AUTOMATIC CONTINUITY, UNIQUE POLISH TOPOLOGIES, AND ZARISKI TOPOLOGIES ON MONOIDS AND CLONES

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ABSTRACT. In this paper we explore the extent to which the algebraic structure of a monoid M determines the topologies on M that are compatible with its multiplication. Specifically we study the notions of automatic continuity; minimal Hausdorff or T_1 topologies; Polish semigroup topologies; and we formulate a notion of the Zariski topology for monoids and inverse monoids.

If M is a topological monoid such that every homomorphism from M to a second countable topological monoid N is continuous, then we say that M has *automatic continuity*. We show that many well-known, and extensively studied, monoids have automatic continuity with respect to a natural semigroup topology, namely: the full transformation monoid $\mathbb{N}^{\mathbb{N}}$; the full binary relation monoid $B_{\mathbb{N}}$; the partial transformation monoid $P_{\mathbb{N}}$; the symmetric inverse monoid $I_{\mathbb{N}}$; the monoid $\mathrm{Inj}(\mathbb{N})$ consisting of the injective transformations of $\mathbb{N}$; and the monoid $C(2^{\mathbb{N}})$ of continuous functions on the Cantor set $2^{\mathbb{N}}$.

The monoid $\mathbb{N}^{\mathbb{N}}$ can be equipped with the product topology, where the natural numbers $\mathbb{N}$ have the discrete topology; this topology is referred to as the *pointwise topology*. We show that the pointwise topology on $\mathbb{N}^{\mathbb{N}}$, and its analogue on $P_{\mathbb{N}}$, are the unique Polish semigroup topologies on these monoids. The compact-open topology is the unique Polish semigroup topology on $C(2^{\mathbb{N}})$, and on the monoid $C([0,1]^{\mathbb{N}})$ of continuous functions on the Hilbert cube $[0,1]^{\mathbb{N}}$. The symmetric inverse monoid $I_{\mathbb{N}}$ has at least 3 Polish semigroup topologies, but a unique Polish inverse semigroup topology. The full binary relation monoid $B_{\mathbb{N}}$ has no Polish semigroup topologies, nor do the partition monoids. At the other extreme, $\mathrm{Inj}(\mathbb{N})$ and the monoid $\mathrm{Surj}(\mathbb{N})$ of all surjective transformations of $\mathbb{N}$ each have infinitely many distinct Polish semigroup topologies.

We prove that the Zariski topologies on $\mathbb{N}^{\mathbb{N}}$, $P_{\mathbb{N}}$, and $\mathrm{Inj}(\mathbb{N})$ coincide with the pointwise topology; and we characterise the Zariski topology on $B_{\mathbb{N}}$.

Along the way we provide many additional results relating to the Markov topology, the small index property for monoids, and topological embeddings of semigroups in $\mathbb{N}^{\mathbb{N}}$ and inverse monoids in $I_{\mathbb{N}}$.

Finally, the techniques developed in this paper to prove the results about monoids, are applied to function clones. In particular, we show that: the full function clone has a unique Polish topology; the Horn clone, the polymorphism clones of the Cantor set and the countably infinite atomless Boolean algebra all have automatic continuity with respect to second countable function clone topologies.

CONTENTS

1. Introduction

1.1. **Summary of main theorems and background.** A *semigroup* is a set together with an associative binary operation, and a *monoid* is a semigroup with an identity. Broadly speaking, the topic of this paper is to explore the extent to which the algebraic structure of a semigroup, or monoid, S determines the topologies on S that are compatible with its multiplication. If S is a semigroup and $\mathcal{T}$ is a topology on the set S, then $\mathcal{T}$ is called a *semigroup topology* if the multiplication function from $S \times S$, with the product topology, to S is continuous. A semigroup together with a semigroup topology is referred to as a *topological semigroup*. For many semigroups and monoids, the problem of determining the admissible semigroup topologies seems to be difficult. In this article, we consider the following well-known, and extensively studied, infinite monoids: the full transformation monoid X^X; the full binary relation monoid B_X; the partial transformation monoid P_X; the symmetric inverse monoid I_X; $\mathrm{Inj}(X)$ consisting of the injective transformations of X; $\mathrm{Surj}(X)$ consisting of the surjective transformations of X; and the monoids $C(2^{\mathbb{N}})$ and $C([0,1]^{\mathbb{N}})$ of continuous functions on the Cantor set $2^{\mathbb{N}}$ and the Hilbert cube $[0,1]^{\mathbb{N}}$, respectively. The general aim being to classify the minimal and maximal semigroup topologies, under some natural additional assumptions, such as being T_1, second countable, or Polish, for example. Along the way we will encounter a number of natural topologies that can be defined for arbitrary semigroups (the Fréchet-Markov, Hausdorff-Markov, and Zariski topologies), and discuss there interrelations (Section 2); we develop some general machinery (property **X**, Section 3); study the related notions of automatic continuity and the small index property for semigroups (Section 4); and characterise when a topological semigroup topologically embeds in $\mathbb{N}^{\mathbb{N}}$ (Section 5.1) and when an inverse monoid topologically embeds in $I_{\mathbb{N}}$ (Section 5.4). Finally, we foray slightly into the land of clones, defined in Section 7, and prove analogues of several of the main results in the previous sections.

This paper arose out of a previous version [53] which considered topologies on the full transformation monoid X^X.

The main theorems in this paper are summarised in Table 1 and Table 2; the relevant definitions are given in the later sections of the paper. Tables 1 and 2 are not exhaustive, in particular, many of the results in the paper apply to sets of arbitrary cardinality, rather than just $\mathbb{N}$.

The questions that are the focus of this paper originally arose in the context of groups; we will briefly, and probably incompletely, discuss the history of these questions for groups, and then for semigroups. A topological semigroup that happens to be a group is called a *paratopological group*, i.e. multiplication is continuous but inversion need not be. If G is a paratopological group and inversion is also continuous, then the topology is a *group topology*, and the group G is a *topological group*.

The problem of determining what topologies are compatible with the multiplication and inversion in a group has an extensive history that can be traced back to Cauchy and Markov. Markov [49] asked whether there exists an infinite group whose only group topologies are the trivial and discrete topologies; such a group is called *non-topologizable*. Shelah [71] showed that a non-topologizable group exists assuming the continuum hypothesis; Hesse [32] showed that the assumption of the continuum hypothesis in Shelah's construction can be avoided. Olshanskii [58] showed that an infinite family of the Adian groups (constructed by Adian [1] as a counter-example to the Burnside problem) are non-topologizable. A more recent paper on this topic is [45].

Given that there exist groups where the only group topologies are trivial and discrete, it is natural to ask if there are groups that admit a unique non-trivial non-discrete group topology. The only results in this direction, that we are aware of, require additional assumptions on the topology.

A topological space X is *Polish* if it is completely metrizable and separable. A *Polish semigroup* is just a topological semigroup where the topology is Polish. *Polish groups* are defined analogously to Polish semigroups. The fundamental results of R. M. Solovay [74] and S. Shelah [72] show that it is consistent with ZF without choice that any Polish group has a unique Polish group topology. The same is not true

		AC	# PST	Min	Z	F-M	H-M	# PIST	
full transformations	$\mathbb{N}^{\mathbb{N}}$	✓	1	✓	✓	✓	✓	-	§5.1
binary relations	$B_{\mathbb{N}}$	✓	0	✓	✓	✓	✓	-	§5.2
partial transformations	$P_{\mathbb{N}}$	✓	1	✓	✓	✓	✓	-	§5.3
partial bijections	$I_{\mathbb{N}}$	✓	≥ 3	✓	?	✓	✓	1	§5.4
injective transformations	$\mathrm{Inj}(\mathbb{N})$	✓	$\geq \aleph_0$	?	✓	?	✓	-	§5.5
surjective transformations	$\mathrm{Surj}(\mathbb{N})$	?	$\geq \aleph_0$	?	?	?	?	-	§5.5
continuous functions on $[0,1]^{\mathbb{N}}$	$C([0,1]^{\mathbb{N}})$	?	1	?	?	?	?	-	§6.1
continuous functions on $2^{\mathbb{N}}$	$C(2^{\mathbb{N}})$	✓	1	?	?	?	?	-	§6.2

AC	=	automatic continuity with respect to a second countable topology on the semigroup and the class of second countable topological semigroups
# PST	=	number of Polish semigroup topologies
Min	=	the minimum T_1 semitopological topology is known
Z	=	the semigroup Zariski topology is known
F-M	=	Fréchet-Markov topology is known
H-M	=	Hausdorff-Markov topology is known
# PIST	=	number of Polish inverse semigroup topologies

TABLE 1. A summary of the main results in this paper, and where they can be found in the document.

		AC	# PCT	
full function clone	$\mathscr{O}_{\mathbb{N}}$	✓	1	§7.2
Horn clone	$\mathscr{H}_{\mathbb{N}}$	✓	$\geq \aleph_0$	§7.3
polymorphisms on $2^{\mathbb{N}}$	$\mathrm{Pol}(2^{\mathbb{N}})$	✓	1	§7.4
polymorphisms on B_{∞}	$\mathrm{Pol}(B_{\infty})$	✓	1	§7.4

AC	=	automatic continuity with respect to the class of second countable topological function clones
# PCT	=	number of Polish clone topologies

TABLE 2. A summary of the main results for clones in this paper, and where they can be found in the document.

in ZFC: the additive group of real numbers $\mathbb{R}$ is a Polish group with the usual topology on $\mathbb{R}$, as too is the additive group $\mathbb{R}^2$. The two groups $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic, since they are vector spaces of equal dimension over the rationals $\mathbb{Q}$, but are not homeomorphic, since $\mathbb{R}^2$ with any point removed is connected and $\mathbb{R}$ with any point removed is not.

The Baire space $\mathbb{N}^{\mathbb{N}}$ is equipped with the product topology, where the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ have the discrete topology. This topology on $\mathbb{N}^{\mathbb{N}}$, and the subspace topology that it induces on any subset of $\mathbb{N}^{\mathbb{N}}$, will be referred to as the *pointwise topology* throughout this paper. The pointwise topology on $\mathbb{N}^{\mathbb{N}}$ is Polish. In addition to being a topological space, $\mathbb{N}^{\mathbb{N}}$ is a monoid under composition of functions; called the *full transformation monoid*. Indeed, $\mathbb{N}^{\mathbb{N}}$ is a topological semigroup with respect to the pointwise topology; see Section 5.1.

The analogue, in the context of groups, of the full transformation monoid $\mathbb{N}^{\mathbb{N}}$ is the symmetric group $\mathrm{Sym}(\mathbb{N})$. Since $\mathrm{Sym}(\mathbb{N})$ is a G_δ subset of $\mathbb{N}^{\mathbb{N}}$, the pointwise topology on $\mathrm{Sym}(\mathbb{N})$ is also Polish (see [42, Theorem 3.11]). Furthermore, $\mathrm{Sym}(\mathbb{N})$ with the pointwise topology is a topological group. In problem number 96 of the famous Scottish Book [50], Ulam asked if the symmetric group $\mathrm{Sym}(\mathbb{N})$ on the natural numbers has a locally compact Polish group topology. Ulam's problem was answered in the negative by Gaughan [26], who also showed that every T_1 group topology on $\mathrm{Sym}(\mathbb{N})$ contains the pointwise topology. It can be shown that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are Polish group topologies on the same group and $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $\mathcal{T}_1 = \mathcal{T}_2$. It therefore follows by Gaughan's result in [26] that $\mathrm{Sym}(\mathbb{N})$ has a unique Polish group topology. This result was strengthened in [66, Theorem 6.26], where it was shown that the pointwise topology is the only non-trivial separable group topology on $\mathrm{Sym}(\mathbb{N})$.

Many further examples of groups are known to have unique Polish group topologies: the groups of isometries of the Urysohn space and of the Urysohn sphere [68]; homeomorphism groups of a wide class of metric spaces such as any separable metric manifold [40] (including the Hilbert cube $[0,1]^{\mathbb{N}}$) or the Cantor set $2^{\mathbb{N}}$; the automorphism groups of many countable relational structures, in particular, Fraïssé limits, such as the rational numbers $\mathbb{Q}$ under the usual linear order, the countable random graph R [34, 35],

and the group of Lipschitz homeomorphisms of the Baire space [43]. Some further references on the uniqueness of Polish group topologies are [14, 25]. On the other hand, many groups have been shown to have no non-discrete Polish group topologies, for example: free groups [18]; the homeomorphism groups of the irrational and of the rational numbers and the group of Borel automorphisms of $\mathbb{R}$ [65]. Additional references include [15, 36, 37, 38, 39, 41, 66].

A topological group G is said to have *automatic continuity* with respect to a class of topological groups, if every homomorphism from G to a member of that class is continuous. Several of the topological groups mentioned in the previous paragraph have automatic continuity with respect to a natural class of topological groups, such as, for example, the separable topological groups. The group $\mathrm{Aut}([0, 1], \lambda)$ was shown to have automatic continuity by Yaacov, Berenstein, and Melleray [83], as was the infinite-dimensional unitary or orthogonal group to a separable group; see Tsankov [79]; further references are [3, 10, 23, 29, 47, 48, 59, 60, 67, 78]. If G is a Polish group that has automatic continuity with respect the class of Polish groups, then G has a unique Polish group topology.

Among many other interesting results in [43], it is shown that a Polish group G has automatic continuity with respect to the class of separable groups whenever there is a comeagre orbit of the action by conjugation of G on G^n for every $n \in \mathbb{N}$; such a group G is said to have *ample generics*. Many permutation groups have ample generics: the symmetric group $\mathrm{Sym}(\mathbb{N})$; the automorphisms of the random graph [34, 35]; the free group on countably many generators [13]; some further references are [31, 69, 73]. Some groups do not have ample generics, for instance: $\mathrm{Aut}(\mathbb{Q}, \leq)$ [46], and the homeomorphism group $\mathrm{Homeo}(\mathbb{R})$ of the reals $\mathbb{R}$.

Another notion that is implied by automatic continuity for a group (with respect to the class of separable groups) is the so-called small index property. A topological group G has the *small index property* if every subgroup of index at most $\aleph_0$ is open; some authors assume this condition for subgroups of index less than $2^{\aleph_0}$. Small index property does not necessarily imply automatic continuity; see Example 4.6. A topological group G has the small index property if and only if every homomorphism from G to $\mathrm{Sym}(\mathbb{N})$ is continuous. Many groups were shown to have the small index property before they were shown to have automatic continuity: $\mathrm{Sym}(\mathbb{N})$ [17, 64, 70], the automorphism groups of countable vector spaces over finite fields [22]; $\mathrm{Aut}(\mathbb{Q}, <)$ and the automorphism group of the countably infinite atomless Boolean algebra [77]; the automorphism group of the random graph and all automorphism groups of ω-categorical ω-stable structures [34]; the automorphism groups of the Henson graphs [30].

Analogously to the definition for groups, we will say that a topological semigroup S has *automatic continuity* with respect to a class of topological semigroups, if every homomorphism from S to a member of that class is continuous. A related notion is that of *automatic homeomorphicity*: a topological semigroup S has automatic homeomorphicity with respect to a class $\mathcal{C}$ of topological semigroups if every isomorphism from S to a member of $\mathcal{C}$ is necessarily a homeomorphism. In the literature, such as [6, 8, 9, 61, 62, 63], the terms automatic continuity and automatic homeomorphicity are exclusively used for the case where S is a submonoid of the full transformation monoid $\mathbb{N}^{\mathbb{N}}$ with the pointwise topology, and the class $\mathcal{C}$ consists of submonoids of $\mathbb{N}^{\mathbb{N}}$ with the pointwise topology.

Examples of monoids with automatic homeomorphicity with respect to the class of closed submonoids of $\mathbb{N}^{\mathbb{N}}$ include the full transformation monoid $\mathbb{N}^{\mathbb{N}}$ and the monoids of: order-endomorphisms or order-embeddings of the rational numbers $\mathbb{Q}$; injective functions $\mathrm{Inj}(\mathbb{N})$ on $\mathbb{N}$; self-embeddings of the countable random graph (the Rado graph); endomorphisms of the countable random graph; self-embeddings of the countable universal homogeneous digraph (see [6, 9]). Many of the results in [6, 9] are extended to the corresponding clones of polymorphisms; see also [8, 61, 62, 63].

1.2. Preliminaries.
We aim to give the relevant definitions within the sections where they are used whenever possible. In this section, we collect some technical definitions that are required throughout the paper.

In this paper, functions are written to the right of their arguments, and composed from left to right. If S is a semigroup, then we use S^1 to denote the monoid obtained by adjoining an identity 1 to S. A set X is *countable* if it is finite or has cardinality $\aleph_0$.

Suppose that S is a semigroup and that $\mathcal{T}$ is a topology on the set S. Then we will say that $\mathcal{T}$ is *left semitopological* for S if for every $s \in S$ the function $\lambda_s : S \longrightarrow S$ defined by $(x)\lambda_s = sx$ is continuous. If every function $\rho_s : S \longrightarrow S$ defined by $(x)\rho_s = xs$ is continuous, then we say that $\mathcal{T}$ is *right semitopological* for S. If $\mathcal{T}$ is left and right semitopological for S, then we say that $\mathcal{T}$ is *semitopological* for S. Every semigroup topology is semitopological. We refer to a semigroup with a left semitopological

topology as a *left semitopological semigroup*. Analogous definitions can be made for *right semitopological* and *semitopological*.

If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X (or indeed any collections of subsets of X), then the least topology on X containing $\mathcal{T}_1$ and $\mathcal{T}_2$ will be referred to as *the topology generated by* $\mathcal{T}_1$ and $\mathcal{T}_2$. If X and Y are topological spaces and $\mathscr{B}$ is a subbasis for Y, then $f : X \longrightarrow Y$ is continuous if and only if $(B)f^{-1}$ is open for all $B \in \mathscr{B}$. Hence if $\mathcal{T}_1$ and $\mathcal{T}_2$ are topological for S, then so too is the topology generated by $\mathcal{T}_1$ and $\mathcal{T}_2$. The analogous statement holds if "topological" is replaced by "semitopological", "right semitopological", or "left semitopological".

An *inverse semigroup* is a semigroup S such that for every $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$; y is usually denoted by x^{-1}. *Inverse semigroup topologies* and *topological inverse semigroups* are defined analogously to group topologies and topological groups. In other words, a topological inverse semigroup is a topological semigroup with continuous inversion. A semitopological group or inverse semigroup is a semitopological semigroup that happens to be a group or inverse semigroup, i.e. inversion is not assumed to be continuous.

2. Fréchet-Markov, Hausdorff-Markov, and Zariski topologies

In this section, we introduce, and prove some results about, three topologies that can be defined for any semigroup, that, in some sense, arise from the algebraic structure of that semigroup.

2.1. **Definitions.** The *Fréchet-Markov topology* of a semigroup S is the intersection of all T_1 semigroup topologies on S. Similarly, the *Hausdorff-Markov topology* of a semigroup S is the intersection of all Hausdorff semigroup topologies for S. The *inverse Fréchet-Markov topology* and *inverse Hausdorff-Markov topologies* of an inverse semigroup S are similarly defined to be the intersections of all T_1 and Hausdorff inverse semigroup topologies on S respectively. Clearly, the Fréchet-Markov topology on a semigroup is contained in the Hausdorff-Markov topology. The intersection of T_1 topologies is T_1 and the intersection of semigroup topologies is semitopological. Hence the Fréchet-Markov and Hausdorff-Markov topologies of a semigroup S are T_1 and S is semitopological with respect to both.

It is well-known that if G is a topological group, then every T_0 group topology is also $T_{3\frac{1}{2}}$ and so, in particular, there is no distinction in the theory of topological groups between the notions of inverse Fréchet-Markov and inverse Hausdorff-Markov topologies. In Example 2.6 we show that these notions are distinct in the context of topological semigroups and topological inverse semigroups.

By definition, the least T_1 topology that is semitopological for S is contained in the Fréchet-Markov topology of S. These two topologies may coincide, for example, if S is a semigroup of right zeros, then every topology on S is a semigroup topology. Hence the least T_1 topology on a semigroup S of right zeros is the cofinite topology, and the Hausdorff-Markov topology is the intersection of all the Hausdorff topologies on the set S, and it is straightforward to verify that this is also the cofinite topology. We will show that for several well-known examples of semigroups the Fréchet-Markov and Hausdorff-Markov topologies coincide. In particular, if S is a semigroup where the least T_1 topology $\mathcal{T}$ that is semitopological for S happens to be a Hausdorff semigroup topology, then $\mathcal{T}$ is both the Fréchet-Markov and Hausdorff-Markov topology for S.

Recall that the *group Zariski* topology on a group G is defined as the topology with subbasis consisting of the sets
$$\{g \in G : (g)\phi \neq 1_G\}$$
where 1_G is the identity of G and $\phi : G \longrightarrow G$ satisfies $(g)\phi = h_0 g^{i_0} h_1 g^{i_1} \cdots h_{k-1} g^{i_{k-1}}$ for every $g \in G$ and for some fixed $h_0, \ldots, h_{k-1} \in G$ and $i_0, \ldots, i_{k-1} \in \{-1, 1\}$. The group Zariski topology has been extensively studied in the literature of topological groups; see, for example, [12, 16, 49].

