On the infinite differentiability of the right edge in the supercritical oriented percolation

Yu Zhang

June 26, 2004

Abstract

Consider the oriented percolation model. Let r_n be the right edge. By a subadditive argument, it is known that

$$\lim_{n\to\infty} \frac{r_n}{n} = \alpha(p) \text{ a.s. and in } L_1.$$

In this paper, we show $\alpha(p)$ is infinitely differentiable in p for all $1 > p > p_c$.

Key words and phrases: The right edge, oriented percolation, supercritical phase and differentiability.

Mathematics subject classification: 60K 35.

1 Introduction and statement of results.

Consider the graph with vertices

$$\mathcal{L} = \{(m,n) \in \mathbf{Z}^2 : m+n \text{ is even } n \geq 0\}$$

and oriented edges from (m,n) to $(m+1,n+1)$ and to $(m-1,n+1)$. Each edge is independently open or closed with probability p or $1-p$. Let P_p be the measure on the sample space $\prod_{\text{edge}}\{\text{open, closed}\}$ and E_p be the expectation with respect to P_p. For two vertices u and v we say $u \to v$ if there is a sequence $v_0 = u, v_1, \cdots, v_m = v$ of points of $\mathcal{L}$ with the vertices $v_i = (x_i, y_i)$ and $v_{i+1} = (x_{i+1}, y_{i+1})$ for $0 \leq i \leq m-1$ such that $y_{i+1} = y_i + 1$ and v_i and v_{i+1} are connected by an open edge. If there is no such a sequence, we say $u \not\to v$. Define the oriented percolation cluster at (x,y) by

$$C^{(x,y)} = \{(x',y') \in \mathcal{L} : (x,y) \to (x',y')\}.$$

Let

$$\Omega_\infty^{(x,y)} = \{|C^{(x,y)}| = \infty\}.$$

The critical point is defined by

$$p_c = \sup\{p : P_p(\Omega_\infty^{(0,0)}) = 0\}.$$

It is well known that

$$0 < p_c < 1.$$

For $A \subset (-\infty, \infty)$, we denote a random subset by

$$\xi_n^A = \{x : \ \exists \ x' \in A \text{ such that } (x', 0) \to (x, n) \text{ for } n > 0\}.$$

The right edge for this set is defined by

$$r_n = \sup \xi_n^{(-\infty,0]} \qquad\qquad (\sup \emptyset = -\infty).$$

We know (see page 1004 in Durrett, 1984) by using Kingman's subadditive ergodic theorem that there exists a nonrandom constant $\alpha(p)$ such that

$$\lim_{n\to\infty} \frac{r_n}{n} = \inf_n \left\{\frac{E_p r_n}{n}\right\} = \alpha(p) \text{ a.s. and in } L_1. \tag{1.1}$$

Similar to most subadditive random sequences, Kingman's theorem tells us almost nothing about the behavior of $\alpha(p)$. A large amount of work on oriented percolation has been focused on the limit $\alpha(p)$. At this moment, it has been proved (see Durrett, 1984 and Bezuidenhout and Grimmett, 1991) that

$$\alpha(p) = -\infty \text{ if } p < p_c, \text{ and } \alpha(p_c) = 0, \text{ and } 1 \geq \alpha(p) > 0 \text{ if } p > p_c.$$

Although it seems to be impossible to write $\alpha(p)$ explicitly when $p > p_c$, we would like to understand the qualitative behavior of $\alpha(p)$ as a function of p. One can carry out the same proof in Liggett's book (Theorem 3.36 in Liggett 1985) to show the continuity of $\alpha(p)$ for $p \in [p_c, 1]$. With the continuity it is natural to ask the smoothness of $\alpha(p)$. It is conjectured that

$$\lim_{p \downarrow p_c} \frac{\alpha(p)}{(p - p_c)} = \infty$$

so $\alpha(p)$ should not have the left-hand derivative at p_c. In this paper we will investigate the differentiability for $\alpha(p)$ as $1 > p > p_c$.

Theorem. $\alpha(p)$ is infinitely differentiable for all $1 > p > p_c$.

Our method in the proof of the Theorem depends on the fact (see Kuczek, 1989) that r_n can be almost decomposed into an i.i.d. sequence. With this decomposition we can represent $\alpha(p)$ as a ratio of two expected values (see Lemma 1 below). Both terms in the ratio can be factored out as two parts: $P_p(\Omega_\infty^{(0,0)})$ and an infinite series with terms i times a probability that depends only on the edges in the finite triangle with corners at $(0,0)$, (i,i) and $(-i,i)$. The smoothness of the first part $P_p(\Omega_\infty^{(0,0)})$ is well known in Durrett (1984). The probability in the second part is just a polynomial with a degree about i^2 so the k-th derivative of the probability contributes at most i^{2k}. Then exponential decay of the probability allows us to add up the smoothness of the second part.

