path contains proportionally many 1^+-edges or proportionally many vertices adjacent to M-broken bridges. If we change 1^+ edge in γ_n to 1-edge, or recover the bridge by changing the time of the 1^+-edges from 1^+ to 1, we have saved a positive passage time for γ_n. Therefore, we can also use Newman and Piza's (1995) martingale method, but with a large square construction, to show the following theorem.

Theorem 2. *Let F be a distribution such that infsupp$(F) = 1$, $\vec{p}_c \leq F(1)$ and satisfying the tail assumption in (1.5). Then there exists $C = C(F)$ such that for all n*

$$\sigma^2(a_{0,n}) \geq C \log n.$$

Remark 4. Together with Newman and Piza's result (1995) in (1.12), we have

$$\sigma^2(a_{0,n}) \geq C \log n$$

for all n whenever $F(0) < p_c$. Together with (1.7), (1.8), and (1.21), the whole picture of convergence or divergence for $\sigma^2(a_{0,n})$ is complete.

Remark 5. Under the same assumptions as in Theorem 2, the same proof can be carried out to show that

$$\sigma^2(b_{0,n}) \geq C \log n.$$

Remark 6. We are unable to show Theorem 2 for the passage time $T((0,0),(n,\theta))$ for all $0 < \theta < \theta_1$ even though we believe it is true. In fact, if one can show (1.20) in Remark 6 for all $n^{2/3} \leq m \leq n/2$, then the same proof of Theorem 2 can be carried out to show

$$\sigma(T((0,0),(n,\theta))) \geq C \log n$$

for all $0 < \theta < \theta_1$.

Remark 7. As we mentioned in (1.11), there exists $C = C(F,\theta)$ for $\theta_1 < \theta < \theta_2$,

$$\sigma^2(T((0,0),(n,\theta))) \leq C \text{ for all } n.$$

This result can be generalized for $\theta = \theta_1$ and $\theta = \theta_2$ without too many difficulties.

2 Preliminaries for the proof of Theorem 1.

Before presenting the proofs of the theorems we would like to introduce a few lemmas.

Lemma 1. *If γ is a path with $|\gamma| \leq 2M$ and without M-broken bridge, then γ is either northeast or southeast.*

Proof. Denote by $u = (u_1, u_2)$ and $v = (v_1, v_2)$ the extremities of γ. By symmetry, we can assume that $v_1 \geq u_1$ and $v_2 \geq u_2$. Let us show that in this case γ is northeast.

If γ is not northeast, there exist, along γ, two successive vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ such that

- either $y_1 = x_1 - 1$ and $y_2 = x_2$
- or $y_1 = x_1$ and $y_2 = x_2 - 1$.

Let us consider the first case. By the continuity of the first coordinate along γ, since γ is simple, there necessarily exists $k \in \{-2M+1, -2M+2, \cdots, -1, 1, \cdots, 2M-2, 2M-1\}$ such that $y' = (y_1, y_2 + k)$ and $x' = (x_1, x_2 + k)$ are successive vertices along γ. But now two cases occur:

- if the couple (x, y) appears in γ before the couple (y', x'), then the vertical segment between x and x' contains an M-broken bridge for γ.
- if the pair (y', x') appears in γ before the pair (x, y), then the vertical segment between y, and y' contains an M-broken bridge for γ. The second case can be proven by a similar argument. □

If we rotate our lattice counterclockwise by 45° and extend each edge by a factor of $\sqrt{2}$, the new graph is denoted by $\mathcal{L}$ with oriented edges from (m,n) to $(m+1,n+1)$ and to $(m-1,n+1)$. Each edge is independently open or closed with probability $p = F(1)$ or $1-p$. For two vertices u and v in $\mathcal{L}$, we say $u \to v$ if there is a sequence $v_0 = u, v_1, \cdots, v_m = v$ of points of $\mathcal{L}$ with the vertices $v_i = (x_i, y_i)$ and $v_{i+1} = (x_{i+1}, y_{i+1})$ for $0 \le i \le m-1$ such that $y_{i+1} = y_i + 1$ and v_i and v_{i+1} are connected by an open edge. For $A \subset (-\infty, \infty)$, we denote a random subset by

$$\xi_n^A = \{x : \exists\ x' \in A \text{ such that } (x', 0) \to (x, n)\} \text{ for } n > 0.$$

The right edge for this set is defined by

$$r_n = \sup \xi_n^{(-\infty,0]} \ \ (\sup \emptyset = -\infty).$$

We know (see Section 3 (7) in Durrett (1984)) that there exists a non-random constant α_p such that

$$\lim_{n\to\infty} \frac{r_n}{n} = \lim_n \frac{Er_n}{n} = \alpha_p \text{ a.s. and in } L_1, \tag{2.1}$$

where $\alpha_p > 0$ if $p > \vec{p}_c$ and $\alpha_p = 0$ if $p = \vec{p}_c$. Now we need to investigate the large deviation for the upper tail of r_n. The exponential bound, when $p = F(1) > \vec{p}_c$, has been obtained by Durrett (1984) in his Section 11. However, his proof will not apply for $p = \vec{p}_c$. We present a new proof, also independently interesting, to cover the case $p = \vec{p}_c$.

Lemma 2. If $p = F(1) \ge \vec{p}_c$, for every $\eta > 0$, there exist $C_i = C_i(p, \eta)$ for $i = 1, 2$ such that

$$P(r_n \ge n(\alpha_p + \eta)) \le C_1 \exp(-C_2 n). \tag{2.2}$$

Proof. We observe that r_n can be embedded in a two-parameter process (see Section 3 in Durrett (1984)). For $0 \le m < n$, let

$$r_{m,n} = \sup\{x - r_m : (x, n) \in \mathcal{L} \text{ and } \exists\ y \le r_m \text{ such that } (y, m) \to (x, n)\}.$$

In particular, we denote by

$$r_{m,n}(j) = \sup\{x - j : (x, n) \in \mathcal{L} \text{ and } \exists\ y \le j \text{ such that } (y, m) \to (x, n)\}.$$

It follows from Section 3 (3) and (4) in Durrett (1984) that

$$r_{m,n} \stackrel{d}{=} r_{n-m} \text{ and } r_n \le r_m + r_{m,n} \text{ for } 0 \le m < n. \tag{2.3}$$

By (2.1), we take M such that

$$Er_M \le M(\alpha_p + \eta/2). \tag{2.4}$$

Without loss of generality, we may assume that $n/M = l$ is an integer. By (2.4), we have

$$P(r_n \geq n(\alpha_p + \eta)) \leq P\left(r_n - lEr_M \geq n\eta/2\right). \tag{2.5}$$

By (2.5) and Markov's inequality, for any $t > 0$,

$$P(r_n \geq n(\alpha_p + \eta)) \leq \exp(-tn\eta/2) E \exp\left[t\left(r_n - lEr_M\right)\right]. \tag{2.6}$$

By (2.3),

$$E \exp\left[t\left(r_n - lEr_M\right)\right] \leq E \exp\left[t\left(r_{n-M} + r_{n-M,n} - lEr_M\right)\right].$$

By our definition, r_{n-M} only depends on the open and closed edges in the region between $\{y = 0\}$ and $\{y = n - M\}$. On the other hand, on $\{r_{n-M} = j\}$ for some j, $r_{n-M,n} = r_{n-M,n}(j)$ only depends on the open and closed edges in the region between $\{y = n - M\}$ and $\{y = n\}$. In addition, for any j,

$$r_{n-M,n}(j) \stackrel{d}{=} r_M.$$

With these observations,

$$\begin{aligned}
& E \exp\left[t\left(r_n - lEr_M\right)\right] \\
\leq\ & E \exp\left[t\left(r_{n-M} + r_{n-M,n} - lEr_M\right)\right] \\
=\ & \sum_j E \exp\left[t\left(r_{n-M} - (l-1)Er_M + r_{n-M,n} - Er_M\right)\right] I(r_{n-M} = j) \\
=\ & \sum_j E \exp\left[t\left(r_{n-M} - (l-1)Er_M\right)\right] I(r_{n-M} = j) \exp[t(r_{n-M,n}(j) - Er_M)] \\
=\ & \{E \exp\left[t\left(r_{n-M} - (l-1)Er_M\right)\right]\} \{E \exp[t(r_M - Er_M)]\},
\end{aligned}$$

where $I(A)$ is the indicator for event A. We iterate this way l times to have

$$E \exp\left[t\left(r_n - lEr_M\right)\right] \leq \left[E \exp t(r_M - Er_M)\right]^l. \tag{2.7}$$

Note that $(r_M - Er_M) < \infty$ almost surely, so we use Taylor's expansion for $\exp t(r_M - Er_M)$ to have

$$E \exp t(r_M - Er_M) = E \sum_{i=0}^{\infty} \frac{[t(r_M - Er_M)]^i}{i!}. \tag{2.8}$$