Analogously to the definition for groups, we define the *semigroup Zariski* topology on a semigroup S as the topology with subbasis consisting of
$$\{s \in S : (s)\phi_0 \neq (s)\phi_1\}$$
where $\phi_0, \phi_1 : S \longrightarrow S$ are any functions such that $(s)\phi_0 = t_0 s t_1 s \cdots t_{k-1} s$, $k \geq 1$ for every $s \in S$ and for some fixed $t_0, \ldots, t_{k-1} \in S^1$, and ϕ_1 is defined analogously for some fixed $u_0, \ldots, u_{l-1} \in S^1$. Similar to the notion for groups used in the literature, the complement of a set in the basis for the Zariski topology:
$$\{s \in S : (s)\phi_0 = (s)\phi_1\}$$
will be referred to as an *elementary algebraic* set.

The *inverse Zariski* topology for an inverse semigroup is defined analogously, where the functions are of the form $(s)\phi = t_0 s^{i_0} t_1 s^{i_1} \cdots t_{k-1} s^{i_{k-1}}$, where $i_0, \ldots, i_{k-1} \in \{-1, 1\}$. The inverse Zariski topology on

a group coincides with the usual notion of the Zariski topology on a group; the inverse semigroup and semigroup Zariski topologies on an inverse semigroup do not always coincide, see Corollary 5.16; the dual question comparing the group and semigroup Zariski topology on group is open, see Question 2.8.

We are primarily concerned with the semigroup Zariski topology in this paper, which we will often refer to as the Zariski topology, if no ambiguity will arise (such as when the semigroup under consideration is not inverse).

2.2. **Relations between the topologies.**

In this section we prove some results about the Zariski, Fréchet-Markov, and Hausdorff-Markov topologies, and their relationships to each other, for an arbitrary semigroup.

Proposition 2.1. *The Zariski topology on any semigroup S is contained in every Hausdorff semigroup topology on S. Similarly, the inverse Zariski topology on any inverse semigroup S is contained in every inverse Hausdorff semigroup topology on S.*

Proof. We prove the result for semigroups without inversion, the inverse semigroup proof is dual. Recall that if X is a Hausdorff topological space and $f, g : X \longrightarrow X$ are continuous functions, then the set

$$\{x \in X : (x)f = (x)g\}$$

is closed in X. Let $\mathcal{T}$ denote a Hausdorff semigroup topology on S. Let $\phi_0 : S \longrightarrow S$ be from the definition of a subbasic open set for the Zariski topology. In other words, $(s)\phi_0 = t_0 s t_1 s \cdots t_{k-1} s$, $k \geq 1$ for every $s \in S$ and for some fixed $t_0, \ldots, t_{k-1} \in S^1$. If $m = 2k$, then the function $\psi_0 : S \longrightarrow S^m$ defined by

$$(s)\psi_0 = (t_0, s, t_1, s, t_2, \ldots, t_{k-1}, s)$$

is continuous in every coordinate, and hence is continuous with respect to $\mathcal{T}$. The function ϕ_0 is then the composite of ψ_0 and the multiplication function from S^m to S, and is hence continuous with respect to $\mathcal{T}$. The function ϕ_1 is continuous by an analogous argument. It follows that every subbasic open set $\{s \in S : (s)\phi_0 \neq (s)\phi_1\}$ for the Zariski topology is open in $\mathcal{T}$. □

Proposition 2.2. *The Zariski topology is semitopological on any semigroup.*

Proof. Let S be any semigroup, and let $y \in S$ be arbitrary. We will show that $\lambda_y : S \longrightarrow S$ defined by $(x)\lambda_y = yx$ is continuous. Suppose that $\phi_0, \phi_1 : S \longrightarrow S$ are defined by

$$
\begin{aligned}
(s)\phi_0 &= t_0 s t_1 s \cdots t_{k-1} s \\
(s)\phi_1 &= u_0 s u_1 s \cdots u_{l-1} s
\end{aligned}
$$

for some $k, l \geq 1$, for every $s \in S$, and for some fixed $t_0, \ldots, t_{k-1}, u_0, \ldots, u_{l-1} \in S^1$. Then

$$
\begin{aligned}
\{s \in S : (s)\phi_0 \neq (s)\phi_1\}\lambda_y^{-1} &= \{x \in S : (yx)\phi_0 \neq (yx)\phi_1\} \\
&= \{x \in S : t_0(yx) \cdots t_{k-1}(yx) \neq u_0(yx) \cdots u_{l-1}(yx)\}.
\end{aligned}
$$

If

$$
\begin{aligned}
(x)\phi_0' &= t_0(yx)t_1(yx) \cdots t_{k-1}(yx) \\
(x)\phi_1' &= u_0(yx)u_1(yx) \cdots u_{l-1}(yx),
\end{aligned}
$$

then it is clear that $\{x \in S : (x)\phi_0' \neq (x)\phi_1'\}$ is open and so S is left semitopological with respect to the Zariski topology.

The proof that S is right semitopological with respect to the Zariski topology is dual. □

See Fig. 1 for the Hassé diagram of the containment of the Hausdorff-Markov, Fréchet-Markov, Zariski, and minimal T_1 topology that is semitopological for a given semigroup.

An *anti-homomorphism* from a semigroup S to a semigroup T is a function $\phi : S \longrightarrow T$ such that $(st)\phi = (t)\phi \, (s)\phi$ for all $s, t \in S$. An *anti-automorphism* of a semigroup S is a bijective anti-homomorphism from S to S.

Proposition 2.3. *Let S be a semigroup, let $\phi : S \longrightarrow S$ be an automorphism or anti-automorphism, and let $\mathcal{T}$ be a topology on S. Then the following hold:*

  (i) *if $\mathcal{T}$ is semitopological for S, then $(\mathcal{T})\phi = \{(U)\phi : U \in \mathcal{T}\}$ is semitopological for S;*
  (ii) *if $\mathcal{T}$ is topological for S, then $(\mathcal{T})\phi$ is topological for S;*
  (iii) *if $\mathscr{B}$ is a subbasis for $\mathcal{T}$, then $\{(B)\phi : B \in \mathscr{B}\}$ is a subbasis for $(\mathcal{T})\phi$.*
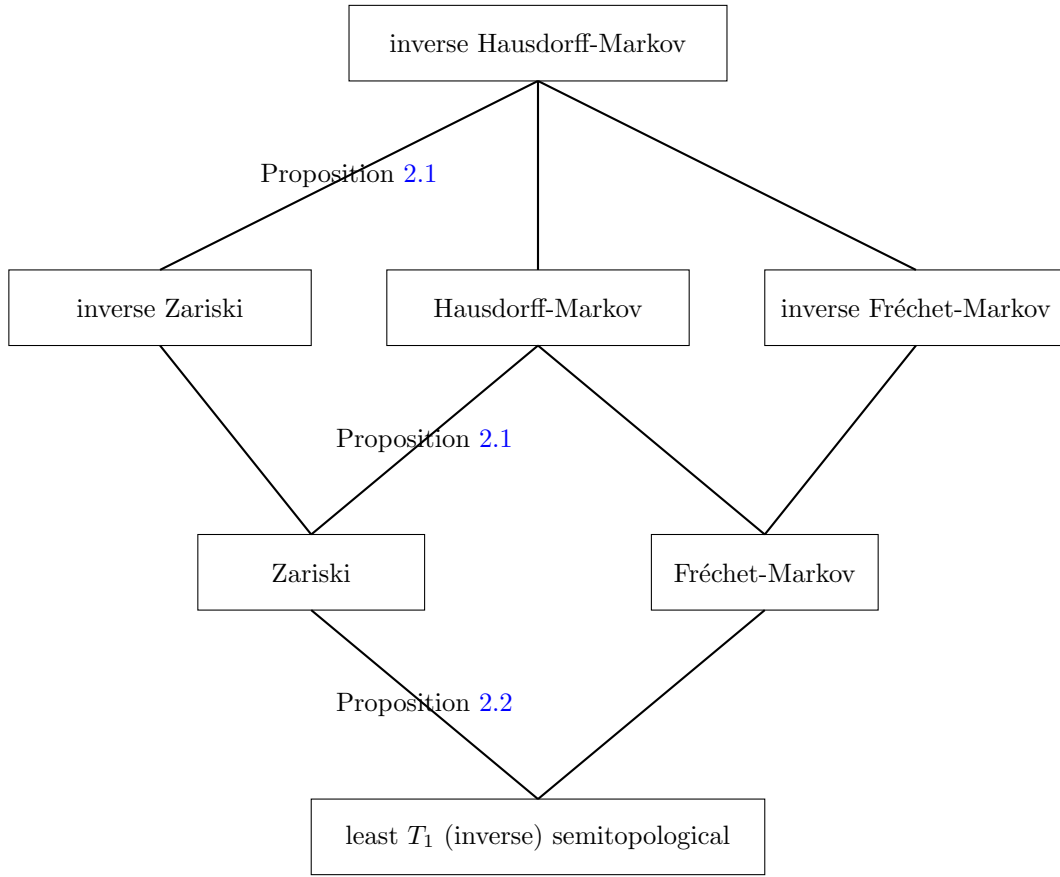
FIGURE 1. A diagram of containments of the Hausdorff-Markov, Fréchet-Markov, and Zariski topologies for a given (inverse) semigroup; a line from one topology to another indicates that for every (inverse) semigroup the topology below is contained in the one above. Note that the absence of a line does not necessarily indicate that one topology is not contained in the other. A label on an edge shows where the indicated containment is shown in the paper, trivial containments are not labelled. It is shown that the least T_1 topology that is semitopological for an inverse semigroup S coincides with the least T_1 topology that is inverse semitopological for S in Corollary 2.5.

Proof. We will prove the proposition in the case when ϕ is an anti-automorphism. The other case is even more straightforward.

(i). We will show that $(\mathcal{T})\phi$ is right semitopological. The proof that $(\mathcal{T})\phi$ is left semitopological follows by symmetry. Let $U \in \mathcal{T}$ be arbitrary. If $(s)\phi, (t)\phi \in S$ are such that $(s)\phi\,(t)\phi \in (U)\phi$, then, since ϕ is an anti-automorphism, $(s)\phi\,(t)\phi = (ts)\phi \in (U)\phi$ and so $ts \in U$. Since $\mathcal{T}$ is left semitopological for S, there exists an open neighbourhood $V_s \in \mathcal{T}$ of s, such that $tV_s \subseteq U$. Thus $(V_s)\phi$ is an open neighbourhood of $(s)\phi$ under $(\mathcal{T})\phi$ and $(V_s)\phi\,(t)\phi = (tV_s)\phi \subseteq (U)\phi$.

(ii). Let $U \in \mathcal{T}$ be arbitrary. If $(s)\phi, (t)\phi \in S$ are such that $(s)\phi\,(t)\phi \in (U)\phi$, then, since ϕ is an anti-automorphism, $(s)\phi\,(t)\phi = (ts)\phi \in (U)\phi$ and so $ts \in U$. Since $\mathcal{T}$ is a semigroup topology, there exist open neighbourhoods $V_s, V_t \in \mathcal{T}$ of s and t, respectively, such that $V_t V_s \subseteq U$. Thus $(V_s)\phi$ and $(V_t)\phi$ are open neighbourhoods of $(s)\phi$ and $(t)\phi$ under $(\mathcal{T})\phi$ and $(V_s)\phi\,(V_t)\phi = (V_t V_s)\phi \subseteq (U)\phi$.

(iii). By definition, $\{(B)\phi : B \in \mathscr{B}\} \subseteq (\mathcal{T})\phi$. Furthermore, ϕ is a bijection, so the unions and (finite) intersections of images are images of unions and (finite) intersections. $\square$

Proposition 2.4. *Every automorphism, or anti-automorphism, of a semigroup S is continuous with respect to each of the Zariski, Fréchet-Markov, and Hausdorff-Markov topologies for S, as well as the least T_1 topology that is semitopological for S.*

Proof. We first consider the Zariski topology. It suffices to show that the image of an elementary algebraic set under an automorphism or anti-automorphism is also elementary algebraic. Let

$$F = \{s \in S : t_0 s t_1 s \cdots t_{k-1} s = u_0 s u_1 s \cdots u_{l-1} s\}$$

be an elementary algebraic set. Let $\phi : S \longrightarrow S$ be an anti-automorphism. Then

$$
\begin{aligned}
(F)\phi &= \{(s)\phi \in S : t_0 s t_1 s \cdots t_{k-1} s = u_0 s u_1 s \cdots u_{l-1} s\} \\
&= \{s \in S : t_0(s\phi^{-1})t_1(s\phi^{-1}) \cdots t_{k-1}(s\phi^{-1}) = u_0(s\phi^{-1})u_1(s\phi^{-1}) \cdots u_{l-1}(s\phi^{-1})\} \\
&= \{s \in S : (t_0(s\phi^{-1})t_1(s\phi^{-1}) \cdots t_{k-1}(s\phi^{-1}))\phi = (u_0(s\phi^{-1})u_1(s\phi^{-1}) \cdots u_{l-1}(s\phi^{-1}))\phi\} \\
&= \{s \in S : s t_{k-1} \phi s \cdots t_1 \phi s t_0 \phi = s u_{l-1} \phi s \cdots u_1 \phi s u_0 \phi\},
\end{aligned}
$$

which is elementary algebraic. Similarly if ϕ is an automorphism of S, then $(F)\phi$ is also elementary algebraic.

Let ϕ be an automorphism or anti-automorphism of S. If $\mathcal{T}$ is any T_1 semigroup topology on S, then, by Proposition 2.3, $(\mathcal{T})\phi$ is also a T_1 semigroup topology for S. If $\mathfrak{T}$ is the collection of all T_1 semigroup topologies on S, then

$$\left(\bigcap_{\mathcal{T} \in \mathfrak{T}} \mathcal{T}\right)\phi = \bigcap_{\mathcal{T} \in \mathfrak{T}}(\mathcal{T}\phi) = \bigcap_{\mathcal{T} \in \mathfrak{T}} \mathcal{T}$$

and thus ϕ is continuous with respect to the Fréchet-Markov topology, which equals $\bigcap_{\mathcal{T} \in \mathfrak{T}} \mathcal{T}$. The proofs for the Hausdorff-Markov topology and the least T_1 topology that is semitopological for S are similar (the latter using Proposition 2.3(i). $\qquad \square$

The analogue of Proposition 2.4 holds for the inverse Zariski, inverse Fréchet-Markov, and inverse Hausdorff-Markov, and the proof is similar.

Corollary 2.5. *If S is an inverse semigroup, then inversion is continuous in each of the Zariski, Fréchet-Markov, and Hausdorff-Markov topologies on S, as well as the least T_1 topology that is semitopological for S.*

As mentioned above, if a topological group is T_0, then it is $T_{3\frac{1}{2}}$, and hence being T_0 and $T_{3\frac{1}{2}}$ is equivalent for such topologies. On the other hand, every topology is a semigroup topology, and so no such implication holds for topological semigroups, in general. It is natural to ask if there is any implication among separation axioms for certain classes of semigroups, such as the inverse semigroups. We show, in Example 2.6, that there is a topological inverse semigroup which is T_1 but not T_2. In fact, we find a inverse semigroup such that its Fréchet-Markov and its Hausdorff-Markov topologies are not equal.

Let $\leq$ be a total order on a set X. Then the *order topology* on X is the topology with subbasis consisting of the sets $\{y \in X : x < y\}$ and $\{y \in X : x > y\}$. Moreover, X together with the operation max forms a commutative inverse semigroup.

Example 2.6. Let X be an infinite set, let $\leq$ be a total order on X, and endow X with the structure of a semigroup by taking multiplication to be max. Then:
  (i) the Hausdorff-Markov topology and the Zariski topology on X coincide with the order topology on X;
  (ii) the sets of the form

$$B_{x,U} := \{y \in U : y > x\}$$

  where U is a cofinite subset of X and $x \in X^1$ is arbitrary, form a basis for the Fréchet-Markov topology on X.

Proof. **(i).** The elementary algebraic sets for the Zariski topology are precisely $V_{a,b} := \{x \in X : \max\{x, a\} = \max\{x, b\}\}$ and $W_{a,b} := \{x \in X : \max\{x, a\} = b\}$ where $a, b \in X^1$ are arbitrary.

If $a, b \in X$ are arbitrary, then the following hold:
  (1) if $a = b$, then $V_{a,b} = X$ and $W_{a,b} = \{x \in X : x \leq a\}$;
  (2) if $a < b$, then $V_{a,b} = \{x \in X : x \geq b\}$ and $W_{a,b} = \{b\}$;
  (3) if $a > b$, then $V_{a,b} = \{x \in X : x \geq a\}$ and $W_{a,b} = \varnothing$.

Hence the complements of the sets $V_{a,b}$ and $W_{a,b}$ are subbasic open sets for the order topology on X and are a basis for the Zariski topology, and so these two topologies coincide.

The order topology is also a Hausdorff semigroup topology for X. It is routine to verify that X is Hausdorff. We now show that the order topology is indeed a semigroup topology. Suppose that $\max\{a, b\} = b \in U$ for some subbasic open U. If $a, b \in U$, then $UU = U$. If $a \notin U$, then $U = \{y \in X : x < y\}$ for

some $x < b$, and so if $V = \{y \in X : b > y\}$, then $VU \subseteq U$. Finally, the order topology, being equal to the Zariski topology, is contained in the Hausdorff-Markov topology, which is contained in the order topology by above.

(ii). Let $\mathcal{T}$ be the topology on X with the sets $B_{x,U}$ as a basis. Note that $B_{1,U} = U$ for all U where $1 \in X^1$ is the adjoined identity. Hence the cofinite sets are basic open in $\mathcal{T}$, and so $\mathcal{T}$ contains the cofinite topology, and it is T_1. Let $\mathcal{S}$ be a T_1 semigroup topology for X. We show that $\mathcal{T} \subseteq \mathcal{S}$. Since $\mathcal{S}$ is T_1, the singletons are closed in $\mathcal{S}$ and hence

$$(X \backslash \{x\}) \lambda_x^{-1} = \{y \in X : \max\{x, y\} \neq x\} = \{y \in X : y > x\} \in \mathcal{S}$$

for all $x \in X$. If $x \in X$ is arbitrary and U is a cofinite subset of X, then

$$B_{x,U} = \{y \in X : y > x\} \cap U,$$

and so $\mathcal{T} \subseteq \mathcal{S}$. It follows that $\mathcal{T}$ is contained in the Fréchet-Markov topology for X, and hence it suffices to show that $\mathcal{T}$ contains the Fréchet-Markov topology. We will show that $\mathcal{T}$ is a semigroup topology for X.

Let $B_{s,U} \in \mathcal{T}$ be an arbitrary basic open set and let $a, b \in X$ be such that $\max\{a, b\} \in B_{s,U}$. We show that there are open neighbourhoods U_a, U_b of a and b respectively, such that $U_a U_b \subseteq B_{s,U}$. If $a = b$ then we choose $U_b = U_a = B_{s,U}$. Otherwise assume without loss of generality that $a < b$. Let $U_b := B_{\max\{a,s\},U}$ and $U_a := U \cup \{a\}$. If $a' \in U_a$ and $b' \in U_b$ then either $\max\{a', b'\} = b' \in U_b \subseteq B_{s,U}$ or $a' > b' > \max\{a, s\}$, in which case $\max\{a', b'\} = a' \in B_{\max\{a,s\},U} \subseteq B_{s,U}$ as required. $\qquad\square$

The topology defined in Example 2.6(i) can be distinct from the topology defined in Example 2.6(ii). For example, if $X = \mathbb{Z}$, then every open set in the Fréchet-Markov topology is unbounded above, whereas some of the basic open sets in the Hausdorff-Markov topology are bounded above by definition. It follows that Fréchet-Markov topology for a semigroup can be strictly contained in its Hausdorff-Markov topology, even if we only consider commutative inverse semigroups.

2.3. **Open questions.** We end this section with some open problems.

Question 2.7. *Is the Fréchet-Markov topology always contained in the Zariski topology?*

Question 2.8. *Is the semigroup version of the Zariski topology applied to a group equal to the group version of the Zariski topology applied to the same group?*

We will determine the Zariski topologies on several well-known classes of semigroups. Notably absent from this list is the symmetric inverse monoid, and so we ask the following question also.

Question 2.9. *What is the Zariski topology of the symmetric inverse monoid?*

3. Property **X**

In this section, we introduce a property that will be central to this paper. If S is a semitopological semigroup and A is a subset of S, then we say that S satisfies *property* **X** *with respect to A* if the following holds:

> for every $s \in S$ there exists $f_s, g_s \in S$ and $t_s \in A$ such that $s = f_s t_s g_s$ and for every neighbourhood B of t_s the set $f_s(B \cap A)g_s$ is a neighbourhood of s.

Although somewhat technical, property **X** is the crucial ingredient in many of the proofs in this paper where we determine unique or maximal Polish topologies on a semigroup S, or show automatic continuity for S with respect to the class of second countable topological semigroups.