The method may not be applied for other subadditive processes such as the first passage percolation model. In this model, the time constant $\mu_x(p)$, similar to $\alpha(p)$, is the limit of a first passage time process, where p is the probability that an edge is open and x is a unit vector as the direction. The continuity of $\mu_x(p)$ took many years to be solved by Cox and Kesten (1981). But the differentiability for $\mu_x(p)$ still remains open. Moreover, Yukich and Zhang (2004) showed that the time constant is not three times differentiable for some p and x.

2 A representation of $\alpha(p)$.

We use the notations in Kuczek (1989) in this section. Let us denote

$$\xi'_0 = \xi_0^{(0,0)},$$

and for $n \geq 0$,

$$\xi'_{n+1} = \begin{cases} \{x : (y,n) \to (x,n+1) \text{ for some } y \in \xi'_n\} & \text{if this set is non empty} \\ \{n+1\} & \text{otherwise.} \end{cases}$$

Let

$$r'_n = \sup \xi'_n.$$

On $\{\xi_n^{(0,0)} \neq \emptyset\}$, we know that $r'_n = r_n$. A vertex $(x,n) \in \mathcal{L}$ is said to be a percolation point if and only if

$$I(\Omega_\infty^{(x,n)}) = 1.$$

Let $T_0 = 0$ and for $m \geq 1$

$$T_m = \inf\{n \geq T_{m-1} + 1 : (r'_n, n) \text{ is a percolation point}\}.$$

Define

$$\tau_1 = T_1, \tau_2 = T_2 - T_1, \cdots, \tau_m = T_m - T_{m-1},$$

where $\tau_i = 0$ if T_i and T_{i-1} are infinity. Also define

$$X_1 = r'_{T_1}, X_2 = r'_{T_2} - r'_{T_1}, \cdots, X_m = r'_{T_m} - r'_{T_{m-1}},$$

where $X_i = 0$ if $T_i = \infty$ and $T_{i-1} = \infty$. With these definitions, Kuczek (1989) proved the following proposition.

Proposition. (Kuczek) Conditioned on $\Omega_\infty^{(0,0)}$, $\{(\tau_i, X_i)\}$ are independently identically distributed with all finite moments.

In fact, Kuczek (1989) shows τ_1 has all moments without the condition $\Omega_\infty^{(0,0)}$. Moreover, if we use Kuczek's generating function argument (see page 1328 in Kuczek 1989), we can even show that if $p > p_c$, there exist C_1 and C_2 depending only on p such that

$$P_p(\tau_1 \geq n) \leq C_1 \exp(-C_2 n)$$

for all n. But we will not use this strong argument in this paper.

With these observations we can give a representation of $\alpha(p)$. We denote by

$$\bar{P}_p(\cdot) = P_p(\cdot \mid \Omega_\infty^{(0,0)}) \text{ and } \bar{E}_p \text{ the expected value respect to } \bar{P}_p.$$

Lemma 1. If $p > p_c$, then

$$\alpha(p) = \frac{\bar{E}_p X_1}{\bar{E}_p \tau_1}.$$

Proof. By the Proposition and a simple computation, we have

$$0 \leq \bar{E}_p X_1 \leq \bar{E}_p \tau_1 < \infty \text{ and } 0 < p P_p(\Omega_\infty^{(0,0)}) \leq \bar{E}_p \tau_1. \tag{2.1}$$

By (2.1) and the ergodic theorem, we have

$$\lim_{m\to\infty} \frac{\sum_{i=1}^m \tau_i}{m} = \bar{E}_p \tau_1 > 0 \ \bar{P}_p\text{-a.s. and } \lim_{m\to\infty} \frac{\sum_{i=1}^m X_i}{m} = \bar{E}_p X_1 \ \bar{P}_p\text{-a.s.} \tag{2.2}$$

By (1.1) and $P_p(\Omega_\infty^{(0,0)}) > 0$ we see that

$$\lim_{n\to\infty} \frac{r_n}{n} = \alpha(p) \ \bar{P}_p\text{-a.s} \tag{2.3}$$

By the definitions of X_i and τ_i the sequence

$$\frac{\sum_{i=1}^m X_i}{\sum_{i=1}^m \tau_i} \text{ is a subsequence of } \frac{r_n}{n} \ \bar{P}_p\text{-a.s.}$$

