Calculus 235—Exam III solutions

1. Use Type I (remark: Type II will not work)

$$\int\int_R \frac{\sin x}{x} dA = \int_0^1 dx \int_0^x \frac{\sin x}{x} dy = \int_0^1 \sin x = 1 - \cos 1.$$

2.

$$\text{Area} = \int\int dA = \int_0^5 dx \int_{-x}^{4x-x^2} dy = 125/6.$$

3.

$$\int\int_S e^{(x^2+y^2)} dA = \int_{-2}^{2\pi} d\theta \int_0^{\sqrt{2}} e^{r^2} r dr = \pi(e^2 - 1).$$

4.

$$\text{volume} = \int\int\int_E dv = \int_{-2}^{2} dx \int_1^{x^2} dy \int_0^{4-y} dz = 14/15.$$

5. (a) R is a rectangle with vertices $(0,2), (0,4), (-2,4), (-2,2)$.
(b) $\partial(x,y)/\partial(u,v) = 0.5$.
(c)

$$\int\int_D (x-y)/(x+y) dx dy = \int\int_R u/v 0.5 dv du = -\ln(2).$$

6. (a) The integral $\int_0^{2\pi} d\theta \int_0^1 dr$ represents the area of disk with radial one. (F needs r).
(b) $\int_0^1 \int_y^1 f(x,y) dx dy = \int_0^1 \int_0^x f(x,y) dy dx$ (T since we can change from type 2 to type 1).
(c) If $E = \{0 \leq x^2 + y^2 \leq 1\}$,, then $\int\int_E = \pi$. (T since it is the area of unit disk).

Calculus 235—Exam II solutions

1. (10 pts) Find and sketch the domain of function $z = \frac{\log x^2 - y}{1-2x}$.
$D = \{(x,y) | x \neq 0.5 \text{ and } x^2 > y\}$
2. (10 pts) If $g(x,y,z) = x + y + z$, $x = \sin(t)$ and $y = e^t$ and $z = s - t$, use the chain rule to find $\partial g/\partial t$ and $\partial g/\partial s$.
$\partial g/\partial t = \cos t + e^t - 1$, so $g/\partial t_{|t=0} = 1$. $\partial g/\partial s = x = sint_{|t=0} = 0$.
3. (10 pts) Write the equation of the tangent plane to the equation

$$x^2 y + y^2 z + z^2 x = 3 \text{ at } (1,1,1).$$

The tangent plane: $x + y + z = 3$.
4.(15 pts) Let $f(x,y) = \frac{2xy}{x^2 - y^2}$.
(a) Find $\lim_{(x,y)\to(2,1)} f(x,y)$. $\lim_{(x,y)\to(2,1)} f(x,y) = f(2,1) = 4/3$
(b) Use the two path test to show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
If $l_1 : x = 0$, then $\lim_{(0,y)\to(0,0)} f(0,y) = 0$. If $l_1 : x = 2y$, then $\lim_{(2y,y)\to(0,0)} f(y,y) = 4/3$. By the two path test, the limit does not exist.
(c) Find what points (sketching a graph on the XY-plane) is the function defined above continuous? $y \neq \pm x$.

5. (10 pts) Let $f(x, y, z) = x^2z + y^2 + xz^2$.
(a) Find the gradient at (1,1,1). $\bigtriangledown f =< 3, 2, 3 >$.
(b) Find the directional derivative at (1,1,1) in the direction $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$. $u = a/|a| =< 1/3, 2/3, 1/3 > .D_u f = 1/3$.
(c) In what direction is f increasing most rapidly at $(1, 1, 1)$, and what is the rate along the direction? Direction: $\bigtriangledown f =< 3, 2, 3 >$. Rate is $\sqrt{22}$.
6. (10 pts) If $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, use linear approximation to estimate $f(1.01, 1.99, 1.97)$.
$f_x(1, 2, 2) = 1/3, f_y(1, 2, 2) = 2/3, f_z(1, 2, 2) = 2/3$, so $f(1.01, 1.99, 1.98) = 3 + 1/3(0.01) + 2/3(-0.01) + 2/3(-0.03) = 2.977$.
7. (10 pts) Find the local maximum or minimum values of $f(x, y) = 9 - 2x + 4y - x^2 - 4y^2$.
$f_x = -2-2x, f_y = 4-8y$. So the critical pt is $(-1, 1/2)$. $f_{xx} = -2, f_{yy} = -8, f_{xy} = 0$. So $D = 16$ Thus $(-1, 1/2)$ is a local max pt and $f(-1, 1/2) = 11$.
8. (15 pts) True or false questions.
(a) If $f(x, y) = x^2$, then $\bigtriangledown f = 2x$. False since $\bigtriangledown f =< 2x, 0 >$.
(b) If (a, b) is a critical point of $f(x, y)$, i.e., $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then $f(x, y)$ has a local maxima or minimum at (a, b). False since (a, b) may be a saddle point.
(c) If $f(x, y)$ is a continuous function on $[0, 5]^2$, then $\lim_{(x,y)\to(1,2)} f(x, y) = f(1, 2)$. Ture since $(1, 2)$ is a continuous pt.