If we can show that

$$\sum_{i=2}^{\infty} t^{i-2} \frac{E|r_M - Er_M|^i}{i!} \text{ converges uniformly for } t \in [0, 1/(4M)], \tag{2.9}$$

then by (2.8), for all $t \in [0, 1/(4M)]$,

$$E \exp t(r_M - Er_M) \leq 1 + t^2 \sum_{i=2}^{\infty} t^{i-2} \frac{E|(r_M - Er_M)|^i}{i!} \leq (1 + O(t^2)) \leq \exp(Ct^2) \tag{2.10}$$

for some constant $C = C(M, F)$. By (2.6), (2.7), and (2.10), if we take $t \leq 1/(4M)$ small enough, then there exist $C_i = C_i(F, \eta)$ for $i = 1, 2$, such that

$$P(r_n \geq n(\alpha_p + \eta)) \leq C_1 \exp(-C_2 n), \tag{2.11}$$

so Lemma 2 follows. Now it remains to show (2.9). Note that r_1 is of geometric type with $p \geq \vec{p}_c > 0$, so by (2.3), there exists $C_3(M, F)$ such that

$$E|r_M|^i \leq E(M|r_1|)^i \leq C_3 M^i i!. \tag{2.12}$$

By (2.12), there exists a constant $C_4 = C_4(M, F)$ such that

$$\begin{aligned} E|r_M - Er_M|^i &\leq E(|r_M| + E|r_M|)^i \\ &= E[(|r_M| + E|r_M|)^i; |r_M| \geq E|r_M|] + E[(|r_M| + E|r_M|)^i; |r_M| < E|r_M|] \\ &\leq 2^i[E(|r_M|^i) + (E|r_M|)^i] \\ &\leq C_4(2M)^i i! \end{aligned} \tag{2.13}$$

By (2.13), (2.9) follows when $0 \leq t \leq 1/(4M)$. □

Given two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ with $u_1 \leq v_1$ and $u_2 \leq v_2$, we define $u \xrightarrow{1} v$ as the event that there exists a northeast 1-path from u to v, and define the *slope* between them by

$$sl(u, v) = \frac{v_2 - u_2}{v_1 - u_1}.$$

With these definitions, we show the following lemma.

Lemma 3. *For $0 < a < \tan(\theta_1)$, if (1.10) holds, then there exist $C_i = C_i(F, a)$ for $i = 1, 2$ such that for all $u, v \in \mathbf{Z^2}$,*

$$P(u \xrightarrow{1} v \text{ with } sl(u, v) \leq a) \leq C_1 \exp(-C_2(v_1 - u_1)).$$

Proof. Suppose that there exists a northeast 1-path from u to v with $sl(u, v) \leq a$ for some $a < \tan(\theta_1)$. Since we need to use the estimate of Lemma 2, we introduce the lowest northeast 1-path. Let $\gamma_{u,v'}$ be the lowest northeast 1-path from u to $\{x = v_1\}$, where the lowest 1-path means that all such northeast 1-paths have no vertex below $\gamma_{u,v'}$. Let the last vertex of $\gamma_{u,v'}$ be $v' = (v_1, v_2')$. To show Lemma 3, we need to show

$$P(u \xrightarrow{1} v \text{ with } sl(u, v) \leq a) \leq P(\exists\, \gamma_{u,v'}, sl(u, v') \leq a) \leq C_1 \exp(-C_2(v_1 - u_1)). \tag{2.14}$$

The first inequality in (2.14) follows from the definition of the lowest northeast 1-path, so we only show the second inequality in (2.14). By translation invariance, we may assume that

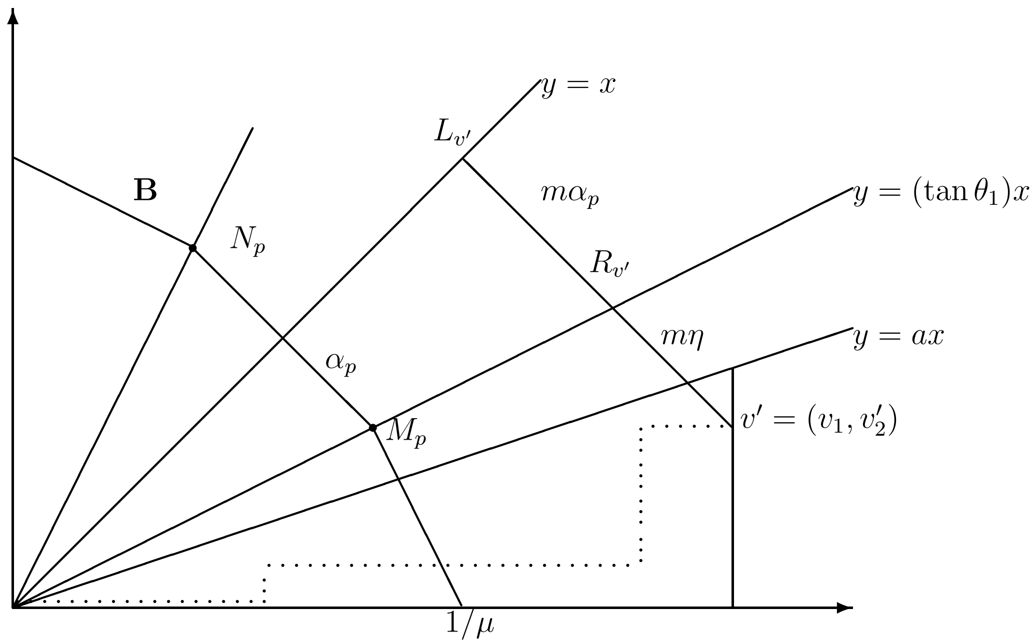

Figure 3: *The graph shows the relationship between* r_n *and* $sl((0,0),v')$. *The dotted line is* $\gamma_{(0,0),v'}$, *the lowest northeast 1-path. Two lines,* $y = x$ *and the line perpendicular to* $y = x$ *passing through* v', *intersect at* $L_{v'}$. *The length of the segment from the origin to* $L_{v'}$ *is* m. *By our construction,* $m \geq v_1/\sqrt{2}$. *We draw a line,* $y = (\tan\theta_1)x$, *passing through the origin and* $M_p = (\frac{1}{2} + \frac{\alpha_p}{\sqrt{2}}, \frac{1}{2} - \frac{\alpha_p}{\sqrt{2}})$. *The line passes through the above perpendicular line at* $R_{v'}$. *By Marchand's result (see Fig. 1), the length of the segment from* $L_{v'}$ *to* $R_{v'}$ *is* $m\alpha_p$. *We also draw a line* $y = ax$ *for some* $a < \tan(\theta_1)$ *defined in Lemma 3. By our assumption,* $\gamma_{(0,0),v'}$ *is below the line* $y = ax$. *We denote by* r'_m *the length of the segment from* $L_{v'}$ *to* v'. *Then* $r'_m \geq m\alpha_p + m\eta$. *If we rotate our lattice counterclockwise by* 45° *and extend each edge by a factor of* $\sqrt{2}$, *then* m *becomes* $\sqrt{2}m$ *and* $r_{\sqrt{2}m} = r'_m$ *for the right-most edge* r_n.

$u = (0,0)$. Fig. 3 shows the relationship between r_n and $sl((0,0),v')$. By using Fig. 3, if $\{\exists\ \gamma_{u,v'}$ with $sl((0,0),v') \leq a\}$ for some $a < \tan(\theta_1)$, then there exists $\eta = \eta(a) > 0$ such that, for some $m \geq v_1/\sqrt{2}$,

$$r'_m \geq m(\alpha_p + \eta) \Rightarrow r_n \geq n(\alpha_p + \eta) \text{ for } n = \sqrt{2}m. \tag{2.15}$$

Here, if n is not an integer, we may define r_n as $r_{\lfloor n \rfloor}$. This implies that

$$P(\exists\ \gamma_{u,v'} \text{ with } sl((0,0),v') \leq a) \leq \sum_{n \geq v_1/2} P(r_n \geq n(\alpha_p + \eta)). \tag{2.16}$$

By (2.16) and Lemma 2, Lemma 3 follows. □

Now we show the following two lemmas in order to explore the passage times in different directions.