Thus if $p > p_c$,

$$\alpha(p) = \frac{\bar{E}_p X_1}{\bar{E}\tau_1}. \quad \square$$

Let

$$\bar{\tau}_1 = \begin{cases} \tau_1 & \text{if } |C^{(0,0)}| = \infty \\ \{0\} & \text{otherwise.} \end{cases}$$

Similarly,

$$\bar{X}_1 = \begin{cases} X_1 & \text{if } |C^{(0,0)}| = \infty \\ \{0\} & \text{otherwise.} \end{cases}$$

Note that if $p > p_c$, $P_p(\Omega_\infty^{(0,0)}) > 0$ so by Lemma 1

$$\alpha(p) = \frac{E_p \bar{X}_1}{E_p \bar{\tau}_1}, \tag{2.4}$$

where $E_p\bar{\tau}_1 > 0$. Since $\alpha(p) > 0$ for $p > p_c$ (see Durrett 1984),

$$\infty > E_p\bar{\tau}_1 > 0. \tag{2.5}$$

3 Proof of the Theorem

First we show a few lemmas before the proof of the Theorem.

Lemma 2. (Durrett 1984) If $p > p_c$, $P_p(\Omega_\infty^{(0,0)})$ is infinitely differentiable.

For a given positive integer j we denote a polynomial $f(p)$ by

$$f(p) = \sum_{m,n \leq j} a_{m,n} p^n (1-p)^m,$$

where $a_{m,n}$ is a positive constant.

Lemma 3. If $0 < p < 1$,

$$|f^{(k)}(p)| \leq \left(\frac{j}{p(1-p)}\right)^k f(p).$$

Proof.

$$\frac{d^k}{dp^k}(p^n(1-p)^m) = \sum_{r=0}^{k} \binom{k}{r} n_r m_{k-r} p^{n-r} (-1)^{k-r} (1-p)^{m-(k-r)},$$

where $x_r = x(x-1)\cdots(x-r+1)$. By the binomial theorem,

$$\begin{aligned} \left|\frac{d^k}{dp^k}(p^n(1-p)^m)\right| &= \left|\sum_{r=0}^{k}\binom{k}{r} n_r m_{k-r} p^{n-r}(-1)^{k-r}(1-p)^{m-(k-r)}\right| \\ &\leq p^n(1-p)^m\left(\frac{n}{p}+\frac{m}{1-p}\right)^k. \end{aligned}$$

Therefore,

$$\left|\frac{d^k}{dp^k}f(p)\right| \leq f(p)\left(\frac{j}{p}+\frac{j}{1-p}\right)^k \leq f(p)\left(\frac{j}{p(1-p)}\right)^k. \qquad \square$$

Next we will show a lemma regarding the differentiability of $E_p\bar{X}_1$.

Lemma 4. If $1 > p > p_c$, $E_p\bar{X}_1$ is infinitely differentiable.

Proof. We know that

$$E_p\bar{X}_1 = \sum_{|j|=1}^{\infty} jP_p(\bar{X}_1 = j).$$

By the definition of $\bar{X}_1$ and $\bar{\tau}_1$ we have

$$\bar{X}_1 = r_{\bar{\tau}_1}.$$

We also know that

$$\bar{\tau}_1 \geq |\bar{X}_1| = |j|, \tag{3.1}$$

since X_1 is the right edge on the level τ_1. By (3.1)

$$E_p\bar{X}_1 = \sum_{i=1}^{\infty}\sum_{j=-i}^{i} jP_p(\bar{X}_1 = j, \bar{\tau}_1 = i). \tag{3.2}$$

For $i > 1$, define the event

$$\mathcal{E}_i = \{(r_l, l) \not\to (x, i) \text{ for any } x \text{ and for all } 0 < l \leq i-1\}.$$

We also define the events

$$\begin{aligned} \mathcal{A} &= \{(0,0) \to (1,1)\}, \\ \mathcal{B} &= \{(0,0) \not\to (1,1), (0,0) \to (-1,1)\}. \end{aligned}$$

With these events we have for $i \geq 2$

$$\{\bar{X}_1 = j, \bar{\tau}_1 = i\} = \{(j,i) \to \infty\} \cap \{(0,0) \to (j,i)\} \cap \mathcal{E}_i. \tag{3.3}$$

For $i = 1$ we have

$$\{\bar{X}_1 = 1, \bar{\tau}_1 = 1\} = \{(1,1) \to \infty\} \cap \mathcal{A} \tag{3.4}$$

and

$$\{\bar{X}_1 = -1, \bar{\tau}_1 = 1\} = \{(-1,1) \to \infty\} \cap \mathcal{B}. \tag{3.5}$$

By (3.3)-(3.5)

$$E_p\bar{X}_1 = P_p(\Omega_\infty^{(0,0)})[p + p(1-p) + \sum_{i=2}^{\infty}\sum_{j=-i}^{i} jP_p((0,0) \to (j,i), \mathcal{E}_i)]. \tag{3.6}$$

By Lemma 2 we only need to show that

$$\sum_{i=2}^{\infty}\sum_{j=-i}^{i} jP_p((0,0) \to (j,i), \mathcal{E}_i)$$

is infinitely differentiable.