Theorem 3.1. *Let S be a semigroup, let $\mathcal{T}$ be a topology with respect to which S is semitopological, and let $A \subseteq S$. If S has property **X** with respect to A, then the following hold:*

(i) *if T is a semitopological semigroup and $\phi : S \longrightarrow T$ is a homomorphism such that $\phi|_A$ is continuous, then ϕ is continuous;*

(ii) *if $\mathcal{T}'$ is a topology with respect to which S is semitopological and $\mathcal{T}'$ induces the same topology on A as $\mathcal{T}$, then $\mathcal{T}'$ is contained in $\mathcal{T}$;*

(iii) *if $\mathcal{T}$ is Polish and A is a Polish subgroup of S, then $\mathcal{T}$ is maximal among the Polish topologies with respect to which S is semitopological;*

(iv) *if A is a semigroup which has automatic continuity with respect to a class $\mathcal{C}$ of topological semigroups, then the semigroup S has automatic continuity with respect to $\mathcal{C}$ also.*

To prove Theorem 3.1, we require the notion of a Borel measurable function between topological spaces. Recall that a *σ-algebra* on a set X is a collection of subsets of X containing $\varnothing$ and which is closed under complements and countable unions (and hence also closed under countable intersections). If X is a topological space, then a set B is *Borel* if it belongs to the least σ-algebra containing the open sets in X. If X and Y are topological spaces, then $f : X \longrightarrow Y$ is *Borel measurable* if the pre-image of every Borel set is Borel. We require the following propositions which follows from Theorem 9.10, Proposition 11.5, and Corollary 15.2 in [42].

Proposition 3.2. *If G and H are Polish semitopological groups and $f : G \longrightarrow H$ is a Borel measurable homomorphism, then f is continuous.*

Proposition 3.3. *If X and Y are Polish spaces and $f : X \longrightarrow Y$ is a Borel measurable bijection, then f^{-1} is Borel measurable also.*

Proof of Theorem 3.1. **(i).** We denote the topology on T by $\mathcal{T}'$. We will show that ϕ is continuous at an arbitrary $s \in S$. Suppose that $U \in \mathcal{T}'$ is an open neighbourhood of $(s)\phi$. By property **X**, there are $f_s, g_s \in S$ and $t_s \in A$ such that $s = f_s t_s g_s$. Since ϕ is a homomorphism, $(s)\phi = (f_s t_s g_s)\phi = (f_s)\phi \, (t_s)\phi \, (g_s)\phi$ and so $(t_s)\phi \in V$ where

$$V = (U)(\lambda_{(f_s)\phi} \circ \rho_{(g_s)\phi})^{-1}.$$

In particular, since T is semitopological, V is an open neighbourhood of $(t_s)\phi$ in $\mathcal{T}'$ and

$$((f_s)\phi)V((g_s)\phi) \subseteq U.$$

Since $\phi|_A$ is continuous, $(V)\phi^{-1} \cap A$ is open in the subspace topology on A induced by $\mathcal{T}$. Hence $(V)\phi^{-1} \cap A = W \cap A$ for some $W \in \mathcal{T}$. By property **X**, there exists an open neighbourhood B of s such that $B \subseteq f_s(W \cap A)g_s$. Then

$$(B)\phi \subseteq (f_s(W \cap A)g_s))\,\phi = \left(f_s((V)\phi^{-1} \cap A)g_s\right)\phi \subseteq \left(f_s((V)\phi^{-1})g_s\right)\phi \subseteq ((f_s)\phi)V((g_s)\phi) \subseteq U,$$

and so ϕ is continuous at s.

(ii). Let $\mathcal{T}'$ be a semitopological semigroup topology for S that induces the same subspace topology as $\mathcal{T}$ on A. Then the restriction of the identity homomorphism $\mathrm{id} : (S, \mathcal{T}) \to (S, \mathcal{T}')$ to A is continuous. Thus $\mathrm{id} : (S, \mathcal{T}) \to (S, \mathcal{T}')$ is continuous by part (i) and so $\mathcal{T}' \subseteq \mathcal{T}$, as required.

(iii). Suppose that $\mathcal{T}'$ is a Polish semitopological semigroup topology on S and that $\mathcal{T} \subseteq \mathcal{T}'$. We will show that $\mathcal{T}' \subseteq \mathcal{T}$, and so $\mathcal{T}$ is maximal. As in the previous part, it suffices to show that the restriction $\mathrm{id}|_A$ of the identity function $\mathrm{id} : (S, \mathcal{T}) \longrightarrow (S, \mathcal{T}')$ is continuous.

Since A is a Polish subspace of S with respect to $\mathcal{T}$, it follows that A is G_δ in $\mathcal{T}$, and so A is G_δ in $\mathcal{T}'$ also. Hence A is a Polish subspace of S with respect to $\mathcal{T}'$. Since $\mathrm{id}|_A^{-1}$ is a Borel measurable bijection between Polish spaces, it follows from Proposition 3.3 that $\mathrm{id}|_A$ is a Borel measurable function between $(S, \mathcal{T})$ and $(S, \mathcal{T}')$. Therefore, by Proposition 3.2, $\mathrm{id}|_A$ is continuous.

(iv). Suppose that T is a topological semigroup belonging to the class $\mathcal{C}$ and that $\phi : S \longrightarrow T$ is a homomorphism. It follows that $\phi|_A$ is a homomorphism from A to T which is therefore continuous by automatic continuity. It follows from part (i) that ϕ is continuous. $\square$

We end this section by observing that "having property **X** with respect to a subset" is a transitive relation.

Lemma 3.4. *Let S be a topological semigroup and let $A \leq T \leq S$. If S has property **X** with respect to T and T has property **X** with respect to A, then S has property **X** with respect to A.*

Proof. Let $s \in S$. Since S has property **X** with respect to T, there exist $f_s, g_s \in S$ and $t_s \in T$ such that $f_s t_s g_s = s$ and $f_s(B' \cap T)g_s$ is a neighbourhood of s for every neighbourhood B' of t_s. Similarly, there exist $t_{t_s} \in A$ and $f_{t_s}, g_{t_s} \in T$ such that $t_s = f_{t_s} t_{t_s} g_{t_s}$ and $f_{t_s}(B \cap A)g_{t_s}$ is a neighbourhood of t_s in T for every neighbourhood B of t_{t_s}.

If B is any neighbourhood of t_{t_s} in S, then $f_{t_s}(B \cap A)g_{t_s}$ is a neighbourhood of t_s in T. Hence there is a neighbourhood B' of t_s in S such that $B' \cap T = f_{t_s}(B \cap A)g_{t_s}$. In particular, $f_s f_{t_s}(B \cap A)g_{t_s}g_s = f_s(B' \cap T)g_s$ is a neighbourhood of s and $s = f_s f_{t_s} t_{t_s} g_{t_s} g_s$, and S has property **X** with respect to A. $\square$

4. AUTOMATIC CONTINUITY AND THE SMALL INDEX PROPERTY

The notions of automatic continuity for semigroups and groups, as defined in the introduction, are superficially different. The former is defined with respect to a class of topological semigroups, and the latter with respect to a class of topological groups. Of course, a class of topological groups is also a class of topological semigroups. We will use the automatic continuity of the symmetric group, and the homeomorphisms of the Cantor space, with respect to the class of second countable semigroups in Sections 5 and 6, and so we require the following proposition.

Proposition 4.1. *Let G be a semitopological group. Then G has automatic continuity with respect to the class of second countable topological semigroups if and only if G has automatic continuity with respect to the class of second countable topological groups.*

Proof. ($\Leftarrow$) Let S be a second countable topological semigroup and let $\phi : G \longrightarrow S$ be a homomorphism. Then $H := (G)\phi$ is a subgroup of S and the induced topology on H is a second countable paratopological group topology for H; we denote this topology by $\mathcal{T}$. If $\mathcal{T}^{-1} := \{U^{-1} : U \in \mathcal{T}\}$, then, by Proposition 2.3, $\mathcal{T}^{-1}$ is a semigroup topology on H and, since $\mathcal{T}^{-1}$ is homeomorphic to $\mathcal{T}$, it is second countable. It then follows that the topology $\mathcal{T}'$ generated by $\mathcal{T}$ and $\mathcal{T}^{-1}$ is also a second countable paratopological group topology for H. Since the topology $\mathcal{T}'$ is generated by an inverse closed collection of sets, it follows that the inverse operation of H is continuous under this topology, and thus $(H, \mathcal{T}')$ is a second countable topological group. Since ϕ is a homomorphism, ϕ is continuous with respect to $\mathcal{T}'$ and the topology on G. If $U \in \mathcal{T}$, then $U \in \mathcal{T}'$ and so $(U)\phi^{-1}$ is open in G by the continuity of $\phi : G \longrightarrow (H, \mathcal{T}')$.

($\Rightarrow$) This is immediate as second countable topological groups are second countable topological semigroups. $\qquad\square$

Corollary 4.2. *The symmetric group $\mathrm{Sym}(\mathbb{N})$ and the group $H(2^{\mathbb{N}})$ of homeomorphisms of the Cantor space $2^{\mathbb{N}}$ have automatic continuity with respect to the class of second countable topological semigroups.*

Proof. It is known that the given groups both have automatic continuity with respect to the class of second countable topological groups (see [43] and [66]), and so the result follows immediately by Proposition 4.1. $\qquad\square$

A topological group G has the *small index property* if every subgroup of index at most $\aleph_0$ is open; some authors assume this condition for subgroups of index less than $2^{\aleph_0}$. A topological group G has the small index property if and only if every homomorphism from G to $\mathrm{Sym}(\mathbb{N})$ with the pointwise topology is continuous. However, having the small index property is strictly weaker than having automatic continuity with respect to the class of the second countable topological groups. In particular, in Example 4.6, we give an example of a group satisfying the small index property but that does not have automatic continuity with respect to the class of second countable topological groups.

We define a notion analogous to small index property for topological semigroups. A topological semigroup S is said to have the *right small index property* if every right congruence on S with countably many classes is open in $S \times S$ with the product topology. Similarly, S is said to have the *left small index property* if every left congruence on S with countably many classes is open. If G is a topological group, then G has the left small index property if and only if G has the right small index property.

In Proposition 5.18, we show that the notions of left and right small index property are distinct, by exhibiting an example which has one property but not the other. If a topological semigroup S has automatic continuity, then S has both left and right small index property (Corollary 4.5). In Proposition 4.3, we will show that a topological monoid S has the right small index property if and only if every homomorphism from S to $\mathbb{N}^{\mathbb{N}}$ with the topology of pointwise convergence is continuous.

Proposition 4.3. *A topological monoid M has the right small index property if and only if every homomorphism from M to $\mathbb{N}^{\mathbb{N}}$ with the pointwise topology is continuous.*

Proof. First note that an equivalence relation on a topological space is open if and only if all of its equivalence classes are open.

We start by assuming that M has the right small index property. Let $\phi : M \longrightarrow \mathbb{N}^{\mathbb{N}}$ be a semigroup homomorphism. Recall that the pointwise topology on $\mathbb{N}^{\mathbb{N}}$ has a subbasis consisting of the sets $U_{i,j} = \{f \in \mathbb{N}^{\mathbb{N}} : (i)f = j\}$ over all $i, j \in \mathbb{N}$. Note that $U_{i,j}$ is an equivalence class under the right congruence

$$\rho := \{(f, g) \in \mathbb{N}^{\mathbb{N}} : (i)f = (i)g\}.$$

Since ϕ is a homomorphism, the relation $\rho' := \{(f, g) \in M : (f\phi, g\phi) \in \rho\}$ on M is a right congruence on the monoid M and ρ' has at most as many classes as ρ. Since M has the right small index property, ρ' is

open. The preimage of $U_{i,j}$ under ϕ is an equivalence class of ρ' and hence open. Since $U_{i,j}$ is an arbitrary subbasic open set, it follows that ϕ is continuous.

Now suppose that every semigroup homomorphism $\phi : M \longrightarrow \mathbb{N}^{\mathbb{N}}$ is continuous. Let ρ be a right congruence on M with countably many classes. We define a homomorphism $\phi : M \longrightarrow (M/\rho)^{(M/\rho)}$ by

$$(g/\rho)((f)\phi) = gf/\rho.$$

The semigroup $(M/\rho)^{(M/\rho)}$, together with the pointwise topology, is topologically isomorphic to $\mathbb{N}^{\mathbb{N}}$ and thus ϕ is continuous with respect to these topologies. Let m/ρ be an arbitrary equivalence class of ρ. It suffices to show that m/ρ is an open subset of M. The set $\{(f)\phi \in (M)\phi : (1_M/\rho)(f)\phi = m/\rho\}$ is open in $(M)\phi$ under the subspace topology inherited from $M/\rho^{M/\rho}$ and

$$\{(f)\phi \in (M)\phi : (1_M/\rho)((f)\phi) = m/\rho\}\phi^{-1} = \{(f)\phi \in (M)\phi : f/\rho = m/\rho\}\phi^{-1} = m/\rho.$$

Since ϕ is continuous, it follows that m/ρ is indeed open as required. $\square$

By symmetry, we obtain the following corollary to Proposition 4.3.

Corollary 4.4. *A topological monoid M has the left small index property if and only if every homomorphism from M to $\mathbb{N}^{\mathbb{N}}$ with left actions is continuous.*

The following corollary follows straight from the definition of automatic continuity, Proposition 4.3, and Corollary 4.4.

Corollary 4.5. *If a topological monoid has automatic continuity with respect to the class of second countable topological semigroups, then it has both the left and right small index properties.*

The converse of Corollary 4.5 is not true. We give an example which demonstrates that the small index property is strictly weaker than automatic continuity even in the case of abelian Hausdorff topological groups.

Example 4.6. Define the topology $\mathcal{T}$ on the group of real numbers $\mathbb{R}$ under addition by choosing the cosets of all countable index subgroups as a subbasis. This subbasis is actually a basis as the intersection of finitely many subgroups of countable index is another subgroup of countable index. Moreover, $\mathcal{T}$ is a group topology since for every countable index subgroup G of $\mathbb{R}$ we have $(G + x) + (G + y) \subseteq (G + x + y)$ and $-(G + x) = (G - x)$ for all $x, y \in \mathbb{R}$. By its definition, $\mathcal{T}$ clearly gives $(\mathbb{R}, +)$ the small index property.

Since $\mathcal{T}$ is a group topology, to show that $\mathcal{T}$ is Hausdorff, it suffices to show that $\mathcal{T}$ is T_0. We will show that for all $x \in \mathbb{R}\backslash\{0\}$ there is a countable index subgroup of $\mathbb{R}$ which does not contain x. Let $x \in \mathbb{R}\backslash\{0\}$. By Zorn's Lemma we can extend the set $\{x\}$ to a basis $\mathscr{B}$ for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Let G be the subspace spanned by $\mathscr{B}\backslash\{x\}$. As $\mathscr{B}$ is linearly independent, $x \notin G$. But as G is a codimension 1 subspace of a vector space over a countable field, it follows that G has countable index as required.

To see that $(\mathbb{R}, +)$ under $\mathcal{T}$ does not have automatic continuity, we will show that the identity map from $\mathbb{R}$ under $\mathcal{T}$ to $\mathbb{R}$ with the standard topology is not continuous. The interval $(-1, 1)$ is not open in $\mathcal{T}$ since all non-empty open sets in $\mathcal{T}$ contain a translation of a non-trivial group, and are thus unbounded.

In Proposition 5.18 we give an example of a topological semigroup which has the right small index property but not the left small index property, so in particular it also does not have automatic continuity. Moreover, this shows that the right and left small index properties are not equivalent in general.

Although Example 4.6 shows that the small index property is strictly weaker than automatic continuity for topological groups in general, we ask if these notions are equivalent for Polish groups.

Question 4.7. *If G is a Polish group and G has the small index property, then does G have automatic continuity with respect to the class of second countable groups?*

5. CLASSICAL MONOIDS

In this section, we consider several monoids that have been extensively studied in the literature which we refer to collectively as *classical monoids*.

Let X be any set. A *binary relation* on X is just a subset of $X \times X$. We will denote the set of all binary relations on X by B_X. If $f, g \in B_X$, then their composition $f \circ g$ is defined by

$$(x, y) \in f \circ g \quad \text{if } (x, z) \in f \text{ and } (z, y) \in g \text{ for some } z \in X.$$

The set B_X with composition of binary relations is the *full binary relation monoid* on X. If $f \in B_X$, then we define the *inverse* of f to be $f^{-1} = \{(y, x) : (x, y) \in f\}$. The relation f^{-1} is not always an inverse for f, in the sense of inverse semigroups or groups, since $f \subseteq ff^{-1}f$, and this containment may be strict.

If $\Sigma \subseteq X$ and $f \in B_X$, then the *image of Σ under f* is the set

$$(\Sigma)f = \{y \in X : (x, y) \in f \text{ for some } x \in \Sigma\}.$$

The *domain* and *image* of f are $\mathrm{dom}(f) = (X)f^{-1}$ and $\mathrm{im}(f) = (X)f$.

We will also consider the following natural subsemigroups of B_X: the *partial transformation monoid*

$$P_X = \{f \in B_X : |(\{x\})f| \leq 1 \text{ for all } x \in X\};$$

the *full transformation monoid*

$$X^X = \{f \in P_X : |(\{x\})f| = 1 \text{ for all } x \in X\};$$

the *symmetric inverse monoid*

$$I_X = \{f \in P_X : |(\{x\})f^{-1}| \leq 1 \text{ for all } x \in X\};$$

the *monoid of injective functions*

$$\mathrm{Inj}(X) = \{f \in X^X : |(\{x\})f^{-1}| \leq 1 \text{ for all } x \in X\};$$

the *monoid of surjective functions*

$$\mathrm{Surj}(X) = \{f \in X^X : (X)f = X\}.$$

These monoids have been extensively studied in the literature in both the finite and infinite cases; see, for example, [19, 33, 54, 55, 56, 57] and the references therein. Of course, (partial) functions are sets of pairs, and so we may write $(x, y) \in f \in P_X$ to denote that $x \in \mathrm{dom}(f)$ and $(x)f = y$. Similarly, we may write that $f \subseteq g$ where $f, g \in P_X$ and $g|_{\mathrm{dom}(f)} = f$.

In Section 5.6, we will also briefly consider certain, so-called, diagram monoids. The definition of these monoids is more involved, and as it turns out, studying them as topological monoids is less fruitful than the monoids defined above. For these reasons we delay the definitions of these monoids to Section 5.6.

5.1. **The full transformation monoid.** Recall that a subbasis for the pointwise topology on X^X, where X is any infinite set, consists of the sets

$$U_{x,y} = \{f \in X^X : (x)f = y\}$$

for all $x, y \in X$. This semigroup topology on X^X is essentially the only such topology considered in the literature. While it is possible to define further semigroup topologies on X^X, we will show that every such semigroup topology on X^X either fails to be T_1 or contains the pointwise topology. If X is countable, then we will show that the pointwise topology is the unique T_1 second countable semigroup topology on X^X; this may account, at least in part, for the absence of alternative topologies in the literature. Furthermore, X^X has automatic continuity with respect to the pointwise topology and the class of second countable topological semigroups. We show that the pointwise topology coincides with the Zariski topology and characterise those topological semigroups that topologically embed into X^X when X is countable.

We begin this subsection with a technical lemma that we will use repeatedly later on.

Lemma 5.1. *Let X be an infinite set, and let S be a subsemigroup of P_X such that S contains all of the constant transformations (defined everywhere on X), and for every $x \in X$ there exists $f_x \in S$ such that $(x)f_x^{-1} = \{x\}$ and $(X)f_x$ is finite. If $\mathcal{T}$ is a topology which is semitopological for S, then the following are equivalent:*

(i) *$\mathcal{T}$ is Hausdorff;*
(ii) *$\mathcal{T}$ is T_1;*
(iii) *$\{f \in S : (y, z) \in f\}$ and $\{f \in S : t \notin \mathrm{dom}(f)\}$ are open with respect to $\mathcal{T}$ for all $y, z, t \in X$;*
(iv) *$\{f \in S : (y, z) \in f\}$ and $\{f \in S : t \notin \mathrm{dom}(f)\}$ are closed with respect to $\mathcal{T}$ for all $y, z, t \in X$.*

Proof. **(i)** $\Rightarrow$ **(ii).** This follows since every Hausdorff space is T_1.

(ii) $\Rightarrow$ **(iii).** Suppose that $y, z \in X$, and $g, h \in S$ are such that h is the constant transformation with value y, and g is arbitrary. Then $y \notin \mathrm{dom}(g)$ or $(y)g \neq z$ if and only if hgf_z is $\varnothing$ or a constant with value belonging to $(X)f_z \setminus \{z\}$. Since $(X)f_z$ is finite by assumption, it follows that $\{f \in S : (y, z) \notin f\}$ is the preimage under left multiplication by h and right multiplication by f_z of the finite set consisting of the empty function and those constant transformations whose image belongs to $(X)f_z \setminus \{z\}$. Hence $\{f \in S : (y, z) \notin f\}$ is closed.

Similarly, $y \in \mathrm{dom}(g)$ if and only if hgf_y is a constant with the value belonging to $(X)f_y$. In this case, we obtain $\{f \in S : y \in \mathrm{dom}(f)\}$ as the preimage under left multiplication by h and right multiplication by f_y of the finite set consisting of those constant transformations whose image belongs to $(X)f_y$.