Lemma 4. (Marchand (2002)) *Under (1.10),*

$$1/\mu > 1/2 + \alpha_p/\sqrt{2}. \tag{2.19}$$

Recall that the two polar coordinates, for the flat edge on the shape in the first quadrant, are denoted by $(\sqrt{1/2 + \alpha_p^2}, \theta_i)$ for $i = 1, 2$, where

$$\theta_i = \arctan\left(\frac{1/2 \mp \alpha_p/\sqrt{2}}{1/2 \pm \alpha_p/\sqrt{2}}\right).$$

Lemma 5. *If F satisfies (1.10), there exists $\eta = \eta(F) > 0$ such that*

$$2 \geq 1 + \tan(\theta_1) = \mu + \eta.$$

Proof. Since

$$\tan(\theta_1) = \frac{1/2 - \alpha_p/\sqrt{2}}{1/2 + \alpha_p/\sqrt{2}}, \tag{2.20}$$

then

$$1 + \tan(\theta_1) \leq 2.$$

By (2.20),

$$(1 + \tan(\theta_1))(1/2 + \alpha_p/\sqrt{2}) = 1. \tag{2.21}$$

By Lemma 4, we take $\eta > 0$ such that

$$1/2 + \alpha_p/\sqrt{2} = \frac{1}{\mu + \eta}, \tag{2.22}$$

so Lemma 5 follows from (2.22). □

Now we will introduce two lemmas regarding the rate of convergence of point-point and point-line passage times. Kesten (1993) proved that if F satisfies (1.5) and $F(0) < p_c$, there exist $C_i = C_i(F)$ for $i = 1, 2$ such that for all $0 < n$ and all $0 < x \leq C_1 n$,

$$P(|\theta_{0,n} - E\theta_{0,n}| \geq x n^{1/2}) \leq C_1 \exp(-C_2 x) \tag{2.23}$$

for $\theta = a, b$. Alexander (1993) used (2.23) to show that there exist $C = C(F)$ such that for all $0 < n$

$$n\mu \leq Ea_{0,n} \leq n\mu + C\sqrt{n}\log n. \tag{2.24}$$

If we combine (2.23) and (2.24) together, we have the following Lemma.

Lemma 6. (Alexander (1993) and Kesten (1993)) *If F satisfies (1.5) and $F(0) < p_c$, there exist $C_i = C_i(F)$ for $i = 1, 2$ such that for all $0 < n$ and all $n^{0.01} \leq x \leq n^{0.99}$,*

$$P(|a_{0,n} - \mu n| \geq x n^{1/2}) \leq C_1 \exp(-C_2 x).$$

Lemma 7. *If F satisfies (1.5) and $F(0) < p_c$, there exist $C_i = C_i(F)$ for $i = 1, 2$ such that for all $0 < n$ and $n^{0.01} \leq x \leq n^{0.99}$,*

$$P(|b_{0,n} - \mu n| \geq x n^{1/2}) \leq C_1 \exp(-C_2 x).$$

Proof. Lemma 7 was proved in Zhang (2005), but the paper was not published, so here we reprove it. We may select an optimal path, denoted by γ_n^b, for $b_{0,n}$ in a unique way. Then $\gamma_n^b \cap \{x = n\}$ contains only one vertex denoted by $(n, h_n(\gamma_n^b))$. Smythe and Wierman (1978) proved in their Corollary 8.16 that

$$\limsup_n \frac{h_n(\gamma_n^b)}{n} \leq 1 \text{ a.s.}$$

Proposition 5.8 in Kesten (1986) tells that, under $F(0) < p_c$, there exist positive numbers $C_i = C_i(F, \delta)$ for $i = 1, 2, 3$ such that

$$P(\exists \text{ a path } \gamma \text{ from the origin with } |\gamma| \geq n \text{ and } T(\gamma) \leq C_1 n) \leq C_2 \exp(-C_3 n). \tag{2.25}$$

Note that if $h_n(\gamma_n^b) \geq Mn$ for some $M > 0$, then $|\gamma_n^b| \geq Mn$. It also follows from a large deviation estimate (see Kesten (1986)) that there exist $C_i = C_i(F)$ for $i = 3, 4$ such that

$$P(T(\gamma_n^b) \geq 2n\mu) \leq C_3 \exp(-C_4 n). \tag{2.26}$$

With these observations, there exist $M = M(F)$ and $C_i(F, M) = C_i$ for $i = 5, 6$ such that

$$P(h_n(\gamma_n^b) \geq Mn) \leq P(|\gamma_n^b| \geq Mn, T(\gamma_n^b) \leq 2\mu n) + C_3 \exp(-C_4 n) \leq C_5 \exp(-C_6 n). \tag{2.27}$$

With (2.27),

$$P(b_{0,n} \leq n\mu - x n^{1/2}) \leq P(b_{0,n} \leq n\mu - x n^{1/2}, h_n(\gamma_n^b) \leq Mn) + C_1 \exp(-C_2 n). \tag{2.28}$$

$$P(b_{0,n} \leq n\mu - x n^{1/2}, h_n(\gamma_n^b) \leq Mn) \leq \sum_{i=-Mn}^{Mn} P(b_{0,n} \leq n\mu - x n^{1/2}, h_n(\gamma_n^b) = i). \tag{2.29}$$

From (2.28) and (2.29), there exists $\bar{i}$ such that

$$P(b_{0,n} \leq n\mu - x n^{1/2}) \leq 2(M+1) n P(b_{0,n} \leq n\mu - x n^{1/2}, h_n(\gamma_n^b) = \bar{i}) + C_1 \exp(-C_2 n). \tag{2.30}$$

Let $\bar{b}_{0,n}$ be the passage time from $(2n, 0)$ to the line $\{x = n\}$. We also select an optimal path $\bar{\gamma}_n^b$ for $\bar{b}_{0,n}$ in a unique way and denote

$$(n, \bar{h}_n(\bar{\gamma}_n^b)) = \bar{\gamma}_n^b \cap \{x = n\}.$$

If

$$\{b_{0,n} \leq n\mu - xn^{1/2}, h_n(\gamma_n^b) = \bar{i}, \bar{b}_{0,n} \leq n\mu - xn^{1/2}, \bar{h}_n(\bar{\gamma}_n^b) = \bar{i}\},$$

then

$$a_{0,2n} \leq 2n\mu - 2xn^{1/2} = 2n\mu - \sqrt{2}x(2n)^{1/2}.$$

Note that $\{b_{0,n} \leq n\mu - xn^{1/2}, h_n(\gamma_n^b) = \bar{i}\}$ and $\{\bar{b}_{0,n} \leq n\mu - xn^{1/2}, \bar{h}_n(\bar{\gamma}_n^b) = \bar{i}\}$ only depend on the configurations of the edges in $-\infty < x < n$ and $n < x < \infty$, respectively, so the two events are independent and have the same probability. By Lemma 6,

$$\begin{aligned}
& P(b_{0,n} \leq n\mu - xn^{1/2})^2 \\
\leq\ & (2(M+1))^2 n^2 P(b_{0,n} \leq n\mu - xn^{1/2}, h_n(\gamma_n^b) = \bar{i})^2 + C_1 \exp(-C_2 n) \\
\leq\ & (2(M+1))^2 n^2 P(b_{0,n} \leq n\mu - xn^{1/2}, h_n(\gamma_n^b) = \bar{i}, \bar{b}_{0,n} \leq n\mu - xn^{1/2}, \bar{h}_n(\bar{\gamma}_n^b) = \bar{i}) + C_1 \exp(-C_2 n) \\
\leq\ & (2(M+1))^2 n^2 P(a_{0,2n} \leq 2n\mu - \sqrt{2}x(2n)^{1/2}) + C_1 \exp(-C_2 n) \\
\leq\ & C_3 \exp(-C_4 x). \qquad (2.31)
\end{aligned}$$

On the other hand, note that $b_{0,n} \leq a_{0,n}$, so Lemma 7 follows from (2.31). □

3 Proof of Theorem 1.

In this section, we show Theorem 1. For the optimal path γ_n for $a_{0,n}$ from the origin to $(n, 0)$, we denote by $\gamma_n'(m)$ the piece of γ_n from the origin to first meet the line $\{x = m\}$ for $n^{2/3} \leq m \leq n/2$. Suppose that $\gamma_n'(m) \cap \{x = m\} = v_n(m)$. The path γ_n then goes from $v_n(m)$ to $(n, 0)$. We denote the last piece by $\gamma_n''(m)$. Clearly,

$$\gamma_n = \gamma_n'(m) \cup \gamma_n''(m) \text{ and } a_{0,n} = T(\gamma_n'(m)) + T(\gamma_n''(m)). \qquad (3.1)$$

If we denote by $\bar{b}_{m,n}$ the passage time from $(n, 0)$ to the line $\{x = m\}$, then

$$T(\gamma_n''(m)) \geq \bar{b}_{m,n}.$$

Note that $\bar{b}_{m,n}$ has the same distribution as $b_{0,n-m}$, so by Lemma 7,

$$P(T(\gamma_n''(m)) \leq (n-m)\mu - n^{4/7}) \leq P(b_{0,n-m} \leq (n-m)\mu - n^{4/7}) \leq C_1 \exp(-C_2 n^{1/14}). \qquad (3.2)$$