The event $\{(0,0) \to (j,i)\} \cap \mathcal{E}_i$ only depends on the edges in the triangle with corners at $(0,0)$, $(-i,i)$ and (i,i). We know that the number of the edges in this triangle is less than $2i^2$. By this observation, we can decompose the probability of the event into

$$P_p((0,0) \to (j,i), \mathcal{E}_i) = \sum_{n,m \le 2i^2} a_{n,m}(i,j)p^n(1-p)^m, \tag{3.7}$$

where $a_{n,m}$ is the number of subevents of

$$\{(0,0) \to (j,i)\} \cap \mathcal{E}_i$$

such that the triangle contains only n open and m closed edges. By lemma 3 for $p_c < p < 1$ there exists a positive constant $C(p,k) = C$ such that

$$|\frac{d^k}{dp^k}\sum_{n,m \le 2i^2} a_{m,n}(i,j)p^n(1-p)^m| \le \left(\frac{2i^2}{p(1-p)}\right)^k \frac{P_p(\bar{X}_1 = j, \bar{\tau}_1 = i,)}{P_p(\Omega_\infty^{(0,0)})} \le Ci^{2k}P_p(\bar{X}_1 = j, \bar{\tau}_1 = i). \tag{3.8}$$

Since $\bar{\tau}_1$ has all finite moments, by (3.8) and Markov's inequality there exist constants $C_1(k,p) = C_1$ and $C_2(k,p) = C_2$ such that

$$\sum_{i=2}^{\infty}\sum_{j=-i}^{i} |j\frac{d^k}{dp^k}\sum_{n,m \le 2i^2} a_{n,m}(i,j)p^n(1-p)^m| \le C_1\sum_{i=2}^{\infty} i^{2k+2}P_p(\bar{\tau}_1 = i) \le C_2\sum_{i=2}^{\infty}\frac{1}{i^2}. \tag{3.9}$$

Therefore, (3.9) shows that $E_p\bar{X}_1$ is k times differentiable for $p_c < p < 1$. $\square$

Next we show the differentiability of $E_p\bar{\tau}_1$.

Lemma 5. If $1 > p > p_c$, $E_p(\bar{\tau}_1)$ is infinitely differentiable.

Proof. Note that

$$E_p\bar{\tau}_1 = \sum_{i=1}^{\infty} iP_p(\bar{\tau}_1 = i) = \sum_{i=1}^{\infty} i \sum_{j=-i}^{i} P_p(\bar{X}_1 = j, \bar{\tau}_1 = i)$$

so the differentiability of $E_p\bar{\tau}_1$ follows from the same proof of Lemma 4. □

Proof the Theorem. We have shown that

$$\alpha(p) = \frac{E_p\bar{X}_1}{E_p\bar{\tau}_1},$$

where both $E_p\bar{X}_1$ and $E_p\bar{\tau}_1$ are infinitely differentiable and $E_p\bar{\tau}_1 > 0$. The Theorem now follows by the calculus rules for taking derivatives.

Acknowledgments The author would like to thank the referees who gave shorter proofs for Lemmas, and comments that improved the original version.

References

Bezuidenhout, C. and Grimmett, G. (1991) Exponential decay for subcritical contact and percolation processes. Ann. of Probab. Vol. **19** 984-1009.

Cox, T. and Kesten H.(1981), On the continuity of the time constant of first-passage percolation, J. App. Probab. **18**, 809-819.

Durrett, R. (1984) Oriented percolation in two dimensions. Ann. of Probab. Vol. **12** 999-1040.

Kuczek, T. (1989) The cental limit theorem for the right edge of supercritical oriented percolation. Ann. of Probab. Vol. **17** 1322-1332.

Liggett, T. (1985) Interacting particle systems. Springer-Verlag.

Yukichi, J. and Zhang, Y. (2004) Singularity points for first passage percolation. Preprint.

Yu Zhang
Department of Mathematics
University of Colorado
Colorado Springs, CO 80933
yzhang@math.uccs.edu