In the two cases above, we showed that the sets $\{f \in S \colon (y, z) \notin f\}$ and $\{f \in S \colon t \in \mathrm{dom}(f)\}$ are closed, and so their complements are open, as required.

(iii) $\Rightarrow$ (i). If $f, g \in S$ are such that $f \neq g$, then there is $y \in X$ with $(y)f \neq (y)g$ or $\mathrm{dom}(f) \neq \mathrm{dom}(g)$. In the first case, $\{s \in S \colon (y)s = (y)f\}$ and $\{s \in S \colon (y)s = (y)g\}$ are the required disjoint neighbourhoods of f and g, respectively. In the second case, we may assume that there is $y \in \mathrm{dom}(f) \setminus \mathrm{dom}(g)$, and so the sets $\{s \in S \colon y \in \mathrm{dom}(s)\} = \bigcup_{z \in X} \{s \in S \colon (y, z) \in f\}$ and $\{s \in S \colon y \notin \mathrm{dom}(s)\}$ are disjoint neighbourhoods of f and g.

(iii) $\Rightarrow$ (iv) This follows from the equalities

$$S \setminus \{f \in S \colon (y, z) \in f\} = \bigcup_{t \in X \setminus \{z\}} \{f \in S \colon (y, t) \in f\} \cup \{f \in S \colon y \notin \mathrm{dom}(f)\}$$

and

$$S \setminus \{f \in S \colon y \notin \mathrm{dom}(f)\} = \bigcup_{t \in X} \{f \in S \colon (y, t) \in f\}.$$

(iv) $\Rightarrow$ (ii). Let $g \in S$ be arbitrary. From the equality

$$\{g\} = \bigcap_{y \in \mathrm{dom}(g)} \{f \in S \colon (y, (y)g) \in f\} \cap \bigcap_{y \in X \setminus \mathrm{dom}(g)} \{f \in S \colon y \notin \mathrm{dom}(f)\}.$$

it follows that the singleton sets are closed in $\mathcal{T}$. $\qquad\square$

Corollary 5.2. *Let X be an infinite set, and let S be a subsemigroup of X^X such that S contains all of the constant transformations, and for every $x \in X$ there exists $f_x \in S$ such that $(x)f_x^{-1} = \{x\}$ and $(X)f_x$ is finite. Then the pointwise topology coincides with the Hausdorff-Markov, Fréchet-Markov, and Zariski topologies for S.*

Proof. Since the pointwise topology is a Hausdorff semigroup topology for S, the Hausdorff-Markov topology is contained in the pointwise topology. By Fig. 1, the Zariski and Fréchet-Markov topologies are contained in the Hausdorff-Markov topology, and hence in the pointwise topology.

On the other hand, Lemma 5.1 implies that the minimum T_1 topology that is semitopological for S contains the pointwise topology, and hence, by Fig. 1, the Hausdorff-Markov, Fréchet-Markov, and Zariski topologies contain the pointwise topology too. $\qquad\square$

Lemma 5.3. *Let X be an infinite set and let S be a subsemigroup of X^X such that for every $x \in X$ there exist $a, b, c_0, \ldots, c_{n-1} \in S$ for some $n \in \mathbb{N}$ such that the following hold:*

(i) *$(y)a = (y)b$ if and only if $y \neq x$;*
(ii) *$x \in \mathrm{im}(c_i)$ for all i;*
(iii) *for every $s \in S$ and every $y \in X \setminus \{(x)s\}$ there is $i \in \{0, \ldots, n-1\}$ so that $\mathrm{im}(c_i) \cap (y)s^{-1} = \varnothing$.*

Then the Zariski topology of S is the pointwise topology.

Proof. The pointwise topology is Hausdorff, and so it contains the Zariski topology by Proposition 2.1. Hence it remains to show that the Zariski topology contains the pointwise topology.

We begin by observing that if $x \in X$ is arbitrary, then there exist $a, b \in S$ satisfying part (i) of the hypothesis and so

$$S \setminus \{f \in S \colon x \in \mathrm{im}(f)\} = \{f \in S \colon x \notin \mathrm{im}(f)\} = \{f \in S \colon fa = fb\}.$$

Hence $\{f \in S \colon x \in \mathrm{im}(f)\}$ is open in the Zariski topology.

If $x \in X$ is arbitrary, then there exist $c_0, \ldots, c_{n-1} \in S$ satisfying part (ii) of the assumption of the lemma. Since the Zariski topology is semitopological, by Proposition 2.2, the map λ_{c_i} is continuous for every $i \in \{0, \ldots, n-1\}$. If $y \in X$ is arbitrary, then the set

$$U_i := \{f \in S \colon \text{ there is } w \in \mathrm{im}(c_i) \text{ with } (w)f = y\} = \{f \in S \colon y \in \mathrm{im}(f)\}\lambda_{c_i}^{-1}$$

is also open in Zariski topology for every $i \in \{0, \ldots, n-1\}$. It suffices to show that $\bigcap_{i=0}^{n-1} U_i$ is equal to the subbasic open set $\{f \in S \colon (x)f = y\}$ in the pointwise topology.

By definition,

$$\bigcap_{i=0}^{n-1} U_i = \{f \in S \colon \text{ there exist } w_i \in \mathrm{im}(c_i) \text{ with } (w_0)f = \cdots = (w_{n-1})f = y\}.$$

Let $f \in \bigcap_{i=0}^{n-1} U_i$, and let $w_0, \ldots, w_{n-1}$ be such that $(w_0)f = \ldots = (w_{n-1})f = y$ where $w_i \in \text{im}(c_i)$ for all i. Then $\text{im}(c_i) \cap (y)f^{-1} \neq \varnothing$ for every $i \in \{0, \ldots, n-1\}$, and so by the assumption of the lemma, it follows that $y = (x)f$. Hence $\bigcap_{i=0}^{n-1} U_i \subseteq \{f \in S : (x)f = y\}$. The converse follows immediately from the fact that $x \in \bigcap_{i=0}^{n-1} \text{im}(c_i)$. □

In the next theorem, we show that the pointwise topology on X^X is the unique Polish (even T_1 second countable) semigroup topology on X^X; and X^X with pointwise topology has automatic continuity with respect to the class of second countable topological semigroups.

Theorem 5.4. *If X is an infinite set, then the following hold:*

(i) X^X *with the pointwise topology has property $\boldsymbol{X}$ with respect to $\text{Sym}(X)$;*

(ii) *the pointwise topology is the only T_1 semigroup topology on X^X which induces the pointwise topology on $\text{Sym}(X)$;*

(iii) *the pointwise topology coincides with the Hausdorff-Markov, Fréchet-Markov, and Zariski topologies for X^X;*

(iv) *if X is countable, then X^X with the pointwise topology has automatic continuity with respect to the class of second countable topological semigroups;*

(v) *if X is countable, then the pointwise topology on X^X is the unique T_1 second countable semigroup topology.*

Proof. **(i).** Let $\psi : X \longrightarrow X \times X$ be a bijection, and let $\pi_1 : X \times X \longrightarrow X$ be defined by $(x, y)\pi_1 = y$. We define $f, g \in X^X$ by
$$(x)f = (x, x)\psi^{-1} \quad \text{and} \quad (x)g = (x)\psi\pi_1.$$
Let $s \in X^X$ and let $t \in \text{Sym}(X \times X)$ be any permutation such that
$$(x, x)t = (x, (x)s)$$
for all $x \in X$. We then define $t_s \in X^X$ to be $\psi t \psi^{-1}$. From the definitions of f, g, and t_s,
$$(x)ft_sg = (x, x)\psi^{-1}\psi t \psi^{-1}\psi\pi_1 = (x, x)t\pi_1 = (x, (x)s)\pi_1 = (x)s$$
for all $x \in X$. Let B be a basic open neighbourhood of t_s. Then there exist $x_0, x_1, \ldots, x_n \in X$ such that $t_s \in \bigcap_{i=0}^{n-1} \{f \in X^X : (x_i)f = (x_i)t_s\} = B$. If
$$V := \{k \in X^X : (x)k = (x)s \text{ for all } x \text{ such that } (x, x) \in \{(x_0)\psi, (x_1)\psi, \ldots, (x_{n-1})\psi\}\},$$
then V is an open neighbourhood of s and so it suffices to show that $V \subseteq f(B \cap \text{Sym}(X))g$. For every $x \in X$, we choose distinct $z_x \in X$ which is also distinct from the first coordinate of every element of $\{(x_0)\psi t, (x_1)\psi t, \ldots, (x_{n-1})\psi t\}$. Let $k \in V$. Then there exists $p \in \text{Sym}(X \times X)$ such that
$$(x_i)\psi p = (x_i)\psi t$$
for all i, and
$$(x, x)p = (z_x, (x)k),$$
for all $x \in X$ for which $(x, x) \notin \{(x_0)\psi, (x_1)\psi, \ldots, (x_{n-1})\psi\}$. We then define t_k to be $\psi p \psi^{-1}$. By the choice of p, it follows that $(x_i)t_k = (x_i)\psi p \psi^{-1} = (x_i)\psi t \psi^{-1} = (x_i)t_s$ for every i, and so $t_k \in B \cap \text{Sym}(X)$. Thus
$$(x)ft_kg = (x, x)\psi^{-1}\psi p \psi^{-1}\psi\pi_1 = (x, x)p\pi_1 = \begin{cases} (x, x)t\pi_1 & \text{if } (x, x) \in \{(x_0)\psi, (x_1)\psi, \ldots, (x_{n-1})\psi\} \\ (z_x, (x)k)\pi_1 & \text{otherwise.} \end{cases}$$

In the first case, $(x)ft_kg = (x, (x)s)\pi_1 = (x)s = (x)k$ and in the second case, $(x)ft_kg = (z_x, (x)k)\pi_1 = (x)k$ also. Hence $k = ft_kg \in f(B \cap \text{Sym}(X))g$ and so, since k was arbitrary, $V \subseteq f(B \cap \text{Sym}(X))g$, as required.

(ii). By Lemma 5.1 every T_1 topology on X^X contains the pointwise topology. By part (i) together with Theorem 3.1(ii), this is also the largest topology which induces the pointwise topology on $\text{Sym}(X)$.

(iii). This follows immediately from Corollary 5.2.

(iv). This follows from part (i), Theorem 3.1(iv), and the automatic continuity of $\text{Sym}(X)$ when X is countable.

(v). By part (iii), every second countable semigroup topology for X^X is contained in the pointwise topology. By Lemma 5.1 every T_1 topology on $\mathbb{N}^{\mathbb{N}}$ contains the pointwise topology. □

In some sense, every group is a group of permutations, since every group can be embedded into the symmetric group $\mathrm{Sym}(X)$ for some set X. Similarly, every semigroup embeds as a subsemigroup of X^X for some set X. Given that X^X has a canonical semigroup topology, it is natural to ask which topological semigroups embed in X^X with the pointwise topology.

It is well-known that a T_1 topological group G is topologically isomorphic to a subgroup of $\mathrm{Sym}(\mathbb{N})$ if and only if G has a countable neighbourhood basis of the identity 1_G consisting of countable index open subgroups. This latter condition is referred to as G being *non-archimedean* by some authors; see, for example, [11, Theorem 1]. In the next theorem, we prove an analogue of this result for T_0 left semitopological semigroups. In [11, Theorem 2], another analogous characterisation is given of those topological monoids that are topologically isomorphic to closed submonoids of $\mathbb{N}^{\mathbb{N}}$. There are many topological monoids that are topologically isomorphic to non-closed submonoids of $\mathbb{N}^{\mathbb{N}}$. For example, if D is a countable dense subset of $\mathbb{N}^{\mathbb{N}}$, then the least submonoid $\langle D \rangle$ of $\mathbb{N}^{\mathbb{N}}$ containing D is countable and dense also. It follows that $\langle D \rangle$ is not G_δ, and hence not Polish. It follows that $\langle D \rangle$ is not topologically isomorphic to any closed submonoid of $\mathbb{N}^{\mathbb{N}}$, but it is obviously (topologically isomorphic to) a submonoid of $\mathbb{N}^{\mathbb{N}}$. Of course, not every Polish semigroup is topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. For example, the reals $\mathbb{R}$ under addition form a connected Polish group, but $\mathbb{N}^{\mathbb{N}}$ is totally disconnected, and so $(\mathbb{R}, +)$ cannot be topology embedded in $\mathbb{N}^{\mathbb{N}}$.

Even though the statement of Theorem 5.5 is superficially different from that of [11, Theorem 2], the proof of Theorem 5.5 is essentially contained in the proof of [11, Theorem 2]. We include Theorem 5.5 because we will use it in Section 5.5 and we include the proof for the sake of completeness.

Theorem 5.5 (cf. Theorem 2 in [11]). *If S is a T_0 left semitopological semigroup, then the following are equivalent:*

(i) *there is a sequence $\{\rho_i : i \in \mathbb{N}\}$ of right congruences of S, each having countably many classes, such that $\{m/\rho_i : m \in S,\ i \in \mathbb{N}\}$ is a subbasis for S;*

(ii) *there is a sequence $\{\sigma_i : i \in \mathbb{N}\}$ of right congruences of S, each having countably many classes, such that $\{m/\sigma_i : m \in S,\ i \in \mathbb{N}\}$ is a basis for S;*

(iii) *S is topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$ (with the pointwise topology and right actions).*

Proof. We will show that (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (i). This implication follows immediately from the definitions of bases and subbases for topological spaces.

(iii) $\Rightarrow$ (ii). Assume without loss of generality that S is a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. If we define $\rho_i = \{(f, g) : (i)f = (i)g\}$ for every $i \in \mathbb{N}$, then clearly every ρ_i is a right congruence with the properties given in (i). If $\sigma_n = \bigcap_{i \le n} \rho_i$, for every $n \in \mathbb{N}$, then $\{x/\sigma_n : x \in S, n \in \mathbb{N}\}$ is a basis for the topology on S with the required properties.

(i) $\Rightarrow$ (iii). We give S^1 the disjoint union topology of S and $\{1\}$. If ρ is any right congruence on S, then $\rho \cup \{(1,1)\}$ is a right congruence on S^1. It follows that S^1 satisfies the hypothesis of (i). Let $\{\rho_i : i \in \mathbb{N}\}$ be a sequence of right congruences of S^1, each having countably many classes, such that $\{m/\rho_i : m \in S^1,\ i \in \mathbb{N}\}$ is a subbasis for S^1.

By assumption, $X = \{m/\rho_i : m \in S^1,\ i \in \mathbb{N}\}$ is countable, and we will show that S^1 is topologically isomorphic to a subsemigroup of X^X with the pointwise topology, from which it will follow that S is also topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. When $i \ne j$, we will consider classes of ρ_i and classes of ρ_j as different elements of X, even if they should happen to be the same subset of S. We define $\phi : S^1 \longrightarrow X^X$ such that $(m)\phi \in X^X$ is defined by

$$(n/\rho_i)(m)\phi = (nm)/\rho_i.$$

It is sufficient to show that ϕ is a well-defined, injective, continuous, homomorphism such that if U is open in S^1, then $(U)\phi$ is open in $(S^1)\phi$ with the subspace topology.

If $(a, b) \in \rho_i$ for some $i \in \mathbb{N}$ and $m \in S^1$, then $(a/\rho_i)(m)\phi = (am)/\rho_i = (bm)/\rho_i = (b/\rho_i)(m)\phi$, since ρ_i is a right congruence. It follows that ϕ is well-defined.

Suppose that $m, n \in S^1$ are such that $(m)\phi = (n)\phi$. It follows that $m/\rho_i = (1/\rho_i)(m)\phi = (1/\rho_i)(n)\phi = n/\rho_i$ for all $i \in \mathbb{N}$. In other words, $(m, n) \in \rho_i$ for all $i \in \mathbb{N}$. But since $\{m/\rho_i : m \in S^1,\ i \in \mathbb{N}\}$ is a subbasis for S^1, and S^1 is T_0 it follows that $\bigcap_{i \in \mathbb{N}} \rho_i = \{(s, s) : s \in S^1\}$. Hence $m = n$ and ϕ is injective.

Let $a, m, n \in S^1$ and $i \in \mathbb{N}$ be arbitrary. Then

$$((a/\rho_i)(m)\phi)(n)\phi = (amn)/\rho_i = (a/\rho_i)(mn)\phi$$

and so ϕ is a homomorphism.

To show that ϕ is continuous it is sufficient to show that the preimage under ϕ of every subbasic open set in X^X is open in S^1. Let $\mathcal{S}$ be the subbasis for X^X consisting of the sets

$$[\alpha, \beta] = \{f \in X^X : (\alpha)f = \beta\}$$

where $\alpha, \beta \in X$. Suppose that $[\alpha, \beta] \in \mathcal{S}$. Then $\alpha = a/\rho_i$ and $\beta = b/\rho_j$ for some $a, b \in S^1$ and $i, j \in \mathbb{N}$. If $[a/\rho_i, b/\rho_j] \cap (S^1)\phi \neq \varnothing$, then $i = j$ by the definition of ϕ. Hence we may suppose that without loss of generality that $\alpha = a/\rho_i$ and $\beta = b/\rho_i$. It follows that

$$
\begin{aligned}
[\alpha, \beta]\phi^{-1} &= \{m \in S^1 : (\alpha)(m)\phi = \beta\} \\
&= \{m \in S^1 : (a/\rho_i)(m)\phi = b/\rho_i\} \\
&= \{m \in S^1 : (am)/\rho_i = b/\rho_i\} \\
&= \{m \in S^1 : am \in b/\rho_i\} \\
&= (b/\rho_i)\lambda_a^{-1}.
\end{aligned}
$$

Since S^1 is left semitopological, λ_a is continuous and since b/ρ_i is open, so too is $[\alpha, \beta]\phi^{-1}$. Hence ϕ is continuous.

If $m \in S^1$ and $i \in \mathbb{N}$ are arbitrary, then

$$
\begin{aligned}
(m/\rho_i)\phi &= \{(n)\phi : n \in S^1, \ m\rho_i = n\rho_i\} \\
&= \{(n)\phi : n \in S^1, \ (1/\rho_i)(n)\phi = m\rho_i\} \\
&= [1/\rho_i, m\rho_i].
\end{aligned}
$$

Hence every subbasic open set in S^1 is mapped to a subbasic open set in $(S^1)\phi$ and so ϕ is open. $\qquad\square$

We prove an analogue of Theorem 5.5 for semitopological inverse monoids and the symmetric inverse monoid in Theorem 5.21.

Uspenskiĭ's Theorem [42, Theorem 9.18] and [80] states that every Polish group is isomorphic to a (necessarily closed) subgroup of the group $H([0,1]^{\mathbb{N}})$ of homeomorphisms of the Hilbert cube $[0,1]^{\mathbb{N}}$. A similar result holds for separable metrizable compactifiable semigroups, every such semigroup is topologically isomorphic to a subsemigroup of $C([0,1]^{\mathbb{N}})$; see [52, Theorem 5.2].

Question 5.6. *Is every countable Polish semigroup topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$?* [1]

5.2. **The full binary relation monoid.** Unlike X^X which is probably better known as a topological space than as a semigroup, there is no obvious candidate for a semigroup topology on the full binary relation monoid B_X. In this section, we prove that a dichotomy exists for semigroup topologies on B_X: either such a topology is so coarse that it is not T_1, or it is so fine that it is not second countable. In particular, B_X possesses no Polish semigroup topologies. We exhibit two semigroup topologies on B_X which are, in some sense, canonical for the poles of the dichotomy.

The natural starting point is to consider a topology which induces the pointwise topology on X^X. This is the subject of the next theorem.

Theorem 5.7. *Let X be an infinite set and let*

$$U_{x,y} = \{f \in B_X : (x, y) \in f\}$$

for all $x, y \in X$. If $\mathcal{B}_1$ is the topology on B_X with subbasis $\{U_{x,y} : x, y \in X\}$, then the following hold:

  (i) *$\mathcal{B}_1$ is a semigroup topology for B_X and inversion of binary relations $f \mapsto f^{-1}$ is continuous;*
 (ii) *$\mathcal{B}_1$ is T_0 but not T_1, and the subspace topology induced by $\mathcal{B}_1$ on the symmetric inverse monoid I_X is not T_1;*
(iii) *the topological semigroup B_X with the topology $\mathcal{B}_1$ has property $\mathbf{X}$ with respect to $\mathrm{Sym}(X)$;*
 (iv) *every topology that is semitopological for B_X and that induces the pointwise topology on $\mathrm{Sym}(X)$ is contained in $\mathcal{B}_1$;*
  (v) *if X is countable, then B_X with the topology $\mathcal{B}_1$ has automatic continuity with respect to the class of second countable topological semigroups.*

---

[1]Addendum: S. Bardyla, L. Elliott, J. D. Mitchell, and Y. Péresse recently showed that the answer to this question is no, see [4].

Proof. Note that a basis for $\mathcal{B}_1$ is given by the collection of sets $\{h \in B_X : f \subseteq h\}$ where $f \in B_X$ is finite, since such sets are precisely the finite intersections of subbasic sets in $\mathcal{B}_1$.