By (3.1) and (3.2),

$$P(a_{0,n} \leq T(\gamma_n'(m)) + (n-m)\mu - n^{4/7}) = P(T(\gamma_n''(m)) \leq (n-m)\mu - n^{4/7}) \leq C_1 \exp(-C_2 n^{1/14}). \qquad (3.3)$$

By (3.3),

$$\begin{aligned} &P(T(\gamma_n'(m)) \geq \mu m + 2n^{4/7}) \\ \leq\ &P(T(\gamma_n'(m)) \geq \mu m + 2n^{4/7}, a_{0,n} \geq T(\gamma_n'(m)) + (n-m)\mu - n^{4/7}) + C_1 \exp(-C_2 n^{1/14}). \end{aligned}$$

Note that $\{T(\gamma_n'(m)) \geq \mu m + 2n^{4/7}, a_{0,n} \geq T(\gamma_n'(m)) + (n-m)\mu - n^{4/7}\}$ implies that $a_{0,n} \geq n\mu + n^{4/7}$, so by Lemma 6,

$$P(T(\gamma_n'(m)) \geq \mu m + 2n^{4/7}) \leq P(a_{0,n} \geq n\mu + n^{4/7}) + C_1 \exp(-C_2 n^{1/14}) \leq C_3 \exp(-C_4 n^{1/14}). \tag{3.4}$$

Now we estimate the length of $\gamma_n'(m)$. Clearly, $|\gamma_n'(m)| \geq m$. Note that $m \geq n^{2/3} > \mu^{-1} n^{4/7}$ for large n, so by (3.4) we have

$$P(T(\gamma_n'(m)) \geq 3\mu m) \leq C_1 \exp(-C_2 n^{1/14}). \tag{3.5}$$

As $t(e) \geq 1$ almost surely for all edges, if $N = 3\mu$, then on $T(\gamma_n'(m)) \leq 3\mu m$,

$$|\gamma_n'(m)| \leq Nm \text{ almost surely.} \tag{3.6}$$

Together with (3.5) and (3.6), we have for all n and $m \geq n^{2/3}$,

$$P(|\gamma_n'(m)| \geq Nm) \leq C_1 \exp(-C_2 n^{1/14}). \tag{3.7}$$

Now we use the method of renormalization in Kesten and Zhang (1990). We define, for integer M and $w = (w_1, w_2) \in \mathbf{Z}^2$, the squares and the vertical strips by

$$B_M(w) = [Mw_1, Mw_1 + M) \times [Mw_2, Mw_2 + M) \text{ and } V_M(w_1) = [Mw_1, Mw_1 + M) \times (-\infty, \infty).$$

We denote the M-squares and the M-strip by $\{B_M(w) : w \in \mathbf{Z}^2\}$ and $\{V_M(w_1) : w_1 \in \mathbf{Z}\}$, respectively. For the optimal path γ_n, we denote a fattened $\gamma_n'(m, M)$ by

$$\gamma_n'(m, M) = \{B_M(w) : B_M(w) \cap \gamma_n'(m) \neq \emptyset\}.$$

By our definition,

$$|\gamma_n'(m, M)| \geq \frac{|\gamma_n'(m)|}{M^2} \geq \frac{m}{M^2}. \tag{3.8}$$

For each M-square $B_M(w)$, there are eight M-square neighbors. We say they are adjacent to $B_M(w)$. Since $\gamma_n(m)'$ is connected, $\gamma_n'(m, M)$ has to be connected through the square connections.

Note that if there are much fewer vertices of $[S_M(\gamma_n) \cup D(\gamma_n)]$ than other vertices in $\gamma_n'(m)$, then there are also fewer strips that contain a vertex in $[S_M(\gamma_n) \cup D(\gamma_n)]$ than the other strips. We say a strip $V_M(w_1)$ is *bad* if

$$B(m) \cap [S_M(\gamma_n) \cup D(\gamma_n)] \cap V_M(w_1) \neq \emptyset.$$

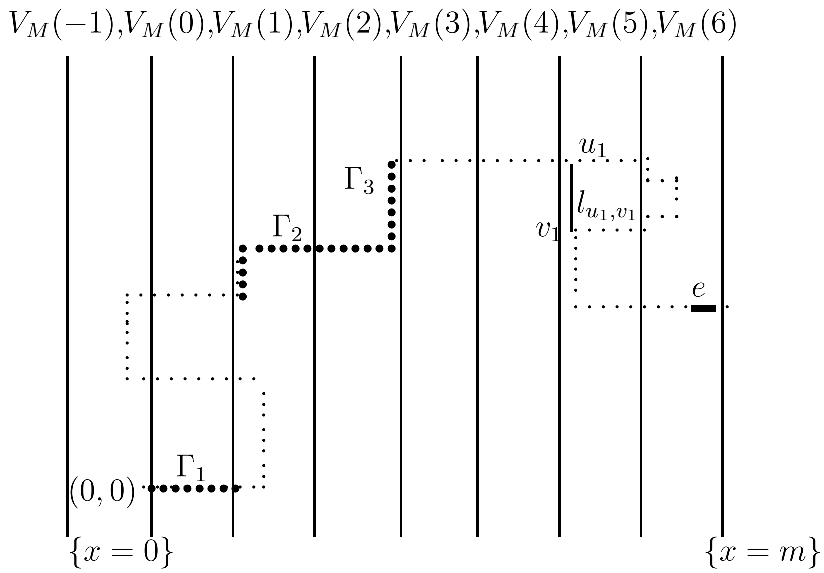

Figure 4: *The graph shows the path $\gamma'_n(m)$ in the strips, where the dotted line is $\gamma'_n(m)$. Γ_m is the path from the origin to u_1 along the dotted line from u_1 to v_1 along l_{u_1,v_1} and then from v_1 along the dotted line to $\{x = m\}$. Γ_m crosses three good strips: $V_M(0)$ with the big dotted line $\Gamma_1 = \Gamma_m(v_M(0,0), v_M(0,1))$; $V_M(1)$ with the big dotted line $\Gamma_2 = \Gamma_m(v_M(1,0), v_M(1,1))$; and $V_M(2)$ with the big dotted line $\Gamma_3 = \Gamma_m(v_M(2,0), v_M(2,1))$. $V_M(0)$ is a good-short-flat strip since the big dotted line Γ_1 is flat. $V_M(1)$ is a good-short-non-flat strip since the slope of its two terminal points in Γ_2 is larger than $\tan(\theta_1) - \delta_1$. $V_M(2)$ is a good-long strip since the number of vertices in Γ_3 is larger than $2M$. $V_M(5)$ is bad because it contains M-broken bridge l_{u_1,v_1} for γ_n. $V_M(3)$ and $V_M(4)$ would be good-short-flat strips, however, they are bad since they are next to $V_M(5)$. $V_M(6)$ is bad since e is a 1^+-edge, and it is also next to bad strip $V_M(5)$. After eliminating the bad strips, $\mathbf{G} = V_M(0) \cup V_M(1) \cup V_M(2)$.*

For each bad strip $V_M(w_1)$, we also say two neighbor strips to its left and two neighbor strips to its right are bad. Otherwise, we say a strip is *good*.

We eliminate all bad strips from $\mathbf{Z}^2$ (see Fig. 4) and denote the remaining vertices by $\mathbf{G}$. Recall our definitions of $\{l_{u_i,v_i}\}$ and $\gamma(u_i, v_i)$ for $i = 1, \cdots, \tau$, in Section 1. We may define Γ_m (see Fig. 4) as the path from the origin that goes along γ_n, meets u_i, then goes along l_{u_i,v_i} from u_i to v_i (not along $\gamma(u_i, v_i)$ in γ_n), and then goes along γ_n from v_i to u_{i+1}, until it meets $\{x = m\}$ for $i = 1, \cdots \tau$.

For a strip with

$$V_M(w_1) \subset \{0 < x < m\},$$

Γ_m will cross the strip $V_M(w_1)$. If we go along Γ_m to cross through $V_M(w_1)$, we will meet vertex $v_M(w_1, 1)$ at the left boundary of $V_M(w_1)$, and then go along Γ_m using the vertices inside $V_M(w_1)$ to meet vertex $v_M(w_1, 2)$ at the right boundary of $V_M(w_1)$. Note that Γ_m

may cross through $V_M(w_1)$ back and forth many times (see Fig. 4). We select one of them (see Fig. 4 for Γ_1) and denote this subpath of Γ_m from $v_M(w_1, 1)$ to $v_M(w_1, 2)$ by $\Gamma_m(v_M(w_1, 1), v_M(w_1, 2))$. If $V_M(w_1) \subset \mathbf{G}$, $\Gamma_m(v_M(w_1, 1), v_M(w_1, 2))$ cannot contain a vertex of l_{u_i,v_i} for $i = 1, \cdots, \tau$, so

$$\Gamma_m(v_M(w_1, 1), v_M(w_1, 2)) \subset \gamma_n(m)'.$$

For each such path $\Gamma_m(v_M(w_1, 1), v_M(w_1, 2))$, the path cannot contain a vertex v such that $v \in [S_M(\gamma_n) \cup D(\gamma_n)]$ by our definition. This tells us that $\Gamma_m(v_M(w_1, 1), v_M(w_1, 2))$ is a 1-path. If

$$|\Gamma_m(v_M(w_1, 1), v_M(w_1, 2))| \geq 2M,$$

we say the strip $V_M(w_1)$ is a *good-long strip*. Otherwise, it is a *good-short strip*.