Calculus 235 — Exam I (Solutions)

1. (10 pts) Let $\mathbf{a} =< a_1, a_2, a_3 >$ and $\mathbf{b} =< b_1, b_2, b_3 >$.
Show that $\mathbf{a} \times (-3\mathbf{b}) = (3\mathbf{b}) \times \mathbf{a}$.

$$\mathbf{a} \times (-2\mathbf{b}) =< -2a_2b_3 + 2a_3b_2, 2a_1b_3 - 2a_3b_1, -2a_1b_2 + 2a_2b_1 >$$

$$(2\mathbf{b}) \times \mathbf{a} =< 2a_3b_2 23a_2b_3, 23a_3b_1 + 2a_1b_3, 2a_2b_1 - 2a_1b_2 > .$$

2. (20 pts) Let $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} =< -2, 2, -1 >$. Find each of the following.
(a) Their lengths. $|\mathbf{a}| = \sqrt{2^2 + (-2)^2 + (-1)^2} = 3$ and $|\mathbf{b}| = \sqrt{(-2)^2 + 2^2 + (-1)^2} = 3$.
(b) The unit vector with the same direction as $\mathbf{a}$. $\mathbf{a}/|\mathbf{a}| =< -2/3, -2/3, -1/3 >$
(c) The angle between $\mathbf{a}$ and $\mathbf{b}$. $\cos\theta = \mathbf{a} \cdot \mathbf{b}/|\mathbf{a}||\mathbf{b}| = (-4 - 4 + 1)/9 = -7/9$, so $\theta = 141°$.
(d) $\mathbf{a} \times \mathbf{b} = 4\mathbf{i} + 4\mathbf{j}$.
3. (15 pts) (a) Find parametric equations for the line through the point $(1, 2, 3)$ and perpendocular to $z = 5 - x - 3y$.

$$x = 1 + t, y = 2 + 3t, z = 3 + t$$

(b) Find the point in which the required line in (a) intersects the XZ plane. $(1/3, 0, 7/3))$
4. (20 pts) A plane through the point $(1, 2, 3)$ is perpendicular to the line joining the points (-1,5, -7) and (4,1,1).
(a) Write the equation of the plane.
(b) Find the distance from point $(1, 1, 1)$ to the plane in (a).

(a) Normal vector $< 4-(-1), 1-5, 1-(-7) > - < 5, -4, 8 >$. So the planar equation

$$5(x-1) - 4(y-2) + 8(z-3) = 0, 5x - 4y + 8z - 21 = 0.$$

(b) $d = |(5)(1) - (4)(1) + (8)(1) - 21|/\sqrt{5^2 + (-4)^2 + 8^2} = 12/\sqrt{105}$.

5. (20 pts) Let $\mathbf{r}(t) =< t^3, t^2, t >$. Find the curvature of $\mathbf{r}(t)$ at $t = 0$.
$r' =< 3t^2, 2t, 1 >$ so $r'' =< 6t, 20 >$. $r'(0) =< 0, 0, 1 >$, so $r''(0) =< 0, 2, 0 >$. $|r'(0) \times r''(0)| = | < 0, 0, 1 > \times < 0, 2, 0 > | = 2$.

$$\kappa(0) = |r' \times r''|/|r'|^3 = 2.$$

6. (10 pts) Find the position verctor of a particle that given velocity $\mathbf{v}(t) =< 2t, 3t^2, 4t^3 >$ and given initial position $\mathbf{r}(0) =< 1, 0, 0 >$.

$$\mathbf{r}(t) =< t^2 + 1, t^3, t^4 > .$$

7. (20 pts) Determine whether the following statement is true or false.
(a) For any vectors $\mathbf{v}$ and $\mathbf{u}$, $(\mathbf{v} \times \mathbf{u}) \cdot \mathbf{v} = 0$. True, since $\mathbf{v} \times u \perp \mathbf{v}$
(b) If $\mathbf{v} \cdot \mathbf{u} = 0$ for $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{u} = \mathbf{0}$. False, since $\mathbf{u} =< 1, 0 >$ and $\mathbf{v} =< 0, 1 >$.
(c) For any vectors $\mathbf{v}$ and $\mathbf{u}$, $(\mathbf{v} + \mathbf{u}) \times \mathbf{v} = \mathbf{u} \times \mathbf{v}$. True, since $\mathbf{v} \times \mathbf{v} = \mathbf{0}$
(d) For any vectors $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{u}$, $(\mathbf{v}+\mathbf{u}) \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} - \mathbf{w} \times \mathbf{u}$. Ture, $-\mathbf{w} \times \mathbf{v} = \mathbf{v} \times \mathbf{w}$ and $-\mathbf{w} \times \mathbf{u} = \mathbf{u} \times \mathbf{w}$.