(i). Let $x, y \in X$ and $f, g \in B_X$ be such that $fg \in U_{x,y}$, i.e. $(x, y) \in fg$. Then there exists $z \in X$ such that $(x, z) \in f$ and $(z, y) \in g$. Conversely, for every $f', g' \in B_X$ with $(x, z) \in f'$ and $(z, y) \in g'$ we have that $(x, y) \in f'g'$. In other words, $f \in U_{x,z}$, $g \in U_{z,y}$ and

$$U_{x,z} U_{z,y} \subseteq U_{x,y}.$$

Hence multiplication is continuous.

The map $f \mapsto f^{-1}$ is a homeomorphism since $(U_{x,y})^{-1} = U_{y,x}$.

(ii). If $f, g \in B_X$ are distinct, then either $f \nsubseteq g$ or $g \nsubseteq f$. Without loss of generality assume that $f \nsubseteq g$. Then there exists $(x, y) \in X \times X$ such that $(x, y) \in f \setminus g$ and so $f \in U_{x,y}$ but $g \notin U_{x,y}$. Hence $\mathcal{B}_1$ is T_0.

On the other hand, suppose that $f, g \in I_X$ and $g \subsetneq f$. Then every subbasic open set $U_{x,y}$ containing g also contains f. Thus every open set containing g contains f and so $\mathcal{B}_1$ is not T_1 (even when restricted to I_X).

(iii). Let $Y \subseteq X$ be such that $|Y| = |X \setminus Y| = |X|$. Enumerate X as $X = \{x_i : i \in |X|\}$ and let $\{X_i : i \in |X|\}$ be a partition of $X \setminus Y$ such that $|X_i| = |X|$ for every $i \in |X|$. Define $f \in B_X$ by

$$f = \{(x_i, y) \in X \times X : i \in |X| \text{ and } y \in X_i\}.$$

Let $s \in B_X$ be arbitrary. A binary relation $t \in B_X$ satisfies $ftf^{-1} = s$ if and only if t has the following property:

(1) $\qquad$ for all $i, j \in |X| : \quad (X_i \times X_j) \cap t \neq \varnothing \quad$ if and only if $\quad (x_i, x_j) \in s$.

Since $|Y| = |X_i| = |X|$, there exists $t_s \in \mathrm{Sym}(X)$ satisfying (1).

Suppose that V is a basic open neighbourhood of t_s. Then there exist finite $k \in B_X$ such that $V = \{h \in B_X : k \subseteq h\}$. If $U := \{h \in B_X : fkf^{-1} \subseteq h\}$, then U is open since fkf^{-1} is finite. We show that $s \in U \subseteq fVf^{-1}$. As $k \subseteq t_s$ and $ft_sf^{-1} = s$ it is clear that $s \in U$. It remains to show that

$$U \subseteq fVf^{-1}.$$

Let $u \in U$ be arbitrary. As in (1) we need only find $t \in V \cap \mathrm{Sym}(X)$ with

for all $i, j \in |X| : \quad (X_i \times X_j) \cap t \neq \varnothing \quad$ if and only if $\quad (x_i, x_j) \in u$.

If $i, j \in |X|$ and $(X_i \times X_j) \cap k \neq \varnothing$, then $(x_i, x_j) \in fkf^{-1} \subseteq u$. Therefore, as k is finite, we can extend k to an element t of $V \cap \mathrm{Sym}(X)$ with the desired property. So $u = ftf^{-1} \in f(V \cap \mathrm{Sym}(X))f^{-1}$ and

$$U \subseteq f(V \cap \mathrm{Sym}(X))f^{-1},$$

as required.

(iv). The subbasis for $\mathcal{B}_1$ induces the usual subbasis for the pointwise topology on $\mathrm{Sym}(X)$, and so the topology on $\mathrm{Sym}(X)$ induced by $\mathcal{B}_1$ is the pointwise topology. Since B_X has property **X** with respect to $\mathrm{Sym}(X)$, by part (iii), it follows from Theorem 3.1(ii) that if $\mathcal{T}$ is a topology that is semitopological for B_X, then $\mathcal{T}$ is contained in $\mathcal{B}_1$.

(v). This follows from part (iii), Theorem 3.1(iv), and the automatic continuity of $\mathrm{Sym}(X)$. $\qquad\square$

As an immediate consequence of Theorem 5.7(iii), there is no T_1 topology that is semitopological for B_X and that induces the pointwise topology on $\mathrm{Sym}(X)$. If instead of trying to extend the pointwise topology, as we did in Theorem 5.7, we look for the weakest T_1 topology that is semitopological for B_X, then we obtain the following theorem.

Theorem 5.8. *Let X be an infinite set and let $\mathcal{B}_2$ be the topology on B_X generated by the sets*

$$U_{x,y} = \{h \in B_X : (x, y) \in h\} \quad \text{and} \quad V_{Y,Z} = \{h \in B_X : (Y)h \subseteq Z\}$$

for all $x, y \in X$ and $Y, Z \subseteq X$. Then the following hold:

(i) *$\mathcal{B}_2$ is a Hausdorff semigroup topology for B_X and inversion of binary relations $f \mapsto f^{-1}$ is continuous;*

(ii) *every T_1 topology that is semitopological for B_X contains $\mathcal{B}_2$;*

(iii) *$\mathcal{B}_2$ is not contained in any second countable topology;*

(iv) *$\mathcal{B}_2$ strictly contains $\mathcal{B}_1$;*

(v) *if X is countable, then $\mathcal{B}_2$ coincides with the Fréchet-Markov, Hausdorff-Markov, and Zariski topologies for B_X.*

Proof. (**i**). If $f, g \in B_X$ are distinct, then without loss of generality, there exist $x, y \in X$ such that $(x, y) \in f$ but $(x, y) \notin g$. Then $f \in U_{x,y}$ and $g \in V_{\{x\}, X \setminus \{y\}}$. Since these two open sets are disjoint, $\mathcal{B}_2$ is Hausdorff.

The topology generated by the sets $U_{x,y}$ is just $\mathcal{B}_1$ as defined in Theorem 5.7. Since $\mathcal{B}_1$ is a semigroup topology, it suffices to show that the pre-images under multiplication and inversion of the subbasic sets $V_{Y,Z}$ are open.

For any $Y \subseteq X$ and $f, g \in B_X$, we clearly have $f \in V_{Y,(Y)f}$ and $g \in V_{(Y)f,(Y)fg}$. Moreover, if $f' \in V_{Y,(Y)f}$ and $g' \in V_{(Y)f,(Y)fg}$, then $(Y)f'g' \subseteq (Y)fg' \subseteq (Y)fg$. Thus if $fg \in V_{Y,Z}$, then $f'g' \in V_{Y,Z}$ and so multiplication is continuous.

Since
$$(V_{Y,Z})^{-1} = V_{X \setminus Z, X \setminus Y}$$
it follows that $f \mapsto f^{-1}$ is also continuous under $\mathcal{B}_2$.

(**ii**). Suppose that $\mathcal{T}$ is a T_1 topology under which B_X is semitopological. We will show that the subbasic open sets of $\mathcal{B}_2$ are open in $\mathcal{T}$ and so, in particular, $\mathcal{B}_2 \subseteq \mathcal{T}$.

For any $Y, Z \subseteq X$ and $f \in B_X$ consider the product
$$(Y \times Y) \circ f \circ ((X \setminus Z) \times (X \setminus Z)) = g.$$
Either $(Y)f \subseteq Z$ and $g = \varnothing$, or $(Y)f \nsubseteq Z$ and $g = Y \times (X \setminus Z)$. Thus
$$(\{Y \times (X \setminus Z)\}) \left(\lambda_{Y \times Y} \circ \rho_{(X \setminus Z) \times (X \setminus Z)} \right)^{-1} = \{f \in B_X : (Y)f \nsubseteq Z\} = B_X \setminus V_{Y,Z}$$
and
$$(\{\varnothing\}) \left(\lambda_{Y \times Y} \circ \rho_{(X \setminus Z) \times (X \setminus Z)} \right)^{-1} = \{f \in B_X : (Y)f \subseteq Z\} = V_{Y,Z}$$
as the continuous pre-images of finite sets, are closed in $\mathcal{T}$. Hence $V_{Y,Z}$ and $B_X \setminus V_{Y,Z}$ are open in $\mathcal{T}$. If $x, y \in X$, we may let $Y = \{x\}$ and $Z = X \setminus \{y\}$, and then
$$B_X \setminus V_{Y,Z} = \{f \in B_X : (Y)f \nsubseteq Z\} = \{f \in B_X : (x, y) \in f\} = U_{x,y}$$
and so $U_{x,y}$ is clopen in $\mathcal{T}$ also.

(**iii**). If X is uncountable, then the collection
$$\{V_{X,\{x\}} \cap U_{x,x} : x \in X\} \subseteq \mathcal{B}_2$$
consists of pairwise disjoint open sets and is uncountable. Hence, if X is uncountable, then $\mathcal{B}_2$ is not contained in any second countable topology.

Suppose that X is countable and that $\{X_i : i \in I\}$ is a family of subsets of X with cardinality $2^{\aleph_0}$ such that $X_i \nsubseteq X_j$ whenever $i \neq j$. For every $i \in I$, choose $f_i \in B_X$ such that $(X)f_i = X_i$. If $\mathcal{U}$ is a basis of any topology containing $\mathcal{B}_2$, then, for every $i \in I$, there exists $U_i \in \mathcal{U}$ such that $f_i \in U_i \subseteq V_{X,X_i}$. If $f_i \in U_j \subseteq V_{X,X_j}$, then $X_i = (X)f_i \subseteq X_j$ and so $i = j$. In other words, $i \neq j$ implies that $f_i \notin U_j$. Thus $|\mathcal{U}| \geq |I| = 2^{\aleph_0}$ and so no topology containing $\mathcal{B}_2$ is second countable.

(**iv**). The subspace topology on $\mathrm{Sym}(X)$ induced by $\mathcal{B}_1$ is the pointwise topology. On the other hand, if Y and Z are two infinite sets with infinite complements, then $V_{Y,Z} \cap \mathrm{Sym}(X)$ is not open in the pointwise topology on $\mathrm{Sym}(X)$.

(**v**). Since the minimal T_1 topology that is semitopological for B_X coincides with the Hausdorff-Markov topology, by parts (i) and (ii), it follows that the Fréchet-Markov, Hausdorff-Markov and Zariski topologies all equal $\mathcal{B}_2$; see Fig. 1. □

We obtain the following corollary to Theorem 5.8(iii).

Corollary 5.9. *Let X be an infinite set. Then no second countable T_1 topology is semitopological for B_X. In particular, B_X possesses no Polish semigroup topologies.*

5.3. **The partial transformation monoid.** A natural way of defining a semigroup topology on the partial transformation monoid P_X, where X is an arbitrary set, is to embed P_X into the full transformation monoid X^X, and use the subspace topology induced by the pointwise topology on X^X. We will show that, when X is infinite, this topology is simultaneously the weakest T_1 semigroup topology on P_X and the finest extension of the pointwise topology of $\mathrm{Sym}(X)$ to P_X.

Roughly speaking, the natural way of embedding P_X into X^X is to add a new element $\blacklozenge$ to X that will represent "not defined". More precisely, if X is a set, $\blacklozenge \notin X$, and $Y = X \cup \{\blacklozenge\}$, then the function $\phi : P_X \longrightarrow Y^Y$ defined by

$$(2) \qquad\qquad (x)((f)\phi) = \begin{cases} (x)f & \text{if } x \in \mathrm{dom}(f) \\ \blacklozenge & \text{if } x \notin \mathrm{dom}(f) \end{cases}$$

is an embedding. Note that, in particular, if $g = (f)\phi$, then $(\blacklozenge)g = \blacklozenge$. We will refer to ϕ as the *natural embedding* of P_X into Y^Y.

Theorem 5.10. *Let X be an infinite set and let $\mathcal{P}$ be the topology on P_X generated by the sets*

$$U_{x,y} = \{h \in P_X : (x,y) \in h\} \quad \text{and} \quad W_x = \{h \in P_X : x \notin \mathrm{dom}(h)\}$$

for all $x, y \in X$. Then the following hold:

- (i) *the topology $\mathcal{P}$ is the subspace topology on P_X induced by the pointwise topology on Y^Y and the natural embedding $\phi : P_X \longrightarrow Y^Y$ defined in (2);*
- (ii) *$\mathcal{P}$ is a Hausdorff semigroup topology for P_X;*
- (iii) *$\mathcal{P}$ coincides with the Hausdorff-Markov, Fréchet-Markov, and Zariski topologies for P_X;*
- (iv) *P_X has property $\boldsymbol{X}$ with respect to $\mathcal{P}$ and $\mathrm{Sym}(X)$;*
- (v) *if $\mathcal{T}$ is a topology that is semitopological for P_X and $\mathcal{T}$ induces the pointwise topology on $\mathrm{Sym}(X)$, then $\mathcal{T}$ is contained in $\mathcal{P}$;*
- (vi) *the topology $\mathcal{P}$ is the unique T_1 topology that induces the pointwise topology on $\mathrm{Sym}(X)$ and that is semitopological for P_X;*
- (vii) *if X is countable, then P_X with the topology $\mathcal{P}$ has automatic continuity with respect to the class of second countable topological semigroups;*
- (viii) *if X is countable, then $\mathcal{P}$ is the unique Polish topology that is semitopological for P_X;*
- (ix) *if X is countable, then $\mathcal{P}$ is the unique T_1 second countable semigroup topology for P_X.*

Proof. **(i).** The image of P_X under the natural embedding ϕ defined in (2) is the set

$$\{h \in Y^Y : (\blacklozenge)h = \blacklozenge\}.$$

The pointwise topology on Y^Y is generated by the sets $A_{x,y} = \{f \in Y^Y : (x,y) \in f\}$ for all $x, y \in Y$. Hence the topology induced on $(P_X)\phi$ is generated by the sets $A_{x,y} \cap (P_X)\phi$. Note that $A_{\blacklozenge,\blacklozenge} \cap (P_X)\phi = (P_X)\phi$ and $A_{\blacklozenge,y} \cap (P_X)\phi = \varnothing$ for all $y \in X = Y \setminus \{\blacklozenge\}$. If $x, y \in X$, then

$$A_{x,y} \cap (P_X)\phi = \{h \in (P_X)\phi : (x,y) \in h\} = (U_{x,y})\phi$$

and

$$A_{x,\blacklozenge} \cap (P_X)\phi = \{h \in (P_X)\phi : (x,\blacklozenge) \in h\} = (W_x)\phi.$$

Hence ϕ is a homeomorphism between $\mathcal{P}$ and the topology generated by $A_{x,y} \cap (P_X)\phi$.

(ii). Since Y^Y is a Hausdorff topological semigroup under the pointwise topology, it follows from part (i) that P_X is a Hausdorff topological semigroup under $\mathcal{P}$.

(iii). The proof of this part is similar to the proof of Corollary 5.2, it is included for the sake of completeness. By part (ii), $\mathcal{P}$ is a Hausdorff semigroup topology for P_X. Hence the Hausdorff-Markov topology is contained in $\mathcal{P}$. By Fig. 1, the Zariski and Fréchet-Markov topologies are contained in the Hausdorff-Markov topology, and hence in $\mathcal{P}$.

On the other hand, Lemma 5.1 implies that every T_1 topology that is semitopological for P_X contains $\mathcal{P}$, and such topologies include the Hausdorff-Markov, Fréchet-Markov, and Zariski topologies.

(iv). We will show that P_X has property $\boldsymbol{X}$ with respect to X^X, from which it will follow, by Lemma 3.4 and Theorem 5.4(i), that P_X has property $\boldsymbol{X}$ with respect to $\mathrm{Sym}(X)$. Let $b \in X$ be fixed and let $\psi : X \longrightarrow X \setminus \{b\}$ be a bijection. If $s \in P_X$ is arbitrary, then we define $f = \psi$, $g = \psi^{-1}$, and define $t_s \in X^X$ by

$$(x)t_s = \begin{cases} b & \text{if } (x)\psi^{-1} \notin \mathrm{dom}(s) \text{ or } x = b \\ (x)\psi^{-1}s\psi & \text{if } (x)\psi^{-1} \in \mathrm{dom}(s). \end{cases}$$

It follows, from the definitions, that $s = ft_s g$ for every $s \in P_X$.

Suppose that $s \in P_X$ is arbitrary but fixed for the remainder of the proof. Let B be an arbitrary basic open neighbourhood of t_s. Then there exist $x_0, x_1, \ldots, x_{n-1} \in X$, $n \geq 1$, such that

$$B = \bigcap_{i=0}^{n-1} \{t \in P_X : (x_i)t = (x_i)t_s\}.$$

Let V be the finite intersection of all of the following subbasic open sets:

(a) $U_{(x_i)\psi^{-1}, (x_i)t_s\psi^{-1}}$ for every $x_i \in \{x_0, x_1, \ldots, x_{n-1}\}\backslash\{b\}$ such that $(x_i)t_s \in \mathrm{im}(\psi)$; and

(b) $W_{(x_i)\psi^{-1}}$ for every $x_i \in \{x_0, x_1, \ldots, x_{n-1}\}\backslash\{b\}$ such that $(x_i)t_s = b$.

Then V is open and we will show that $s \in V$. If $x_i \in \{x_0, x_1, \ldots, x_{n-1}\}\backslash\{b\}$ and $(x_i)t_s \in \mathrm{im}(\psi)$, then $(x_i)\psi^{-1} \in \mathrm{dom}(s)$ and $((x_i)\psi^{-1})s = ((x_i)t_s)\psi^{-1}$. This implies that $s \in U_{(x_i)\psi^{-1}, (x_i)t_s\psi^{-1}}$. Similarly, if $x_i \in \{x_0, x_1, \ldots, x_{n-1}\}\backslash\{b\}$ and $(x_i)t_s = b$, then $(x_i)\psi^{-1} \notin \mathrm{dom}(s)$. Hence $s \in W_{(x_i)\psi^{-1}}$ for such x_i. We conclude that $s \in V$.

It remains to show that $V \subseteq f_s(B \cap X^X)g_s$. Let $k \in V$ be arbitrary. We will show that t_s and t_k agree on $\{x_0, x_1, \ldots, x_{n-1}\}$, or, in other words, $t_k \in B$. By the definitions of t_s and t_k (b)$t_s = b = (b)t_k$. Suppose that $x_i \in \{x_0, x_1, \ldots, x_{n-1}\}\backslash\{b\}$. If $(x_i)t_s = b$, then $(x_i)\psi^{-1} \notin \mathrm{dom}(s)$ and, since $k \in W_{(x_i)\psi^{-1}}$, $(x_i)\psi^{-1} \notin \mathrm{dom}(k)$. Hence $(x_i)t_s = b = (x_i)t_k$. On the other hand, if $(x_i)t_s \neq b$, then $(x_i)t_s \in \mathrm{im}(\psi)$. It follows from the fact that $k \in U_{(x_i)\psi^{-1}, (x_i)t_s\psi^{-1}}$ that $(x_i)\psi^{-1}k = (x_i)t_s\psi^{-1}$, and so $(x_i)t_s = (x_i)\psi^{-1}k\psi = (x_i)t_k$ and so $t_k \in B$. Since $t_k \in X^X$ by definition, and so $k = ft_kg \in f(B \cap X^X)g$, as required.

(v). Suppose that $\mathcal{T}$ is a topology that is semitopological for P_X and that $\mathcal{T}$ induces the pointwise topology on $\mathrm{Sym}(X)$. Since $W_x \cap X^X = \{h \in P_X : x \notin \mathrm{dom}(h)\} \cap X^X = \varnothing$, the topology induced by $\mathcal{P}$ on $\mathrm{Sym}(X)$ is just the pointwise topology. Hence $\mathcal{T}$ and $\mathcal{P}$ induce the same topology on $\mathrm{Sym}(X)$. Since P_X has property **X** with respect to $\mathcal{P}$ and $\mathrm{Sym}(X)$, it follows that $\mathcal{T} \subseteq \mathcal{P}$ by Theorem 3.1(ii).

(vi). This follows from Lemma 5.1 and part (v).

(vii). This follows immediately from part (iv), Theorem 3.1(iv) and the automatic continuity of $\mathrm{Sym}(X)$.

(viii). Let X be countable. By part (i), P_X is homeomorphic to its image under ϕ. It is easy to see that $(P_X)\phi$ is a closed subset of Y^Y under the pointwise topology. Thus P_X under $\mathcal{P}$ is homeomorphic to a closed subspace of a Polish space and is hence Polish.

Suppose that $\mathcal{T}$ is a Polish topology that is semitopological for P_X. Then $\mathcal{T}$ is T_1 and so $\mathcal{P} \subseteq \mathcal{T}$ by Lemma 5.1. Since $\mathcal{P}$ is Polish and $\mathrm{Sym}(X)$ is a Polish subgroup of $\mathcal{P}_X$, it follows from Theorem 3.1(iii) that $\mathcal{T} \subseteq \mathcal{P}$. Hence $\mathcal{T} = \mathcal{P}$, as required.