Now we focus on all good-short strips. Assume $V_M(w_1)$ is a good-short strip. Then by (1.15), $\Gamma_m(v_M(w_1, 1), v_M(w_1, 2))$ has none of its own M-broken bridges. By Lemma 1, $\Gamma_m(v_M(w_1, 1), v_M(w_1, 2))$ is either northeast or southeast.

We say that a good-short strip $V_M(w_1)$ is *good-short-flat* if

$$|sl(v_M(w_1, 1), v_M(w_1, 2))| \leq \tan(\theta_1) - \delta_1,$$

where $0 < \delta_1 < \tan(\theta_1)$ is taken such that

$$\delta_1 = \eta/100 \tag{3.9}$$

for the η defined in Lemma 5. In contrast, $V_M(w_1)$ is *good-short-non-flat* if

$$|sl(v_M(w_1, 1), v_M(w_1, 2))| > \tan(\theta_1) - \delta_1.$$

For a good-short-flat strip $V_M(w_1)$, if $v_M(w_1, 1) \in B_M(w)$ for some $w = (w_1, w_2)$, then we say the square $B_M(w)$ is a *good-short-flat square*. By Lemma 3, for a fixed $B_M(w)$, there exist positive constants $\beta_i = \beta_i(F, \delta_1)$ for $i = 1, 2, 3, 4$ such that for all M,

$$P(B_M(w) \text{ is a good-short-flat square}) \leq \beta_1 M \exp(-\beta_2 M) \leq \beta_3 \exp(-\beta_4 M). \tag{3.10}$$

Now we denote by $F(\gamma_n'(m))$ the number of good-short-flat strips. We shall show that there exist $\delta_2 > 0$ with $\delta_2 \leq \eta/100$ for the η in Lemma 5, $M = M(\delta_2, \beta_3, \beta_4, N)$, and $C_i = C_i(F, N, M, \delta_1, \delta_2)$ for $i = 1, 2$ such that for all m and n with $n^{2/3} \leq m \leq n/2$,

$$P(F(\gamma_n'(m)) \geq \delta_2 m/M) \leq C_1 \exp(-C_2 n^{1/14}). \tag{3.11}$$

To show (3.11), we need to introduce a few basic methods to account for connected squares. By (3.7),

$$P(F(\gamma_n'(m)) \geq \delta_2 m/M) \leq P(F(\gamma_n'(m)) \geq \delta_2 m/M, |\gamma_n'(m)| \leq Nm) + C_1 \exp(-C_2 n^{1/14}). \tag{3.12}$$

As we mentioned, $\gamma_n'(m, M)$ is connected through horizontal, vertical, and diagonal squares. Therefore, there are at most 8^k choices for all M-squares in the path $\gamma_n'(m, M)$ if $|\gamma_n'(m, M)| = k$, where $|\gamma_n'(m, M)|$ represents the number of M-squares in $\gamma_n'(m, M)$. Let $\bar{B}_M(w)$ be the union of $B_M(w)$ and its eight neighbor M-squares. We call this a $3M$-*square*. If $B_M(w) \cap \gamma_n'(m) \neq \emptyset$, then $\bar{B}_M(w)$ contains at least M vertices of $\gamma_n'(m)$ in its interior. We collect all such $3M$-squares $\{\bar{B}_M(w)\}$ such that their center M-squares contain at least a vertex of $\gamma_n'(m)$.

We need to decompose $\gamma_n'(m, M)$ into disjoint $3M$-squares. We select an M-strip, from $\{V_M(w_1) : w_1 \in \mathbf{Z}\}$, with the maximum number of M-squares in $\gamma_n'(m, M)$. We denote by $V_{\max}(1)$ and i_1 the strip and the number of M-squares in $V_{\max}(1) \cap \gamma_n'(m, M)$, respectively. We also denote by $V_{\max}^-(1)$ the two neighboring M-strips on the left side of $V_{\max}(1)$. In addition, we denote by i_1^- the number of M-squares in $V_{\max}^-(1) \cap \gamma_n'(m, M)$. Similarly, $V_{\max}^+(1)$ and i_1^+ are denoted by the two neighboring M-strips on the right side of $V_{\max}(1)$, and the number of the M-squares in $V_{\max}^+(1) \cap \gamma_n'(m, M)$, respectively. By this construction,

$$i_1^+ \leq 2i_1, \text{ and } i_1^- \leq 2i_1.$$

We consider the top square, denoted by $B_M(u_1) \in \gamma_n'(m, M)$, in M-strip $V_{\max}(1)$. Clearly, $\bar{B}_M(u_1)$ contains at least M vertices in $\gamma_n'(m)$. Now we consider the second top M-square in this strip, denoted by $B_M(u_2) \subset \gamma_n'(n, M)$, such that $\bar{B}_M(u_2)$ and $\bar{B}_M(u_1)$ have no common vertices. We continue in this way to find all the disjoint $3M$-squares in this strip such that their center M-squares contain at least a vertex of $\gamma_n'(m)$. Note that it is possible that only one $3M$-square exists in this strip.

Next we select an M-strip, from $\{V_M(w_1) : w_1 \in \mathbf{Z}\} \setminus \{V_{\max}(1) \cup V_{\max}^{\pm}(1)\}$, with the maximum number of M-squares in $\gamma_n'(m, M)$. We denote by $V_{\max}(2)$ and i_2 the strip and the number of M-squares in $V_{\max}(2) \cap \gamma_n'(m, M)$, respectively. We also select the two neighboring M-strips on the left side $V_{\max}(2)$. Note that these two strips might overlap the strips of $V_{\max}^-(1) \cup V_{\max}^+(1)$. If so, we eliminate the overlapped strips from these two strips. We denote the selected strips by $V_{\max}^-(2)$. In addition, we denote by i_2^- the number of M-squares in $V_{\max}^-(2) \cap \gamma_n'(m, M)$. Note that $V_{\max}^-(2)$ might be empty after the overlapped strips are eliminated. If it is empty, then $i_2^- = 0$. Similarly, we select the two neighboring M-strips on the right side of $V_{\max}(2)$. After eliminating the overlapped strips of $V_{\max}^-(1) \cup V_{\max}^+(1)$ from these two strips, let $V_{\max}^+(2)$ and i_2^+ denote the selected strips and the number of M-squares in $V_{\max}^+(2) \cap \gamma_n'(m, M)$, respectively. By this construction,

$$i_2^+ \leq 2i_2, \text{ and } i_2^- \leq 2i_2.$$

With $V_{\max}(2)$, we select $3M$-squares in the same way that we selected for $V_{\max}(1)$. Note that the selected $3M$-squares in $V_{\max}(1)$ and $V_{\max}(2)$ are disjoint.

We then continue this process to select from the third to the last M-strip to find all the disjoint $3M$-squares squares such that their center M-squares contain at least a vertex of

$\gamma_n'(m)$. By these selections, we have

$$i_j^+ \leq 2i_j, \text{ and } i_j^- \leq 2i_j, \text{ for all } j. \tag{3.13}$$

By (3.13),

$$k = \sum_j (i_j + i_j^+ + i_j^-) \leq \sum_j 5i_j.$$

Thus,

$$k/5 \leq \sum_j i_j.$$

By our construction, each strip $V_{\max}(j)$ contains at least $\lceil i_j/3 \rceil$ of these disjoint $3M$-squares. With these observations, if $|\gamma'(m, M)| = k$ for large k, there are at least $k/15$ disjoint $3M$-squares such that their center M-squares contain at least a vertex of $\gamma_n'(m)$.