(ix). Suppose that $\mathcal{T}$ is a T_1 second countable semigroup topology for P_X. By Lemma 5.1, $\mathcal{P}$ is contained in $\mathcal{T}$. Applying (vii) to the identity function from P_X with $\mathcal{P}$ to P_X with $\mathcal{T}$ shows that $\mathcal{T}$ is contained in $\mathcal{P}$ also. $\qquad\square$

Since $P_X \subseteq B_X$, the topologies $\mathcal{B}_1$ and $\mathcal{B}_2$ on B_X, given in Theorem 5.7 and Theorem 5.8, induce semigroup topologies on P_X. It is natural to ask how the topology $\mathcal{P}$ from Theorem 5.10 relates to the semigroup topologies on P_X induced by $\mathcal{B}_1$ and $\mathcal{B}_2$.

Proposition 5.11. *If X is an infinite set, then $\mathcal{P}$ strictly contains the subspace topology on P_X induced by $\mathcal{B}_1$ and $\mathcal{P}$ is strictly contained in the subspace topology on P_X induced by $\mathcal{B}_2$.*

Proof. Clearly from the definitions of $\mathcal{B}_1$ and $\mathcal{P}$, the subspace topology induced by $\mathcal{B}_1$ on P_X is contained in $\mathcal{P}$. This containment is strict because $\mathcal{P}$ is T_1 but $\mathcal{B}_1$ is not T_1 on $I_X \subseteq P_X$ by Theorem 5.7(ii).

The second containment follows since

$$\{h \in P_X : x \notin \mathrm{dom}(h)\} = \{h \in B_X : (\{x\})h \subseteq \varnothing\} \cap P_X \quad \text{and} \quad \{h \in B_X : (\{x\})h \subseteq \varnothing\} \in \mathcal{B}_2$$

for all $x \in X$. The topology $\mathcal{P}$ induces the pointwise topology on $\mathrm{Sym}(X)$. On the other hand, if $\mathcal{B}_2$ induced the pointwise topology on $\mathrm{Sym}(X)$, then $\mathcal{B}_2 \subseteq \mathcal{B}_1$, by Theorem 5.7(iv), contradicting Theorem 5.8(iv). $\qquad\square$

5.4. The symmetric inverse monoid. In this section we consider semigroup and inverse topologies on the symmetric inverse monoid I_X where X is any infinite set. One natural way to obtain a Hausdorff semigroup topology on I_X is to consider it as a subspace of P_X with the topology defined in Section 5.3. In Theorem 5.13, we will show that this topology is minimal among the T_1 semigroup topologies on I_X but that inversion is not continuous. If U is open in the subspace topology, then the homeomorphic topology generated by the sets $U^{-1} = \{f \in I_X : f^{-1} \in U\}$ is also minimal among the T_1 semigroup topologies on I_X and again inversion is not continuous. Moreover, every T_1 semigroup topology on I_X contains one or the other of these two topologies. This contrasts with the results of Sections 5.1, 5.2,

and 5.3 where it was shown that each of the full transformation monoid, the full binary monoid, and the partial transformation monoid have unique minimal T_1 semigroup topologies. On the other hand, I_X has a minimum inverse semigroup topology, the topology generated by the two minimal T_1 semigroup topologies (see Theorem 5.15). Arguably, this topology is the best candidate, among those we exhibit, for a canonical topology on I_X. With this viewpoint in mind, we conclude this section by proving an analogue of Theorem 5.5 which characterises those topological inverse semigroups that are topologically isomorphic to inverse subsemigroups of I_X, when X is countable.

We begin by constructing the least T_1 topology that is semitopological for I_X.

Theorem 5.12. *Let X be an infinite set and let $\mathcal{I}_1$ denote the topology on I_X generated by the sets*

$$U_{x,y} = \{h \in I_X : (x,y) \in h\} \quad and \quad V_{x,y} = \{h \in I_X : (x,y) \notin h\}$$

for all $x, y \in X$. Then the following hold:

(i) *the topology $\mathcal{I}_1$ is compact, Hausdorff, and semitopological for I_X and inversion is continuous;*
(ii) *$\mathcal{I}_1$ is the least T_1 topology that is semitopological for I_X;*
(iii) *if X is countable, then $\mathcal{I}_1$ is Polish.*

Proof. **(i).** The sets $U_{x,y}$ generate the subspace topology on I_X induced by the topology $\mathcal{B}_1$ on B_X defined in Theorem 5.7. Hence, by Theorem 5.7(i), the subspace topology induced by $\mathcal{B}_1$ on I_X is a semigroup topology where inversion is continuous also. Hence to prove that I_X is semitopological and inversion is continuous with respect to $\mathcal{I}_1$, it suffices consider the subbasic open sets $V_{x,y}$, $x, y \in X$.

Suppose that $x, y \in X$ are fixed. If $\iota : I_X \longrightarrow I_X$ is defined by $(f)\iota = f^{-1}$, then $(V_{x,y})\iota = \{h \in I_X : (y,x) \notin h\} = V_{y,x}$ is open in $\mathcal{I}_1$, and hence ι is continuous. Suppose that $f \in I_X$ is arbitrary. If $x \notin \operatorname{dom}(f)$, then $(x,y) \notin fg$ for all $g \in I_X$. Hence $(V_{x,y})\lambda_f^{-1} = I_X$ is open. If $x \in \operatorname{dom}(f)$ and $fg \in V_{x,y}$ for some $g \in I_X$. Then $((x)f, y) \notin g$. Thus $g \in V_{(x)f,y} \subseteq (V_{x,y})\lambda_f^{-1}$ and so $(V_{x,y})\lambda_f^{-1}$ is open. Hence λ_f is continuous for every $f \in I_X$. Since $\rho_f = \iota\lambda_{f^{-1}}\iota$ is a composition of continuous functions, ρ_f is continuous also.

It remains to show that I_X is Hausdorff and compact with respect to $\mathcal{I}_1$. We do this by identifying I_X with a subset of $\{0,1\}^{X \times X}$ with the product topology. It will follow that I_X is homeomorphic to a closed subspace of the compact Hausdorff space $\{0,1\}^{X \times X}$ and is thus compact and Hausdorff. If $f \in I_X$ is arbitrary, then we identify f with the function from $X \times X$ to $\{0,1\}$ defined by

$$(a,b) \mapsto \begin{cases} 1 & \text{if } (a,b) \in f \\ 0 & \text{if } (a,b) \notin f, \end{cases}$$

for all $(a,b) \in X \times X$. We may, therefore, identify I_X with a subset of $\{0,1\}^{X \times X}$. Viewed in this way, the topology $\mathcal{I}_1$ is precisely the subspace topology inherited from the product space $\{0,1\}^{X \times X}$. The complement of I_X in $\{0,1\}^{X \times X}$ is the union of the open sets

$$\left\{f \in \{0,1\}^{X \times X} : (a,b)f = (a,c)f = 1\right\}$$

and

$$\left\{f \in \{0,1\}^{X \times X} : (b,a)f = (c,a)f = 1\right\}$$

for all $a, b, c \in X$ with $b \neq c$.

(ii). Let $\mathcal{T}$ be a T_1 topology that is semitopological for I_X. Let $x, y \in X$ and let $f \in I_X$ be arbitrary. Then

$$\{(x,x)\} \circ f \circ \{(y,y)\} = \varnothing \quad \text{if and only if} \quad f \in V_{x,y}$$

and

$$\{(x,x)\} \circ f \circ \{(y,y)\} = \{(x,y)\} \quad \text{if and only if} \quad f \in U_{x,y}.$$

Since $\mathcal{T}$ is T_1, the singletons $\{\varnothing\}$ and $\{\{(x,y)\}\}$ are closed in $\mathcal{T}$. Thus their respective pre-images $V_{x,y}$ and $U_{x,y}$ under $\lambda_{\{(x,x)\}} \circ \rho_{\{(y,y)\}}$ are closed in $\mathcal{S}$. Hence $U_{x,y}$ and $V_{x,y}$ are both open in $\mathcal{T}$, as they are mutual complements. Thus $\mathcal{I}_1 \subseteq \mathcal{T}$, as required.

(iii). In the proof of part (i) we showed that I_X is homeomorphic to a closed subspace of the Cantor space $\{0,1\}^{X \times X}$. It follows that I_X is a compact metrisable space and is thus Polish.

An explicit complete metric on I_X can be obtained by choosing any metric compatible with the topology on $\{0,1\}^{X \times X}$ and translating it to I_X via the homeomorphism given in part (i). If we choose $X = \mathbb{N}$, then one such example is given by

$$d(f,g) = \frac{1}{m+1}$$

where $m = \min\{n \in \mathbb{N} : (n \times n) \cap f \neq (n \times n) \cap g\}$. $\qquad \square$

We will show that there are precisely two minimal T_1 semigroup topologies on I_X. The first is just the topology induced by the minimal T_1 semigroup topology on P_X (see Theorem 5.10) and the second consists of the inverses $U^{-1} = \{f^{-1} : f \in U\}$ of the open sets U of the first one. In Proposition 5.14, we will show that the two topologies introduced in the next theorem do not coincide with $\mathcal{I}_1$ from the previous theorem. It will follow that the topology $\mathcal{I}_1$ from Theorem 5.12 is not a semigroup topology on I_X.

Theorem 5.13. *Let X be an infinite set, let $\mathcal{I}_2$ be the topology on the symmetric inverse monoid I_X generated by the collection of sets*

$$U_{x,y} = \{h \in I_X : (x,y) \in h\} \quad and \quad W_x = \{h \in I_X : x \notin \mathrm{dom}(h)\}$$

and let $\mathcal{I}_3$ be the topology on I_X generated by the sets

$$U_{x,y} = \{h \in I_X : (x,y) \in h\} \quad and \quad W_x^{-1} = \{h \in I_X : x \notin \mathrm{im}(h)\}$$

for all $x, y \in X$. Then the following hold:

(i) *$\mathcal{I}_2$ and $\mathcal{I}_3$ are distinct and both contain the topology $\mathcal{I}_1$ defined in Theorem 5.12;*
(ii) *I_X with $\mathcal{I}_2$ and I_X with $\mathcal{I}_3$ are homeomorphic Hausdorff topological semigroups;*
(iii) *every T_1 semigroup topology for I_X contains $\mathcal{I}_2$ or $\mathcal{I}_3$;*
(iv) *$\mathcal{I}_2 \cap \mathcal{I}_3$ coincides with the Hausdorff-Markov and Fréchet-Markov topologies for I_X;*
(v) *if X is countable, then $\mathcal{I}_2$ and $\mathcal{I}_3$ are Polish.*

Proof. **(i).** Let $x \in X$ be fixed. We will show that the set W_x^{-1} is not open in $\mathcal{I}_2$. Seeking a contradiction, suppose that W_x^{-1} is open in $\mathcal{I}_2$. It follows that there are finite $f \in I_X$ and $Y \subseteq X$ such that

$$\varnothing \in \left(\bigcap_{(a,b) \in f} U_{a,b}\right) \cap \left(\bigcap_{y \in Y} W_y\right) \subseteq W_x^{-1}$$

and so $f = \varnothing$. Furthermore, if $b \in X$, is distinct from every element of Y, then

$$\{(b,x)\} \in \left(\bigcap_{(a,b) \in f} U_{a,b}\right) \cap \left(\bigcap_{y \in Y} W_y\right),$$

but $\{(b,x)\} \notin W_x^{-1}$, a contradiction.

The fact that $\mathcal{I}_1$ is contained in both $\mathcal{I}_2$ and $\mathcal{I}_3$ follows since

$$V_{x,y} = \left(\bigcup_{z \neq x} U_{z,y}\right) \cup W_x = \left(\bigcup_{z \neq y} U_{x,z}\right) \cup W_y^{-1}.$$

(ii). The Hausdorff semigroup topology $\mathcal{P}$ on P_X defined in Theorem 5.10 induces $\mathcal{I}_2$ on I_X. Hence I_X is a Hausdorff topological semigroup under $\mathcal{I}_2$.

The map $f \mapsto f^{-1}$ defines an anti-automorphism of I_X. The images of the subbasic open sets for $\mathcal{I}_2$ under inversion give the subbasis for $\mathcal{I}_3$. By Proposition 2.3, it follows that $(I_X, \mathcal{I}_3)$ is a topological semigroup homeomorphic to $(I_X, \mathcal{I}_2)$.

(iii). Let $\mathcal{T}$ be any T_1 semigroup topology for I_X. By Theorem 5.12(ii), $\mathcal{T}$ contains the topology $\mathcal{I}_1$ with subbasis

$$U_{x,y} = \{h \in I_X : (x,y) \in h\} \text{ and } V_{x,y} = \{h \in I_X : (x,y) \notin h\}$$

for all $x, y \in X$. It remains to show that either $\{h \in I_X : x \notin \mathrm{dom}(x)\} \in \mathcal{T}$ for all $x \in X$ or $\{h \in I_X : y \notin \mathrm{im}(h)\} \in \mathcal{T}$ for all $y \in X$.

For every $x \in X$, the set $V_{x,x} = \{h \in I_X : (x,x) \notin h\}$ is an open neighbourhood of $\varnothing$ in $\mathcal{T}$. Since $\varnothing \circ \varnothing = \varnothing \in V_{x,x}$ and $\mathcal{T}$ is a semigroup topology, there exists an open neighbourhood U of $\varnothing$ such that $U \circ U \subseteq V_{x,x}$. In other words, $(x,x) \notin uv$ for any $u, v \in U$. If $z \in X$ is arbitrary such that $z \notin \{y \in X : (x,y) \notin u$ for all $u \in U\}$ and $z \notin \{y \in X : (y,x) \notin u$ for all $u \in U\}$, then there exist $u, v \in U$ such that $(x,z) \in u$ and $(z,x) \in v$, and so $(x,x) \in uv$, a contradiction. Hence every $z \in X$ belongs to one of the sets:

$$\{y \in X : (x,y) \notin u \text{ for all } u \in U\} \text{ or } \{y \in X : (y,x) \notin u \text{ for all } u \in U\}$$

and so one of these two sets has cardinality $|X|$.

Suppose that $|\{y \in X : (x,y) \notin u \text{ for all } u \in U\}| = |X|$. It follows that there exists $Y \subseteq X$ with $|Y| = |X \setminus Y|$ such that $(x,y) \notin u$ for all $y \in Y$ and $u \in U$. Let $p \in \mathrm{Sym}(X)$ be any involution such that $(Y)p = X \setminus Y$, and so $(X \setminus Y)p = Y$. Then for any $u \in U$ and any $y \in X \setminus Y$ since $(y)p \in Y$ it follows that $(x, (y)p) \notin u$ and so $(x,y) \notin up$. Let $V = U \cap Up$. Then V is an open neighbourhood of $\varnothing$ and $x \notin \mathrm{dom}(f)$ for all $f \in V$. Let $g \in I_X$ be arbitrary. If $x \in \mathrm{dom}(g)$, then $x \in \mathrm{dom}(\{(x,x)\} \circ g)$ and so $\{(x,x)\} \circ g \notin V$. On the other hand, if $x \notin \mathrm{dom}(g)$, then $\{(x,x)\} \circ g = \varnothing \in V$. Thus

$$(V)\lambda_{\{(x,x)\}}^{-1} = \{g \in I_X : \{(x,x)\} \circ g \in V\} = \{g \in I_X : x \notin \mathrm{dom}(g)\}$$

is open in $\mathcal{T}$. Since $x \in X$ was arbitrary, it follows that $\mathcal{I}_2 \subseteq \mathcal{T}$.

If $|\{y \in X : (y,x) \notin u \text{ for all } u \in U\}| = |X|$, then $\mathcal{I}_3 \subseteq \mathcal{T}$ by an analogous argument.

(iv). These follow directly from parts (ii), (iii), and Fig. 1.

(v). It is shown in Theorem 5.10(ii) and (viii) that the partial transformation monoid P_X forms a Polish semigroup with the topology $\mathcal{P}$ defined in that theorem. The topology induced by $\mathcal{P}$ on I_X is $\mathcal{I}_2$. It is routine to verify that I_X is a closed subset of P_X under the topology $\mathcal{P}$, and so $\mathcal{I}_2$ is Polish also. Thus $\mathcal{I}_3$ is Polish as it is homeomorphic to $\mathcal{I}_2$. □

We remark that complete metrics on $I_\mathbb{N}$, that induce $\mathcal{I}_2$ and $\mathcal{I}_3$ from Theorem 5.13, can be defined using the natural embedding ϕ defined in (2) of Section 5.3:

$$(3) \qquad d_1(f,g) = \begin{cases} 0 & \text{if } f = g \\ \frac{1}{m+1} & \text{if } f \neq g \end{cases} \qquad \text{where } m = \min\{x \in \mathbb{N} : (x)(f)\phi \neq (x)(g)\phi\}$$

and

$$(4) \qquad d_2(f,g) = \begin{cases} 0 & \text{if } f = g \\ \frac{1}{m+1} & \text{if } f \neq g \end{cases} \qquad \text{where } m = \min\{x \in \mathbb{N} : (x)\left(f^{-1}\right)\phi \neq (x)\left(g^{-1}\right)\phi\}.$$

Next, we show that the Hausdorff-Markov and Fréchet-Markov topology $\mathcal{I}_2 \cap \mathcal{I}_3$ is not equal to the least T_1 semitopological semigroup topology $\mathcal{I}_1$.

Proposition 5.14. *If $\mathcal{Z}$ is the Zariski topology on the symmetric inverse monoid I_X, then $\mathcal{Z}$ properly contains $\mathcal{I}_1$ and so $\mathcal{I}_1$ is properly contained in $\mathcal{I}_2 \cap \mathcal{I}_3$.*

Proof. Since $\mathcal{I}_1$ is the least T_1 topology which is semitopological for I_X (Theorem 5.12(ii)) and $\mathcal{I}_2 \cap \mathcal{I}_3$ is the Hausdorff-Markov topology (Theorem 5.13(iv)), it follows from Fig. 1 that

$$\mathcal{I}_1 \subseteq \mathcal{Z} \subseteq \mathcal{I}_2 \cap \mathcal{I}_3.$$

It therefore suffices to show that $\mathcal{Z} \nsubseteq \mathcal{I}_1$. Let $x \in X$ be arbitrary and consider the set

$$U = \{s \in I_X : \{(x,x)\}s^2 \neq \{(x,x)\}\} \in \mathcal{Z}.$$

We will show that the element $\varnothing \in U$ does not have an open neighbourhood in $\mathcal{I}_1$ which is contained in U. If V is a basic open neighbourhood of $\varnothing$ in $\mathcal{I}_1$, then V is of the form $V = \{s \in I_X : h \cap s = \varnothing\}$ for some finite $h \in I_X$. Since h is finite, there exists $y \in X$ such that $y \notin \mathrm{dom}(h) \cup \mathrm{im}(h)$. Then $\{(x,y),(y,x)\} \in V \setminus U$ and so $V \nsubseteq U$, as required. □

In the next theorem we consider the topology generated by the union of the two minimal T_1 semigroup topologies $\mathcal{I}_2$ and $\mathcal{I}_3$ on I_X.

Theorem 5.15. *Let X be an infinite set and let $\mathcal{I}_4$ be the topology on the symmetric inverse monoid I_X generated by the collection of sets*

$$U_{x,y} = \{h \in I_X : (x,y) \in h\}, \ W_x = \{h \in I_X : x \notin \mathrm{dom}(h)\}, \ \text{and } W_x^{-1} = \{h \in I_X : x \notin \mathrm{im}(h)\}.$$

Then the following hold:

(i) *the topology $\mathcal{I}_4$ is a Hausdorff inverse semigroup topology for I_X;*
(ii) *$\mathcal{I}_4$ is the Hausdorff-Markov inverse and the Fréchet-Markov inverse topology for I_X;*
(iii) *the inverse Zariski topology for I_X is $\mathcal{I}_4$;*
(iv) *I_X has property **X** with respect to $\mathcal{I}_4$ and $\mathrm{Sym}(X)$;*
(v) *if $\mathcal{T}$ is a topology that is semitopological for I_X and $\mathcal{T}$ induces the pointwise topology on $\mathrm{Sym}(X)$, then $\mathcal{T}$ is contained in $\mathcal{I}_4$;*
(vi) *$\mathcal{I}_4$ is the unique T_1 inverse semigroup topology on I_X inducing the pointwise topology on $\mathrm{Sym}(X)$;*
(vii) *if X is countable, then I_X with the topology $\mathcal{I}_4$ has automatic continuity with respect to the class of second countable topological semigroups;*

(viii) *if X is countable, then $\mathcal{I}_4$ is the unique T_1 second countable inverse semigroup topology on I_X;*

(ix) *if X is countable, then $\mathcal{I}_4$ is the unique Polish inverse semigroup topology on I_X.*

Proof. The topology $\mathcal{I}_4$ is the topology generated by the union $\mathcal{I}_2 \cup \mathcal{I}_3$ defined in Theorem 5.13.

(i). Since $\mathcal{I}_4$ is generated by the Hausdorff semigroup topologies $\mathcal{I}_2$ and $\mathcal{I}_3$ (by Theorem 5.13(ii)), it follows that $\mathcal{I}_4$ is a Hausdorff semigroup topology for I_X. If U is any of the subbasic open sets defining $\mathcal{I}_4$, then $U^{-1} = \{f^{-1} : f \in U\}$ is also a subbasic open set, and so $\mathcal{I}_4$ is an inverse semigroup topology.

(ii). The topology $\mathcal{I}_4$ is Hausdorff by (i). Thus both the Fréchet-Markov inverse semigroup topology, and the inverse Hausdorff-Markov topology for I_X are contained in $\mathcal{I}_4$.

For the converse, suppose that $\mathcal{T}$ is any T_1 inverse semigroup topology for I_X. By Theorem 5.13(iii), $\mathcal{T}$ contains either $\mathcal{I}_2$ or $\mathcal{I}_3$. Since inversion is continuous with respect to $\mathcal{T}$ and $\{U^{-1} : U \in \mathcal{I}_2\} = \mathcal{I}_3$, it follows that $\mathcal{T}$ contains $\mathcal{I}_4$. Thus the Fréchet-Markov inverse semigroup topology, and the inverse Hausdorff-Markov topology for I_X are contained in $\mathcal{I}_4$.

(iii). By Fig. 1 and part (ii), the inverse Zariski topology is contained in $\mathcal{I}_4$. On the other hand, the subbasic open sets of $\mathcal{I}_4$ of the form $U_{x,y}$ are open in the least T_1 topology $\mathcal{I}_1$ that is semitopological for I_X, as shown in Theorem 5.12(ii). Since $\mathcal{I}_1$ is contained in the inverse Zariski topology by Fig. 1, it only remains to show that the subbasic open sets of the form W_x and W_x^{-1} of $\mathcal{I}_4$ are open in the inverse Zariski topology.

Let $x \in X$ and $s \in I_X$ be arbitrary. Then $\{(x,x)\}ss^{-1} = \{(x,x)\}$ if $x \in \mathrm{dom}(s)$ and $\{(x,x)\}ss^{-1} = \varnothing$ otherwise. Hence

$$W_x = \{s \in I_X : x \notin \mathrm{dom}(s)\} = \{s \in I_X : \{(x,x)\}ss^{-1} \neq \{(x,x)\}\}$$

is open in the inverse Zariski topology. Similarly, $W_x^{-1} = \{s \in I_X : s^{-1}s\{(x,x)\} \neq \{x,x\}\}$ and so the inverse Zariski topology coincides with $\mathcal{I}_4$, as required.

(iv). Suppose that $s \in S$ is arbitrary. Let $f \in I_X$ be any element satisfying $\mathrm{dom}(f) = X$ and $|X \setminus \mathrm{im}(f)| = |X|$. Then $t \in \mathrm{Sym}(X)$ satisfies $ftf^{-1} = s$ if and only if

$$(x, (x)s) \in ftf^{-1} \text{ for all } x \in \mathrm{dom}(s) \text{ and } x \notin \mathrm{dom}(ftf^{-1}) \text{ for all } x \in X \setminus \mathrm{dom}(s),$$

which is equivalent to

(5) $\qquad (x)ft = (x)sf$ for all $x \in \mathrm{dom}(s)$ and $(x)ft \in X \setminus \mathrm{im}(f)$ for all $x \in X \setminus \mathrm{dom}(s)$.

If $\phi_0 : (X \setminus \mathrm{dom}(s))f \longrightarrow X \setminus \mathrm{im}(f)$ is any injection such that $|(X \setminus \mathrm{im}(f)) \setminus \mathrm{im}(\phi_0)| = |X|$, and $\phi_1 : X \setminus \mathrm{im}(f) \longrightarrow (X \setminus \mathrm{im}(sf)) \setminus \mathrm{im}(\phi_0)$ is a bijection, then $t : X \longrightarrow X$ defined by

$$(x)t = \begin{cases} (x)f^{-1}sf & \text{if } x \in (\mathrm{dom}(s))f \\ (x)\phi_0 & \text{if } x \in (X \setminus \mathrm{dom}(s))f \\ (x)\phi_1 & \text{if } x \notin \mathrm{im}(f) \end{cases}$$

belongs to $\mathrm{Sym}(X)$ and, by (5), $ftf^{-1} = s$.

We will define a basis $\mathscr{B}$ for $\mathcal{I}_4$ and then show that if $U \in \mathscr{B}$ contains t, then $f(U \cap \mathrm{Sym}(X))f^{-1} \in \mathscr{B}$, and so property **X** holds. If $k \in I_X$ is finite, and Y and Z are finite subsets of X, then we define

$$R_{k,Y,Z} = \{h \in I_X : k \subseteq h \text{ and } Y \cap \mathrm{dom}(h) = Z \cap \mathrm{im}(h) = \varnothing\}.$$

Since

$$R_{k,Y,Z} = \bigcap_{x \in \mathrm{dom}(k)} U_{x,(x)k} \cap \bigcap_{y \in Y} W_y \cap \bigcap_{z \in Z} W_z^{-1}$$

the sets $R_{k,Y,Z}$ are open in $\mathcal{I}_4$. On the other hand, any finite intersection of subbasic open sets for $\mathcal{I}_4$ is of the form $R_{k,Y,Z}$ for some finite k, Y, and Z. Hence the collection $\mathscr{B}$ of all such sets $R_{k,Y,Z}$ forms a basis for $\mathcal{I}_4$. Let $U \in \mathscr{B}$ be any such basic open neighbourhood of t. By definition, U is one of the sets of the form $R_{k,Y,Z}$ but since $t \in \mathrm{Sym}(X)$, $\mathrm{dom}(t) = \mathrm{im}(t) = X$, it follows that $U = R_{k,\varnothing,\varnothing}$ for some finite k. Hence

$$f(U \cap \mathrm{Sym}(X))f^{-1} = \{fgf^{-1} : g \in \mathrm{Sym}(X) \text{ and } k \subseteq g\}.$$

It suffices to show that

$$f(U \cap \mathrm{Sym}(X))f^{-1} = R_{fkf^{-1},Y,Z}$$

where

$$Y = \{(y)f^{-1} : y \in \mathrm{dom}(k) \text{ and } (y)k \notin \mathrm{dom}(f^{-1})\}$$

and
$$Z = \{(z)f^{-1} : z \in \operatorname{im}(k) \text{ and } (z)k^{-1} \notin \operatorname{dom}(f^{-1})\}.$$

If $fgf^{-1} \in f(U \cap \operatorname{Sym}(X))f^{-1}$, then $fkf^{-1} \subseteq fgf^{-1}$ since $g \in U = R_{k,\varnothing,\varnothing}$. If $(y)f^{-1} \in Y$ is arbitrary, then $y \in \operatorname{dom}(k)$ and $(y)k \notin \operatorname{dom}(f^{-1})$. Hence $(y)f^{-1}fkf^{-1} = (y)kf^{-1}$ is not defined and so $(y)f^{-1}fgf^{-1} = (y)gf^{-1} = (y)kf^{-1}$ is not defined. In other words, $(y)f^{-1} \notin \operatorname{dom}(fgf^{-1})$ and so $Y \cap \operatorname{dom}(fgf^{-1}) = \varnothing$. Similarly, if $(z)f^{-1} \in Z$ is arbitrary, then $z \in \operatorname{im}(k)$ and $(z)k^{-1} \notin \operatorname{dom}(f^{-1})$. Hence $(z)f^{-1}fk^{-1}f^{-1} = (z)k^{-1}f^{-1}$ is not defined, or, in other words, $(z)f^{-1} \notin \operatorname{dom}(fg^{-1}f^{-1}) = \operatorname{im}(fgf^{-1})$. It follows that $Z \cap \operatorname{im}(fgf^{-1}) = \varnothing$. Hence every $fgf^{-1} \in f(U \cap \operatorname{Sym}(X))f^{-1}$ belongs to $R_{fkf^{-1},Y,Z}$, and so $f(U \cap \operatorname{Sym}(X))f^{-1} \subseteq R_{fkf^{-1},Y,Z}$.

Conversely, suppose that $u \in R_{fkf^{-1},Y,Z}$ is arbitrary. We define $\phi_2 : (X \setminus \operatorname{dom}(u))f \longrightarrow (X \setminus \operatorname{im}(f)) \setminus \{(x)k : x \in X \setminus \operatorname{dom}(f)\}$ is an injection which agrees with k when they are both defined and $|(X \setminus \operatorname{im}(f)) \setminus \operatorname{im}(\phi_2)| = |X|$, and $\phi_3 : X \setminus \operatorname{im}(f) \longrightarrow (X \setminus \operatorname{im}(uf)) \setminus \operatorname{im}(\phi_2)$ is a bijection which agrees with k when they are both defined. We define $g : X \longrightarrow X$ by

$$(x)g = \begin{cases} (x)f^{-1}uf & \text{if } x \in (\operatorname{dom}(u))f \\ (x)\phi_2 & \text{if } x \in (X \setminus \operatorname{dom}(u))f \\ (x)\phi_3 & \text{if } x \notin \operatorname{im}(f). \end{cases}$$

Then $g \in \operatorname{Sym}(X)$ and $k \subseteq g$. Hence $g \in U$. Finally, g satisfies condition (5) (where s and t are replaced by u and g, respectively), and so $u = fgf^{-1} \in f(U \cap \operatorname{Sym}(X))f^{-1}$, as required.

(**v**). This follows from (iv) together with Theorem 3.1(ii).

(**vi**). This follows straight from (ii) and (v).

(**vii**). This follows from part (iv), Theorem 3.1(iv) and the automatic continuity of $\operatorname{Sym}(X)$.

(**viii**). If X is countable, then the subbasis for $\mathcal{I}_4$ is countable also. Thus $\mathcal{I}_4$ is second countable. On the other hand, if $\mathcal{T}$ is any T_1 second countable, inverse semigroup topology for I_X, then, by part (ii), $\mathcal{I}_4 \subseteq \mathcal{T}$. By part (vi), we also have $\mathcal{T} \subseteq \mathcal{I}_4$ and so $\mathcal{T} = \mathcal{I}_4$, as required.

(**ix**). Suppose that $X = \mathbb{N}$. We only need to show that $\mathcal{I}_4$ is completely metrizable, since separability and uniqueness then follow from part (viii). Let ϕ be the natural embedding defined in (2) of $I_{\mathbb{N}}$ into $\mathbb{N} \cup \{\blacklozenge\}^{\mathbb{N} \cup \{\blacklozenge\}}$. We define the metric d on $I_{\mathbb{N}}$ by $d(f,f) = 0$ and if $f \neq g$, then $d(f,g) = \frac{1}{m+1}$ where

$$m = \min\{y \in \mathbb{N} : (y)((f)\phi) \neq (y)((g)\phi) \text{ or } (y)((f^{-1})\phi) \neq (y)((g^{-1})\phi)\}.$$

It is routine to show that d is a metric on $I_{\mathbb{N}}$. (In fact, d is the maximum of the metrics d_1 and d_2 defined in (3) and (4).)

We will now show that the topology induced by d is $\mathcal{I}_4$. As in the proof of part (iv), the sets

$$R_{k,Y,Z} = \{f \in I_{\mathbb{N}} : k \subseteq f \text{ and } Y \cap \operatorname{dom}(f) = Z \cap \operatorname{im}(f) = \varnothing\}$$

where $f \in I_{\mathbb{N}}$ is finite and Y and Z are finite subsets of X, form a basis $\mathscr{B}$ for $\mathcal{I}_4$.

For any $f \in I_{\mathbb{N}}$ and $m \in \mathbb{N}$, we have that

$$B\left(f, \frac{1}{m+1}\right) = \{g \in I_{\mathbb{N}} : (x)((f)\phi) = (x)((g)\phi) \text{ and } (x)((f^{-1})\phi) = (x)((g^{-1})\phi) \text{ for all } x \in m\}$$
$$= R_{k,Y,Z}$$

where $k = (f \cap (m \times \mathbb{N})) \cup (f \cap (\mathbb{N} \times m))$, $Y = m \setminus \operatorname{dom}(f)$, and $Z = m \setminus \operatorname{im}(f)$. Hence every open ball under d is open in $\mathcal{I}_4$ and so the topology induced by d is contained in $\mathcal{I}_4$.

Suppose that $f \in I_{\mathbb{N}}$ is finite, Y, Z are finite subsets of X, and

$$M = \max(\operatorname{dom}(f) \cup \operatorname{im}(f) \cup Y \cup Z) \in \mathbb{N}.$$

If $g \in R_{f,Y,Z}$, then $B(g, 1/M) \subseteq R_{f,Y,Z}$ and so $R_{f,Y,Z}$ is open in the topology induced by d. We have shown that $\mathcal{I}_4$ coincides with the topology induced by d.

To show that d is complete, suppose that $f_0, f_1, f_2, \ldots \in I_{\mathbb{N}}$ is a Cauchy sequence. For every $m \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $i, j \geq M$ implies that $d(f_i, f_j) < 1/m$. So if $x \leq m$, then $(x)((f_i)\phi) = (x)((f_j)\phi)$ and $(x)((f_i^{-1})\phi) = (x)((f_j^{-1})\phi)$ for all $i, j \geq M$. In particular, the sequences $((x)((f_0)\phi), (x)((f_1)\phi), (x)((f_2)\phi), \ldots)$ and $((x)((f_0^{-1})\phi), (x)((f_1^{-1})\phi), (x)((f_2^{-1})\phi), \ldots)$ are eventually constant with values $(x)F$ and $(x)F^{-1}$, respectively. This defines $F \in I_{\mathbb{N}}$ and the sequence $f_0, f_1, f_2, \ldots$ converges to F. Therefore the metric d is complete. $\qquad\square$

Corollary 5.16. *The semigroup and inverse semigroup Zariski topologies on the symmetric inverse monoid $I_\mathbb{N}$ are distinct.*

Proof. By Theorem 5.15(iii), the inverse semigroup Zariski topology on $I_\mathbb{N}$ is $\mathcal{I}_4$. On the other hand, the semigroup Zariski topology is contained in $\mathcal{I}_2 \cap \mathcal{I}_3$ by Proposition 5.14. Theorem 5.13(i) implies that $\mathcal{I}_2 \cap \mathcal{I}_3$ is strictly contained in $\mathcal{I}_4$, and so the semigroup and inverse semigroup Zariski topologies must be distinct. $\qquad\square$

By Theorem 5.13(iii), every Polish semigroup topology on $I_\mathbb{N}$ contains $\mathcal{I}_2$ or $\mathcal{I}_3$ and, by Theorem 5.15(iv), is contained in $\mathcal{I}_4$.

Question 5.17. *Are $\mathcal{I}_2, \mathcal{I}_3$ and $\mathcal{I}_4$ the only Polish semigroup topologies on $I_\mathbb{N}$?*[2]

We can now give an example to demonstrate that the right small index property is really distinct from the left small index property.

Proposition 5.18. *The topological semigroup $I_\mathbb{N}$ with $\mathcal{I}_2$ has the right small index property but not the left small index property.*

Proof. We start by showing that $(I_\mathbb{N}, \mathcal{I}_2)$ does not have the left small index property. Let $x \in \mathbb{N}$ be fixed. We define a left congruence on $I_\mathbb{N}$ by

$$\sigma := \left\{(f,g) \in I_\mathbb{N} : (x \notin \operatorname{im}(f) \cup \operatorname{im}(g)) \text{ or } \left(x \in \operatorname{im}(f) \cap \operatorname{im}(g) \text{ and } (x)f^{-1} = (x)g^{-1}\right)\right\}.$$

The classes of this left congruence are the sets $U_{y,x}$ for $y \in \mathbb{N}$ and W_x^{-1}. In the proof of Theorem 5.13(i), it was shown that the set W_x^{-1} is not open in $\mathcal{I}_2$. Hence $I_\mathbb{N}$ with the topology $\mathcal{I}_2$ does not have the left small index property.

We next show that $I_\mathbb{N}$ with $\mathcal{I}_2$ has the right small index property. Let ρ be a right congruence on $I_\mathbb{N}$ with countably many classes. By Theorem 5.15(vii) together with Corollary 4.5, ρ is open with respect to $\mathcal{I}_4$. It suffices to show that f/ρ is open in $\mathcal{I}_2$ for an arbitrary $f \in I_\mathbb{N}$. In particular, we need only find an open neighbourhood U of f in $\mathcal{I}_2$ such that $U \subseteq f/\rho$. Let

$$V = \{g \in I_\mathbb{N} : h \subseteq g, X \cap \operatorname{dom}(g) = \varnothing, Y \cap \operatorname{im}(g) = \varnothing\},$$

where $h \subseteq \mathbb{N} \times \mathbb{N}$ and $X, Y \subseteq \mathbb{N}$ are finite, be a basic open neighbourhood of f in $\mathcal{I}_4$ such that $V \subseteq f/\rho$. We define

$$U := \{g \in I_\mathbb{N} : h \subseteq g, X \cap \operatorname{dom}(g) = \varnothing\}.$$

The set U is an open neighbourhood of f in $\mathcal{I}_2$. It therefore suffices to show that $U \subseteq f/\rho$. Let $g \in U$ be arbitrary. Let $k \in \mathcal{I}_\mathbb{N}$ be such that $k|_{\operatorname{im}(h)}$ is the identity function, $\operatorname{dom}(k) = \mathbb{N}$, and $\operatorname{im}(k) \subseteq \mathbb{N} \backslash Y$. Since $\operatorname{dom}(k) = \mathbb{N}$, kk^{-1} is the identity of $I_\mathbb{N}$. Furthermore, $fk \in V \subseteq f/\rho$ and also $gk \in V \subseteq f/\rho$. It follows that $gk/\rho = f/\rho = fk/\rho$, and so $g/\rho = gkk^{-1}/\rho = fkk^{-1}/\rho = f/\rho$, as required. $\qquad\square$

The symmetric inverse monoid $I_\mathbb{N}$ is the natural analogue of the full transformation monoid $\mathbb{N}^\mathbb{N}$, in that every countable inverse monoid can be embedded in $I_\mathbb{N}$. We will now prove the inverse monoid analogue of Theorem 5.5, characterising those inverse monoids that embed as topological inverse submonoids of $I_\mathbb{N}$ with its unique Polish inverse semigroup topology $\mathcal{I}_4$.

If S is an inverse monoid with identity 1_S and ρ is a right congruence, then we will say that ρ is *Vagner-Preston* if for every $s \in S$ either: $t \in s/\rho$ implies $tt^{-1}/\rho = 1_S/\rho$; or $st/\rho = s/\rho$ for all $t \in S$.

Lemma 5.19. *Let ρ be a Vagner-Preston right congruence on an inverse monoid S with identity 1_S. If $f, g \in S$ satisfy $ff^{-1}/\rho \neq 1_S/\rho$ and $gg^{-1}/\rho \neq 1_S/\rho$, then $f/\rho = g/\rho$.*

Proof. Since ρ is Vagner-Preston, $ff^{-1}/\rho \neq 1_S/\rho_i \neq gg^{-1}/\rho_i$ implies that $fs/\rho_i = f/\rho_i$ and $gs/\rho_i = g/\rho_i$ for all $s \in S$. Hence

$$f/\rho = ff^{-1}gg^{-1}/\rho = gg^{-1}ff^{-1}/\rho = g/\rho_i$$

since $ff^{-1}, gg^{-1} \in S$ are idempotents, and idempotents commute in inverse semigroups. $\qquad\square$

Lemma 5.20. *Let S be a semitopological inverse monoid and let ρ be an open Vagner-Preston right congruence on S. If $a \in S$ satisfies $aa^{-1}/\rho = 1_S/\rho$, then $\{f \in S : a/\rho = aff^{-1}/\rho\}$ is clopen.*

---

[2]Addendum: S. Bardyla, L. Elliott, J. D. Mitchell, and Y. Péresse recently showed that the answer to this question is no, and there are infinitely many Polish semigroup topologies on $I_\mathbb{N}$.

Proof. Throughout this proof, let $a \in S$ satisfy $aa^{-1}/\rho = 1_S/\rho$. If $f \in S$ is arbitrary, then

$$a/\rho = aff^{-1}/\rho \Rightarrow 1_S/\rho = aa^{-1}/\rho = aff^{-1}a^{-1}/\rho = (af)(af)^{-1}/\rho$$

and

$$1_S/\rho = (af)(af)^{-1}/\rho \Rightarrow a/\rho = aff^{-1}a^{-1}a/\rho = aa^{-1}aff^{-1}/\rho = aff^{-1}/\rho.$$

Hence $a/\rho = aff^{-1}/\rho$ if and only if $1_S/\rho = (af)(af)^{-1}/\rho$ and so

$$
\begin{aligned}
\{f \in S : a/\rho = aff^{-1}/\rho\} &= \{f \in S : (af)(af)^{-1}/\rho = 1_S/\rho\} \\
&= \{f \in S : (af)(af)^{-1} \in 1_S/\rho\} \\
&= \{f \in S : ff^{-1} \in 1_S/\rho\}\lambda_a^{-1} \\
&= \{f \in S : ff^{-1}/\rho = 1_S/\rho\}\lambda_a^{-1}.
\end{aligned}
$$

Since S is semitopological, it now suffices to show that $U = \{f \in S : ff^{-1}/\rho = 1_S/\rho\}$ is clopen. We will do so by showing that U is a union of classes of the open equivalence relation ρ. Let $f \in U$, i.e. $ff^{-1}/\rho = 1_S/\rho$. Since ρ is Vagner-Preston, either $tt^{-1}/\rho = 1_S/\rho$ for all $t \in f/\rho$ or $ft/\rho = f/\rho$ for all $t \in S$. In the first case, clearly $f/\rho \subseteq U$. In the second case $t \in f/\rho$ implies that

$$tt^{-1}/\rho = ft^{-1}/\rho = f/\rho_i = ff^{-1}/\rho = 1_S/\rho$$

and so, again, $f/\rho \subseteq U$, as required. $\qquad \square$

Theorem 5.21. *If S is a T_0 semitopological inverse monoid, then the following are equivalent:*

(i) *there is a sequence $\{\rho_i : i \in \mathbb{N}\}$ of right Vagner-Preston congruences on S, each with countably many classes, such that the set $\{s/\rho_i, (s/\rho_i)^{-1} : s \in S, \ i \in \mathbb{N}\}$ is a subbasis[3] for the topology on S;*

(ii) *S is topologically isomorphic to an inverse subsemigroup of $I_{\mathbb{N}}$ with the topology $\mathcal{I}_4$;*

(iii) *S is topologically isomorphic to an inverse submonoid of $I_{\mathbb{N}}$ with the topology $\mathcal{I}_4$.*

Proof. Obviously (iii) implies (ii) . So it suffices to show that (i) and (ii) are equivalent, and that (ii) implies (iii).