If there are at most Nm vertices in $\gamma_n'(m)$, then

$$(Mk/15) \leq Nm.$$

On the other hand, note that $\gamma_n'(m)$ is a path from $(0,0)$ to $\{x = m\}$, so it at least crosses m/M strips. This implies that

$$|\gamma_n'(m, M)| \geq m/M.$$

With these observations, on $|\gamma_n'(m)| \leq Nm$,

$$m/M \leq |\gamma_n'(m, M)| = k \leq 15Nm/M. \tag{3.14}$$

By (3.14),

$$\begin{aligned}
& P(F(\gamma_n'(m)) \geq \delta_2 m/M, |\gamma_n'(m)| \leq Nm) \\
= & \sum_{15Nm/M \geq k \geq m/M} P(F(\gamma_n'(m)) \geq \delta_2 m/M, |\gamma_n'(m)| \leq Nm, |\gamma_n'(m, M)| = k) \\
\leq & \sum_{15Nm/M \geq k \geq m/M} 8^k P(F(\gamma_n'(m)) \geq \delta_2 m/M, |\gamma_n'(m)| \leq Nm, \gamma_n'(m, M) = \Gamma, |\Gamma| = k),
\end{aligned} \tag{3.15}$$

where Γ is a fixed connected M-squares and $|\Gamma| = k$ means that Γ contains k squares. On the event $\{F(\gamma_n'(m)) \geq \delta_2 m/M, |\gamma_n'(m)| \leq Nm, \gamma_n'(m, M) = \Gamma, |\Gamma| = k)\}$, there exists a connected set Γ of k M-squares, with the bounds in (3.14) for k, and that contains at least $\delta_2 m/M$ good-short-flat squares. If there are more than $\delta_2 m/M$ of such good-short-flat strips, note that $v_M(w_1, 1)$ has to stay in the left boundary of an M-square $B_M(w) \subset \gamma_n'(m, M)$, so we select all such $\delta_2 m/M$ good-short-flat squares from Γ to have at most

$$\binom{15Nm/M}{\delta_2 m/M} \leq 2^{15Nm/M} \text{ choices.} \tag{3.16}$$

Therefore, by (3.10), (3.15), and (3.16),

$$P(F(\gamma'_n(m)) \geq \delta_2 m/M, |\gamma_n| \leq Nm) \leq (15Nm)8^{15Nm/M}2^{15Nm/M}\beta_3^{\delta_2 m/M}\exp(-\beta_4 m\delta_2). \tag{3.17}$$

Thus, for η in Lemma 5, $0 < \delta_2 \leq \eta/100$, and N in (3.7), we select $M = M(\eta, \delta_2, \beta_3, \beta_4, N)$ large such that (3.11) follows from (3.12) and (3.17).

Now we show that (3.11) implies Theorem 1. Note that $\gamma'_n(m)$ crosses out from $\{x = 0\}$ to $\{x = m\}$. Without loss of generality, we may assume that γ_n first meets the left or right boundary of $B(m)$. If not, we can always work on the horizontal strips rather than the vertical strips by using the same argument. There are at least m/M strips that have a common vertex with $\gamma'_n(m)$ on $B(m)$. On

$$|B(m) \cap [S_M(\gamma_n) \cup D(\gamma_n)]| \leq \delta m$$

for $\delta = \delta_2/M$ and δ_2 defined above, there are at most $(5\delta_2 m)/M$ bad strips. After eliminating these bad strips, we have at least $(1 - 5\delta_2)m/M$ good strips left. Note that $\gamma'_n(m)$ contains at least m horizontal edges and each of them at least costs time one, so

$$T(\text{horizontal edges of } \gamma'_n(m)) \geq m.$$

Now we account for the vertical edges in $\gamma'_n(m)$. In fact, under $F(\gamma'_n(m)) \leq \delta_2 m/M$ for a small $\delta_2 > 0$, the number of good-short-flat strips is less than $\delta_2 m/M$. The total number of good-long and good-short-non-flat strips is more than $m/M - 5\delta_2 m/M - \delta_2 m/M = (1 - 6\delta_2)m/M$. For each good-long or good-short-non-flat strip, by the definition of δ_1 in (3.9), $\Gamma(v_M(u, 1), v_M(u, 2))$ has to contain at least

$$(\tan(\theta_1) - \delta_1)M - 1 \geq (\tan(\theta_1) - 0.02\eta)M$$

vertical edges, where -1 is a possible error made by a non-integer number $(\tan(\theta_1) - \delta_1)M$. Therefore, on $F(\gamma'_n(m)) \leq \delta_2 m/M$, there are at least

$$[m(1 - 6\delta_2)/M][(\tan(\theta_1) - 0.02\eta)M] = m(1 - 6\delta_2)(\tan(\theta_1) - 0.02\eta)$$

total vertical edges. The total time of vertical edges costs at least $(\tan(\theta_1) - 0.1\eta)m$. Together with horizontal edges, by Lemma 5, on $F(\gamma'_n(m)) \leq \delta_2 m/M$,

$$T(\gamma'_n(m)) \geq m + [\tan(\theta_1) - 0.1\eta]m \geq m(\mu + \eta/2). \tag{3.18}$$

Therefore, for n and m with $m \geq n^{2/3} > 4\eta^{-1}n^{4/7}$, by (3.4), (3.11) and (3.18), there exist $C_i = C_i(F, \delta, \eta)$ for $i = 1, 2, 3, 4$ such that

$$\begin{aligned}
& P(|B(m) \cap [S_M(\gamma_n) \cup D(\gamma_n)]| \leq \delta m) \\
\leq\ & 2P(|B(m) \cap [S_M(\gamma_n) \cup D(\gamma_n)]| \leq \delta m, F(\gamma'_n(m)) \leq \delta_2 m/M) + C_1\exp(-C_2 n^{1/14}) \\
\leq\ & 2P(T(\gamma'_n(m)) \geq m(\mu + \eta/2)) + C_1\exp(-C_2 n^{1/14}) \\
\leq\ & C_3\exp(-C_4 n^{1/14}),
\end{aligned}$$

where factor 2 above is the result of the assumption that $\gamma'_n(m)$ meets the left or right boundary of $B(m)$ first. So Theorem 1 follows.

4 Corollaries of Theorem 1.

In this section, we need to generalize Theorem 1. Let δ_3 be a number such that

$$\delta_3^{\delta/16}(48M)^N = 1/2, \tag{4.0}$$

where δ, N, and M are the numbers selected in Theorem 1, (3.6) and (3.11) in Section 3. Since F is a right continuous function and $t(e)$ is not a constant, we may select $z > 1$ such that

$$P(1 < t(e) \leq z) = F(z) - F(1) \leq \delta_3, \tag{4.1}$$

where δ_3 is the number in (4.0). We say e is a z^+-edge if $t(e) > z$, where $z > 1$ is the number in (4.1). For the optimal path γ_n, we denote by $D(z, \gamma_n)$ all the vertices in γ_n that are adjacent to z^+-edges on γ_n. We also let $S_M(z, \gamma_n)$ be the set of vertices in γ_n that are adjacent to M-broken bridges $\{l_{u_i,v_i}\}$ of γ_n and, in addition, each broken bridge l_{u_i,v_i} contains at least one z^+-edge. With these definitions, we have the following corollary.

Corollary 1. *If F satisfies (1.5) and (1.10), there exist $C_i = C_i(F, M, N, z, \delta_3, \delta)$ for $i = 1, 2, 3$, and M in (3.11), N in (3.7), and δ in Theorem 1 and δ_5 in (4.0), and z in (4.1) such that*

$$P(|B(m) \cap [D(z, \gamma_n) \cup S_M(z, \gamma_n)]| \leq \delta m/4) \leq C_1 \exp(-C_2 n^{1/14})$$

for all m with $n/2 \geq m \geq n^{2/3}$ and

$$E(|B(m) \cap [D(z, \gamma_n) \cup S_M(z, \gamma_n)]|) \geq C_3 m.$$

Proof. By Theorem 1,

$$\begin{aligned}
& P(|B(m) \cap [D(z, \gamma_n) \cup S_M(z, \gamma_n)]| \leq \delta m/4) \\
\leq\ & P(|B(m) \cap [D(z, \gamma_n) \cup S_M(z, \gamma_n)]| \leq \delta m/4, |B(m) \cap [D(\gamma_n) \cup S_M(\gamma_n)]| \geq \delta m) \\
& + C_1 \exp(-C_2 n^{1/14}).
\end{aligned} \tag{4.2}$$

Note that if $|B(m) \cap D(\gamma_n) \cup S_M(\gamma_n)| \geq \delta m$, then either

$$|D(\gamma_n) \cap B(m)| \geq \delta m/2,$$

or

$$|S_M(\gamma_n) \cap B(m)| \geq \delta m/2.$$

We may assume that the first event occurs. For $v \in D(\gamma_n) \cap B(m)$, v is adjacent to e on γ_n with $t(e) > 1$. Thus, there are at least half of these vertices in $D(\gamma_n) \cap B(m)$ such that the edges adjacent to these vertices cannot take a value larger than z under

$$|B(m) \cap [D(z, \gamma_n) \cap S_M(z, \gamma_n)]| \leq \delta m/4.$$

In other words, there are at least $\delta m/4$ vertices in γ_n that are adjacent to edges $\{e\}$ on γ_n with $1 < t(e) \leq z$. Therefore, there are at least $\delta m/16$ edges in $\gamma_n'(m)$ adjacent to these vertices with $1 < t(e) \leq z$, since each vertex is adjacent to at most four edges. Recall that $\gamma_n'(m)$, defined in the last section, is the piece of γ_n from the origin to the line $\{x = m\}$. To fix our path $\gamma_n'(m)$, on $|\gamma_n'(m)| \leq Nm$, we have at most $4 \cdot 3^{Nm}$ choices. After fixing our path $\gamma_n'(m)$, we fix these edges with $1 < t(e) \leq z$, so we have at most

$$\sum_{k=1}^{Nm} \binom{Nm}{k} \leq 2^{Nm} \text{ choices.}$$

With these observations, by (3.7), if we take δ_3 satisfying (4.0),

$$\begin{aligned}
& P(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]| \leq \delta m/4, |B(m) \cap D(\gamma_n)| \geq \delta m/2) \\
\leq\ & P(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]| \leq \delta m/4, |B(m) \cap D(\gamma_n)| \geq \delta m/2, |\gamma_n'(m)| \leq Nm) \\
& + C_1 \exp(-C_2 n^{1/14}) \\
\leq\ & C_3 3^{Nm} 2^{Nm} \delta_3^{\delta m/16} + C_1 \exp(-C_2 n^{1/14}) \\
\leq\ & C_4 \exp(-C_5 n^{1/14}). \qquad (4.3)
\end{aligned}$$

Now we assume that the second event occurs:

$$|S_M(\gamma_n) \cap B(m)| \geq \delta m/2.$$

By (1.18), each M-broken bridge has at least an edge e with $t(e) > 1$. If this edge is not a z^+-edge, then we have $1 < t(e) \leq z$. Note that each M-bridge has at most $2M$ edges. Note also that if u is fixed, then there are at most four choices for $l_{u,v}$, so we use the same estimate as (4.3) to fix the path $\gamma_n'(m)$, the starting vertices in γ_n for M-broken bridges, the M-broken bridges, and the edges with $1 < t(e) \leq z$ in these M-broken bridges, resulting in

$$\begin{aligned}
& P(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]| \leq \delta m/4, |B(m) \cap S_M(\gamma_n)| \geq \delta m/2, |\gamma_n'(m)| \leq Nm) \\
\leq\ & C_3 3^{Nm} 2^{Nm} 4^{Nm} (2M)^{Nm} \delta_3^{\delta m/16}.
\end{aligned}$$

By (4.0), there exist $C_i = C_i(F, M, N, \delta_3, \delta, z)$ for $i = 1, 2$ such that

$$\begin{aligned}
& P(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]| \leq \delta m/4, |B(m) \cap S_M(\gamma_n)| \geq \delta m/2, |\gamma_n'(m)| \leq Nm) \\
\leq\ & C_1 \exp(-C_2 m). \qquad (4.4)
\end{aligned}$$

Together with (4.3) and (4.4),

$$\begin{aligned}
& P(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]| \leq \delta m/4, |B(m) \cap [D(\gamma_n) \cup S_M(\gamma_n)]| \geq \delta m) \\
\leq\ & P(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]| \leq \delta m/4, |B(m) \cap D(\gamma_n)| \geq \delta m/2, |\gamma_n'(m)| \leq Nm) \\
& + P(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]| \leq \delta m/4, |B(m) \cap S_M(\gamma_n)| \geq \delta m/2, |\gamma_n'(m)| \leq Nm) \\
& + 2C_1 \exp(-C_2 n^{1/14}) \\
\leq\ & C_4 \exp(-C_5 n^{1/14}). \qquad (4.5)
\end{aligned}$$

Therefore, the probability estimate in Corollary 1 follows from (4.5). With (4.5), we also have

$$E(|B(m) \cap [D(z,\gamma_n) \cup S_M(z,\gamma_n)]|) \geq C_3 m. \tag{4.6}$$

□

In Corollary 1, we showed that there are proportionally many vertices $\{v\}$ such that $v \in S_M(z,\gamma_n) \cup D(z,\gamma_n)$. If $u \in S_M(z,\gamma_n)$ for the optimal path γ_n, then to show Theorem 2, we need the M-broken bridge $l_{u,v}$ to stay inside a large square. Let us consider $B_M(u)$. The square has the center at $(u_1 + M/2, u_2 + M/2)$. Now we construct a larger square $G(B_M(u))$ with the same center at $(u_1 + M/2, u_2 + M/2)$ and a side length of $7M$. Note that $G(B_M(u))$ contains 49 of these M-squares and $B_M(u)$ is the center M-square among these 49 M-squares. Here we require these G-squares, the same as for the B-squares, to have lower and left boundaries but no top and right boundaries.

$G(B_M((0,0)))$ contains 49 of these M-squares. We denote them by $\{B_M(q_1), \cdots, B_M(q_{49})\}$, where q_s is the left-lower corner vertex of $B_M(q_s)$, the same as before. For example, we may think $q_1 = (0,0)$, $q_2 = (1,0)$, $q_3 = (0,1)$, $q_4 = (-1,0)$, $q_5 = (0,-1)\cdots$. For each vertex q_s, we work on $\{B_M(q_s + (7i,7j))\}$ for all integers i and j. In words, they are the M-square lattice on the plane at $7M$ apart. With this definition,

$$\bigcup_{s=1}^{49} \bigcup_{i,j} B_M(q_s + (7i,7j)) = \mathbf{Z}^2.$$

We also work on $\{G(B_M(q_s + (7i,7j)))\}$ for all i and j. By our definition, for q_s, these $7M$-squares $\{G(B_M(q_s + (7i,7j)))\}$ are disjoint for all the different i or j and the union of all these $7M$-squares is $\mathbf{Z}^2$.

For $n^{2/3} \leq m \leq n/2$ and q_s, we denote by $R_M(q_s,m,n)$ the number of squares of $\{B_M(q_s + (7i,7j))\}$ that contain at least a vertex $v \in B(m) \cap (S_M(z,\gamma_n) \cup D(z,\gamma_n))$ for all possible integers i and j. Note that for each $u \in B_M(q_s + (7i,7j))$, its M-bridge

$$l_{u,v} \subset G(B_M(q_s + (7i,7j))). \tag{4.7}$$

Note also that

$$\sum_{s=1}^{49} R_M(q_s,m,n) \geq |B(m) \cap (S_M(z,\gamma_n) \cup D(z,\gamma_n))|/M^2. \tag{4.8}$$

If m is not an integer, we may define $R_M(q_s,m,n) = R_M(q_s,\lfloor m \rfloor,n)$. With Corollary 1 and (4.8), we have the following corollary.

Corollary 2. *Under the same hypotheses as Theorem 1, there exists $C = C(F,z,M)$ such that*

$$E\left(\sum_{s=1}^{49} R_M(q_s,m,n)\right) \geq Cm.$$

5 Proof of Theorem 2.

Before the proof, we need to introduce a martingale inequality obtained by Newman and Piza (1995). Let $U_1, U_2, \cdots$ be disjoint edge subsets of $\mathbf{Z}^2$. We will express configuration ω for each k as $(\omega_k, \hat{\omega}_k)$, where ω_k (resp. the edges in $\hat{\omega}_k$) is the restriction of ω to U_k (resp. the edges in $\mathbf{Z}^2 \setminus U_k$). We also have, for each k, disjoint events D_k^+ and D_k^- in $\mathcal{F}(U_k)$, where $\mathcal{F}(U_k)$, for each k, is the sigma-field generated by $t(e)$ for $e \in U_k$. With these two events, let

$$H_k(\omega) = a_{0,n}^+(\hat{\omega}_k) - a_{0,n}^-(\hat{\omega}_k), \tag{5.0}$$

where

$$a_{0,n}^+(\hat{\omega}_k) = \inf_{\omega_k \in D_k^+} a_{0,n}(\omega_k, \hat{\omega}_k) \text{ and } a_{0,n}^-(\hat{\omega}_k) = \sup_{\omega_k \in D_k^-} a_{0,n}(\omega_k, \hat{\omega}_k).$$

Using these definitions, Newman and Piza (1995) proved in their Theorem 8 the following lemma.

Lemma 8. (Newman and Piza (1995)) *If U_k, D_k, and H_k satisfy the following:*
(i) Conditional on $\mathcal{F}(\mathbf{Z}^2 \setminus \cup_k U_k)$, then $\mathcal{F}(U_i)$ and $\mathcal{F}(U_j)$ are mutually independent for $i \neq j$.
(ii) There exist positive p and q such that for any k

$$P(\omega_k \in D_k^- | \mathcal{F}(\mathbf{Z}^2 \setminus U_k)) \geq p \text{ and } P(\omega_k \in D_k^+ | \mathcal{F}(\mathbf{Z}^2 \setminus U_k)) \geq q \text{ a.s.}$$

(iii) For every k, $H_k \geq 0$ a.s.
Suppose that, for $\epsilon > 0$ and each k, $F_k \subset \mathcal{F}(\mathbf{Z}^2)$ is a subset of event $\{H_k \geq \epsilon\}$. Then

$$\sigma^2(a_{0,n}) \geq pq\epsilon^2 \sum_k P(F_k)^2.$$

To apply Lemma 8, we set all vertices on $\mathbf{Z}^2$ in a spiral ordering starting from the origin. We denote these vertices by $\{(i_t, j_t)\}$ for $t = 1, 2, \cdots$. Now we define vertex sets $U_1 = G(B_M(q_1 + (7i_1, 7j_1)))$, $U_2 = G(B_M(q_1 + (7i_2, 7j_2)))$, $U_3 = G(B_M(q_1 + (7i_3, 7j_3)))$ $\cdots$, which is a spiral ordering of these $7M$-squares. Recall that our squares are the sets of vertices, but it is easy to reconsider them as the edges in these squares without the edges in the top and right boundaries.

Note that with this ordering, $U_1, \cdots, U_k, \cdots$ eventually cover all $\mathbf{Z}^2$, and

$$B(m) \subset \bigcup_{k=1}^{m^2} U_k. \tag{5.1}$$

Since $U_i \cap U_j = \emptyset$ for $i \neq j$, (i) in Lemma 8 holds. Let D_k^- be the event that all edges in U_k are 1-edges and let D_k^+ be the event that all edges in U_k are z^+-edges. Since U_k is finite, then

$$P(\omega_k \in D_k^- | \mathcal{F}(\mathbf{Z}^2 \setminus U_k)) \geq \vec{p}_c^{\,100M^2} \text{ and } P(\omega_k \in D_k^+ | \mathcal{F}(\mathbf{Z}^2 \setminus U_k)) \geq [1 - P(t(e) = 1) - \delta_3]^{100M^2},$$

where δ_3 is defined in (4.0).

Therefore, (ii) in Lemma 8 is satisfied if δ_3 is small enough. Note that $a_{0,n}$ is a coordinatewise non-decreasing function of ω, so (iii) holds.

Let $F_k(q_1)$ be the event that
(a) γ_n, defined in Section 1, has to use at least a z^+-edge of U_k or
(b) there is an M-broken bridge $l_{u_i,v_i} \subset U_k$ $(1 \leq i \leq \tau)$ for γ_n such that l_{u_i,v_i} contains at least one z^+-edge.
We will show that

$$F_k(q_1) \subset \{H_k \geq \min\{2, z-1\}\}, \tag{5.2}$$

so Lemma 8 can be applied.

On (a), for $\omega = (\omega_k, \hat{\omega}_k) \in F_k(q_1)$, note that if all z^+-edges in U_k are changed to be 1-edges, the passage time $T(\gamma_n)$ is at least saved by $z-1$, so

$$a_{0,n}(\omega) = T(\gamma_n)(\omega) \geq T(\gamma_n)(\omega_k^1, \hat{\omega}_k) + (z-1) \geq a_{0,n}^-(\hat{\omega}_k) + (z-1), \tag{5.3}$$

where ω_k^1 is the configuration in U_k such that all edges in U_k have value one, and $T(\gamma_n)(\omega_k^1, \hat{\omega}_k)$ is the passage time for path γ_n, but with configuration $(\omega_k^1, \hat{\omega}_k)$.

On the other hand, we denote by ω_k^+ the configuration in U_k such that

$$a_{0,n}^+(\hat{\omega}_k) = \inf_{\omega_k \in D_k^+} a_{0,n}(\omega_k, \hat{\omega}_k) = a_{0,n}(\omega_k^+, \hat{\omega}_k).$$

For the configuration ω_k^+, all edges in U_k have values larger than z. With this new configuration $(\omega_k^+, \hat{\omega}_k)$, if an optimal path for $a_{0,n}(\omega_k^+, \hat{\omega}_k)$ never passes through U_k, then

$$a_{0,n}(\omega_k^+, \hat{\omega}_k) = a_{0,n}(\omega). \tag{5.4}$$

By (5.3) and (5.4),

$$a_{0,n}^+(\hat{\omega}_k) = a_{0,n}(\omega_k^+, \hat{\omega}_k) = a_{0,n}(\omega) \geq a_{0,n}^-(\hat{\omega}_k) + z - 1. \tag{5.5}$$

Therefore, by (5.5) we have

$$H_k \geq (z-1). \tag{5.6}$$

If all optimal paths for $a_{0,n}(\omega_k^+, \hat{\omega}_k)$ have to pass through U_k, we denote by γ_n^+ an optimal path for the configuration $(\omega_k^+, \hat{\omega}_k)$. Then we reduce the value of the edges in $U_k \cap \gamma_n^+$ from z to 1 to have

$$a_{0,n}^+(\hat{\omega}_k) = a_{0,n}(\omega_k^+, \hat{\omega}_k) \geq T(\gamma_n^+)(\omega_k^1, \hat{\omega}_k) + (z-1) \geq a_{0,n}^-(\hat{\omega}_k) + (z-1), \tag{5.7}$$

where $T(\gamma_n^+)(\omega_k^1, \hat{\omega}_k)$ is the passage time for path γ_n^+ with configuration $(\omega_k^1, \hat{\omega}_k)$. Therefore, we still have (5.6).

On (b), for $\omega = (\omega_k, \hat{\omega}_k) \in F_k(q_1)$, $l_{u_i,v_i} \subset U_k$ has to contain at least one z^+-edge. If we change all edges in U_k from $z > 1$ to 1, then all the z^+-edges in l_{u_i,v_i} are changed to be

1-edges. If we go along the bridge l_{u_i,v_i} from u_i to v_i, we at least save time two, compared with going along $\gamma(u_i, v_i)$ from u_i to v_i. Therefore,

$$a_{0,n}(\omega) \geq a^-_{0,n}(\hat{\omega}_k) + 2. \tag{5.8}$$

If an optimal path for $a_{0,n}(\omega^+_k, \hat{\omega}_k)$ never passes through U_k, then by (5.8),

$$a_{0,n}(\omega^+_k, \hat{\omega}_k) = a_{0,n}(\omega) \geq a^-_{0,n}(\hat{\omega_k}) + 2. \tag{5.9}$$

If all optimal paths for $a_{0,n}(\omega^+_k, \hat{\omega_k})$ have to pass through U_k, by the same reason in (5.7), we have

$$H_k \geq 2. \tag{5.10}$$

Together with (5.9) and (5.10), on (b), we also have

$$H_k \geq \min(z - 1, 2).$$

Thus, (5.2) follows. It follows from Lemma 8 that there exists $C = C(F, M, \delta_5, z)$ such that

$$\sigma^2(a_{0,n}) \geq C \sum_k [P(F_k(q_1))]^2. \tag{5.11}$$

By Lemma 1 in Newman and Piza (1995), we have

$$\sigma^2(a_{0,n}) \geq C(\log n)^{-1} \left(\sum_{m=1}^{n^2/4} m^{-3/2} \left[\sum_{k=1}^{m} P(F_k(q_1)) \right] \right)^2. \tag{5.12}$$

By (5.2), we have for $n^{2/3} \leq m \leq n/2$,

$$ER_M(q_1, m, n) \leq \sum_{k=1}^{m^2} P(F_k(q_1)). \tag{5.13}$$

This shows that

$$\sigma^2(a_{0,n}) \geq C(\log n)^{-1} \left(\sum_{m=n^{4/3}}^{n^2/4} m^{-3/2} \left[ER_M(q_1, \sqrt{m}, n) \right] \right)^2. \tag{5.14}$$

Similarly, we have the same inequalities corresponding to $q_2, \cdots, q_{49}$, to have

$$\sigma^2(a_{0,n}) \geq C(\log n)^{-1} \left(\sum_{m=n^{4/3}}^{n^2/4} m^{-3/2} \left[ER_M(q_t, \sqrt{m}, n) \right] \right)^2. \tag{5.15}$$

If we sum all t from $t = 1$ to $t = 49$ together, by a standard inequality ($2ab \leq a^2 + b^2$ for positive a and b) we have

$$49\sigma^2(a_{0,n}) \geq C(2^{49}\log n)^{-1}\left(\sum_{m=n^{4/3}}^{n^2/4} m^{-3/2}\left[\sum_{t=1}^{49} ER_M(q_t, \sqrt{m}, n)\right]\right)^2. \tag{5.16}$$

By using Corollary 2 for each $m \geq n^{2/3}$ in (5.16), we have

$$\sigma^2(a_{0,n}) \geq C\log n.$$

So Theorem 2 follows.

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