(ii) $\Rightarrow$ (i). Assume without loss of generality that S is an inverse subsemigroup of $I_{\mathbb{N}}$. We will show that S with the subspace topology induced by $\mathcal{I}_4$ satisfies the conditions in (i). For every $i \in \mathbb{N}$, we define $\rho_i \subseteq S \times S$ so that

$$f/\rho_i = \{g : i \in \mathrm{dom}(g), \ (i)f = (i)g\} = U_{i,(i)f}$$

if $i \in \mathrm{dom}(f)$ and

$$f/\rho_i = \{g : i \notin \mathrm{dom}(g)\} = W_i$$

if $i \notin \mathrm{dom}(f)$, where $U_{i,(i)f}$ and W_i are the subbasic open sets for $\mathcal{I}_4$ defined in Theorem 5.15. It is routine to verify that ρ_i is a right congruence for every $i \in \mathbb{N}$. Clearly,

$$(f/\rho_i)^{-1} = \{g : i \in \mathrm{im}(g), \ (i)f^{-1} = (i)g^{-1}\} = U_{(i)f^{-1},i}$$

if $i \in \mathrm{im}(f)$ and

$$(f/\rho_i)^{-1} = \{g : i \notin \mathrm{im}(g)\} = W_i^{-1}$$

if $i \notin \mathrm{im}(f)$.

It remains to prove that conditions (a) or (b) holds for every $i \in \mathbb{N}$. If 1_S is the identity of S, then 1_S is not necessarily $1_{\mathbb{N}}$. However, 1_S is an idempotent satisfying $\mathrm{dom}(s), \mathrm{im}(s) \subseteq \mathrm{dom}(1_S)$ for all $s \in S$. If $i \in \mathrm{dom}(s)$ and $t \in s/\rho_i$, then $i \in \mathrm{dom}(t)$ and $(i)tt^{-1} = i$. Hence $tt^{-1}/\rho_i = 1_S/\rho_i$, and condition (a) holds. If $i \notin \mathrm{dom}(s)$ and $t \in S$ is arbitrary, then $i \notin \mathrm{dom}(st)$ and so $st/\rho_i = s/\rho_i$, and condition (b) holds.

(i) $\Rightarrow$ (ii). Let $X = \{a/\rho_i : i \in \mathbb{N}, a \in S\}$. When $i \neq j$, we will consider classes of ρ_i and classes of ρ_j as different elements of X, even if they should happen to be the same subset of S.

Since X is countable, it suffices to find a topological isomorphism ϕ from S to an inverse subsemigroup of I_X with the topology $\mathcal{I}_4$. This $\phi : S \longrightarrow I_X$ is defined by

$$
(a/\rho_i)(f)\phi =
\begin{cases}
af/\rho_i & \text{if } aa^{-1}/\rho_i = 1_S/\rho_i \text{ and } a/\rho_i = aff^{-1}/\rho_i; \\
\text{undefined} & \text{otherwise.}
\end{cases}
$$

We need to show that the partial function $(f)\phi \in I_X$ is well-defined and injective, that ϕ is a continuous homomorphism, and that if U is open in S, then $(U)\phi$ is open in $(S)\phi$ with subspace topology.

---

[3]The term "subbasis" cannot be replaced with "basis" here, unlike in Theorem 5.5.

To see that $(f)\phi$ is a well-defined partial function from X to X, let $a, b \in S$ such that $a/\rho_i = b/\rho_i$ for some $i \in \mathbb{N}$. Suppose that $aa^{-1}/\rho_i = 1_S/\rho_i$ and $a/\rho_i = aff^{-1}/\rho_i$. Since ρ_i is a right congruence and $a/\rho_i = b/\rho_i$, $b/\rho_i = bff^{-1}/\rho_i$. By the assumption of the theorem, either $bb^{-1}/\rho_i = 1_S/\rho_i$ or $bs/\rho_i = b/\rho_i$ for all $s \in S$. In the latter case, $as/\rho_i = bs/\rho_i = b/\rho_i = a/\rho_i$ and so taking $s = a^{-1}$, it follows that $a/\rho_i = aa^{-1}/\rho_i = 1_S/\rho_i$. Hence, right multiplying 1_S and a by bb^{-1}, we obtain $bb^{-1}/\rho_i = abb^{-1}/\rho_i$ and applying $as/\rho_i = a/\rho_i$ when $s = bb^{-1}$ yields

$$bb^{-1}/\rho_i = abb^{-1}/\rho_i = a/\rho_i = 1_S/\rho_i.$$

So in either case, $bb^{-1}/\rho_i = 1_S/\rho_i$. We have shown that $a/\rho_i = b/\rho_i$ and $(a/\rho_i)(f)\phi$ being defined implies that $(b/\rho_i)(f)\phi$ is defined also. That ϕ is well-defined follows from $(a/\rho)(f)\phi = af/\rho_i = bf/\rho_i = (b/\rho)(f)\phi$.

To see that $(f)\phi$ is injective, let $a/\rho_i, b/\rho_j \in \text{dom}((f)\phi)$ such that $(a/\rho_i)(f)\phi = (b/\rho_j)(f)\phi$. By the definition of $(f)\phi$, it follows that $i = j$, $aff^{-1}/\rho_i = a/\rho_i$ and $bff^{-1}/\rho_i = b/\rho_i$. Hence

$$(a/\rho_i)(f)\phi = (b/\rho_i)(f)\phi \;\Rightarrow\; af/\rho_i = bf/\rho_i \;\Rightarrow\; aff^{-1}/\rho_i = bff^{-1}/\rho_i \;\Rightarrow\; a/\rho_i = b/\rho_i.$$

We conclude that $(f)\phi$ is injective and hence $(f)\phi \in I_X$.

To prove that ϕ is injective, suppose that $(f)\phi = (g)\phi$ for some $f, g \in S$. We will show that $f/\rho_i = g/\rho_i$ for all $i \in \mathbb{N}$. If $f/\rho_i \neq 1_S/\rho_i$ and $g/\rho_i \neq 1_S/\rho_i$, then $f/\rho_i = g/\rho_i$, by Lemma 5.19. So, without loss of generality, assume that $ff^{-1}/\rho_i = 1_S/\rho_i$. Then $1_S/\rho_i = ff^{-1}/\rho_i \in \text{dom}((f)\phi) = \text{dom}((g)\phi)$ and so $f/\rho_i = (1_S/\rho_i)(f)\phi = (1_S/\rho_i)(g)(\phi) = g/\rho_i$. We have shown that $f/\rho_i = g/\rho_i$ for all $i \in \mathbb{N}$. It follows that $(f/\rho_i)^{-1} = (g/\rho_i)^{-1}$, and since $\{a/\rho_i, (a/\rho_i)^{-1} : a \in S, \; i \in \mathbb{N}\}$ is a subbasis for a T_0 topology on S, we conclude that $f = g$, and so ϕ is injective.

To prove that ϕ is a homomorphism, let $f, g \in S$ be arbitrary. The key step in showing that $(fg)\phi = (f)\phi(g)\phi$ is to show that $\text{dom}((fg)\phi) = \text{dom}((f)\phi(g)\phi)$. By definition, $a/\rho_i \in \text{dom}((fg)\phi)$ if and only if

$$(6) \qquad aa^{-1}/\rho_i = 1_S/\rho_i \quad \text{and} \quad a/\rho_i = a(fg)(fg)^{-1}/\rho_i.$$

On the other hand, $a/\rho_i \in \text{dom}((f)\phi(g)\phi)$ if and only if $a/\rho_i \in \text{dom}((f)\phi)$ and $af/\rho_i \in \text{dom}((g)\phi)$ if and only if

$$(7) \qquad aa^{-1}/\rho_i = 1_S/\rho_i = (af)(af)^{-1}/\rho_i, \quad aff^{-1}/\rho_i = a/\rho_i, \quad \text{and} \quad af/\rho_i = afgg^{-1}/\rho_i.$$

To show that (6) and (7) are equivalent, first assume that $a \in S$ satisfies (6). Then $aa^{-1}/\rho_i = 1_S/\rho_i$ and

$$
\begin{aligned}
aff^{-1}/\rho_i &= (a(fg)(fg)^{-1})ff^{-1}/\rho_i = afgg^{-1}f^{-1}ff^{-1}/\rho_i = afgg^{-1}f^{-1}/\rho_i \\
&= a(fg)(fg)^{-1}/\rho_i = a/\rho_i.
\end{aligned}
$$

Hence

$$1_S/\rho_i = aa^{-1}/\rho_i = (aff^{-1})a^{-1}/\rho_i = (af)(af)^{-1}/\rho_i,$$

and

$$af/\rho_i = (a(fg)(fg)^{-1})f/\rho_i = afgg^{-1}f^{-1}f/\rho_i = aff^{-1}fgg^{-1}/ rho_i = afgg^{-1}/\rho_i.$$

Thus (6) implies (7). Now assume that $a \in S$ satisfies (7). Then $aa^{-1}/\rho_i = 1_S/\rho_i$ and

$$a/\rho_i = aff^{-1}/\rho_i = (afgg^{-1})f^{-1}/\rho_i = a(fg)(fg)^{-1}/\rho_i.$$

Hence (7) implies (6) and so the domains of $(fg)\phi$ and $(f)\phi(g)\phi$ coincide. If $a/\rho_i \in \text{dom}((fg)\phi) = \text{dom}((f)\phi(g)\phi)$, then

$$(a/\rho_i)(f)\phi(g)\phi = (af/\rho_i)(g)\phi = afg/\rho_i = (a/\rho_i)(fg)\phi.$$

It follows that $(fg)\phi = (f)\phi(g)\phi$ and so ϕ is a homomorphism.

To show that ϕ is continuous we need to prove that the preimages of the subbasic open sets $U_{a/\rho_i, b/\rho_j}$, W_{a/ρ_i} and W_{a/ρ_i}^{-1} of $\mathcal{I}_4$ are open in S.

If $U_{a/\rho_i, b/\rho_j} \cap (S)\phi \neq \varnothing$, then $i = j$ and $aa^{-1}/\rho_i = 1_S/\rho_i$. Thus

$$
\begin{aligned}
\left(U_{a/\rho_i, b/\rho_i}\right)\phi^{-1} &= \{f \in S : (a/\rho_i, b/\rho_i) \in (f)\phi\} \\
&= \{f \in S : a/\rho_i \in \text{dom}((f)\phi) \text{ and } af/\rho_i = b/\rho_i\} \\
&= \{f \in S : a/\rho_i \in \text{dom}((f)\phi)\} \cap \{f \in S : af/\rho_i = b/\rho_i\} \\
&= \{f \in S : a/\rho_i = aff^{-1}/\rho_i\} \cap \{f \in S : af \in b/\rho_i\}.
\end{aligned}
$$

The set $\{f \in S : a/\rho_i = aff^{-1}/\rho_i\}$ is clopen by Lemma 5.20 and $\{f \in S : af \in b/\rho_i\} = (b/\rho_i)\lambda_a^{-1}$ is open since b/ρ_i is open and S is semitopological. It follows that $\left(U_{a/\rho_i, b/\rho_i}\right)\phi^{-1}$ is open.

Now consider $(W_{a/\rho_i})\phi^{-1} = \{f \in S : a/\rho_i \notin \mathrm{dom}((f)\phi)\}$. If $aa^{-1}/\rho_i \neq 1_S/\rho_i$, then $(W_{a/\rho_i})\phi^{-1} = S$. If $aa^{-1}/\rho_i = 1_S/\rho_i$, then

$$(W_{a/\rho_i})\phi^{-1} = \{f \in S : a/\rho_i \neq aff^{-1}/\rho_i\} = S \setminus \{f \in S : a/\rho_i = aff^{-1}/\rho_i\}.$$

Hence $(W_{a/\rho_i})\phi^{-1}$ is open as $\{f \in S : a/\rho_i = aff^{-1}/\rho_i\}$ is clopen (again by Lemma 5.20). Since ϕ is a homomorphism

$$(W_{a/\rho_i}{}^{-1})\phi^{-1} = \left((W_{a/\rho_i})\phi^{-1}\right)^{-1},$$

which is open since inversion is continuous in S.

It only remains to prove that if U is open in S, then $(U)\phi$ is open in $(S)\phi$. Since $\phi : S \to (S)\phi$ is a bijection, it is enough to show that the images of the subbasic open sets a/ρ_i and $(a/\rho_i)^{-1}$ are open. Let $(f)\phi \in (a/\rho_i)\phi = \{(g)\phi \in I_X : g \in a/\rho_i\}$ be arbitrary. We need to find an open neighbourhood U of $(f)\phi$ such that $U \cap (S)\phi \subseteq (a/\rho_i)\phi$. Since ϕ is injective, $f \in a/\rho_i$ and so $(a/\rho_i)\phi = (f/\rho_i)\phi = \{(g)\phi \in I_X : g/\rho_i = f/\rho_i\}$. It therefore suffices to show that there exists an open set U in $\mathcal{I}_4$ such that

(8) $\qquad\qquad\qquad (f)\phi \in U$ and $U \cap (S)\phi \subseteq \{(g)\phi \in I_X : g/\rho_i = f/\rho_i\}$.

If $ff^{-1}/\rho_i = 1_S/\rho_i$, then $1_S/\rho_i \in \mathrm{dom}((f)\phi)$ and we let $U = U_{1_S/\rho_i, f/\rho_i}$. Then $(f)\phi \in U$ and if $(g)\phi \in U \cap (S)\phi$, then $g/\rho_i = (1_S/\rho_i)(g)\phi = f/\rho_i$. Hence U satisfies (8).

If $ff^{-1}/\rho_i = 1_S ff^{-1}/\rho_i \neq 1_S/\rho_i$, then $1_S/\rho_i \notin \mathrm{dom}((f)\phi)$. In this case, let $U = W_{1_S/\rho_i}$. Then $(f)\phi \in U$ and if $(g)\phi \in U \cap (S)\phi$, then $1_S/\rho_i \notin \mathrm{dom}((g)\phi)$. Hence $gg^{-1}/\rho_i \neq 1_S/\rho_i$, and so, by Lemma 5.19, $f/\rho_i = g/\rho_i$. It follows that U satisfies (8). We have shown that $(a/\rho_i)\phi$ is open in $(S)\phi$ for all $i \in \mathbb{N}$ and $a \in S$.

Since ϕ is a homomorphism and inversion is continuous (and hence open) in S and in I_X, it follows that

$$((a/\rho_i)^{-1})\phi = ((a/\rho_i)\phi)^{-1}$$

is open in $(S)\phi$. We have shown that ϕ is open, which concludes the proof of this part of the theorem.

(ii) $\Rightarrow$ **(iii).** Without loss of generality, we assume that the inverse monoid S is a subsemigroup of $I_\mathbb{N}$ with the topology $\mathcal{I}_4$, and the subspace topology. We define X to be $\mathrm{dom}(1_S)$ and note that:

$$X = \mathrm{dom}(1_S) = \bigcup_{s \in S} \mathrm{dom}(s) = \bigcup_{s \in S} \mathrm{im}(s) = \mathrm{im}(1_S).$$

It is clear that S is a submonoid of I_X, the symmetric inverse monoid on the set X. There are two cases to consider: when X is finite, and when X is infinite.

Suppose that X is infinite. In this case, since X is countable, I_X is isomorphic to $I_\mathbb{N}$. It is also the case that $I_\mathbb{N}$ and I_X with the subspace topology induced by $\mathcal{I}_4$ (on $I_\mathbb{N}$) are topologically isomorphic. Indeed, the subbasis for $\mathcal{I}_4$ on $I_\mathbb{N}$ given in Theorem 5.15, when restricted to I_X gives the subbasis for $\mathcal{I}_4$ on I_X. Alternatively, the unique T_1 second countable inverse semigroup topology on $I_\mathbb{N}$, and hence I_X, is $\mathcal{I}_4$ (Theorem 5.15(viii)). Hence, by either route, S is topologically isomorphic to an inverse submonoid of $I_\mathbb{N}$.

If X is finite, then S is finite and hence discrete, since $\mathcal{I}_4$ is T_1. The map

$$s \mapsto s \cup \{(x,x) : x \in \mathbb{N} \setminus X\}$$

embeds S as an inverse submonoid of $I_\mathbb{N}$. The embedding is topological, since the domain and range are both finite and hence discrete. $\qquad\square$

5.5. **Injective and surjective transformations.** In this section we consider the submonoids $\mathrm{Inj}(X)$, of injective, and $\mathrm{Surj}(X)$, of surjective, transformations in X^X. If X is countable, then $\mathrm{Inj}(X)$ and $\mathrm{Surj}(X)$ are G_δ subsets of X^X with the pointwise topology, and so these submonoids are Polish semigroups with the subspace topology. However, in contrast to the monoids considered up to this point in the paper, we will show that both of $\mathrm{Inj}(X)$ and $\mathrm{Surj}(X)$ have infinitely many distinct Polish semigroup topologies when X is countably infinite. In the case of $\mathrm{Inj}(X)$, when X is countably infinite, we show that the pointwise topology is the minimum Polish semigroup topology on $\mathrm{Inj}(X)$, and we show that there is a maximum Polish semigroup topology on $\mathrm{Inj}(X)$ given by an explicit subbasis.

Since $\mathrm{Inj}(X)$ is a closed submonoid of the symmetric inverse monoid I_X with its unique Polish inverse semigroup topology $\mathcal{I}_4$ (see Theorem 5.15), one natural Polish topology on $\mathrm{Inj}(X)$ is the subspace topology induced by $\mathcal{I}_4$. A subbasis for this topology comprises the subbasic open sets

(9) $\qquad\qquad \{f \in \mathrm{Inj}(X) : (x)f = y\} \quad \text{and} \quad \{f \in \mathrm{Inj}(X) : y \notin \mathrm{im}(f)\}$ for all $x, y \in X$.

Although natural, this topology does not play a central role in this section. We will show that this topology is one of infinitely many Polish semigroup topologies on $\mathrm{Inj}(X)$ between the minimum and maximum Polish

semigroup topologies on $\mathrm{Inj}(X)$. The topology with subbasis given in (9) coincides with the subspace topology induced by the topology $\mathcal{I}_3$ on I_X from Theorem 5.13. The topology on $\mathrm{Inj}(X)$ induced by $\mathcal{I}_2$ from Theorem 5.13, is the pointwise topology.

Theorem 5.22. *Let X be an infinite set and let $\mathcal{J}$ be the topology on $\mathrm{Inj}(X)$ generated by the pointwise topology and the sets*

$$\{f \in \mathrm{Inj}(X) : x \notin \mathrm{im}(f)\} \quad and \quad \{f \in \mathrm{Inj}(X) : |X \setminus \mathrm{im}(f)| = n\}$$

for every $x \in X$ and for every cardinal $n \leq |X|$. Then the following hold:

 (i) *there are infinitely many distinct semigroup topologies on $\mathrm{Inj}(X)$ containing the pointwise topology and contained in $\mathcal{J}$ including $\mathcal{J}$ itself;*
 (ii) *$\mathrm{Inj}(X)$ has property $\boldsymbol{X}$ with respect to $\mathcal{J}$ and $\mathrm{Sym}(X)$;*
 (iii) *if $\mathcal{T}$ is any semigroup topology on $\mathrm{Inj}(X)$ and $\mathcal{T}$ induces the pointwise topology on $\mathrm{Sym}(X)$, then $\mathcal{T}$ is contained in $\mathcal{J}$;*
 (iv) *the Zariski topology for $\mathrm{Inj}(X)$ is the pointwise topology;*
 (v) *if X is countable, then $\mathrm{Inj}(X)$ has automatic continuity with respect to $\mathcal{J}$ and the class of second countable topological semigroups;*
 (vi) *if X is countable, then there are infinitely many distinct Polish semigroup topologies on $\mathrm{Inj}(X)$ containing the pointwise topology and contained in $\mathcal{J}$ including $\mathcal{J}$ itself;*
 (vii) *if X is countable, then the pointwise topology is the minimum, and $\mathcal{J}$ is the maximum, Polish semigroup topology on $\mathrm{Inj}(X)$.*

Proof. **(i).** If κ is any cardinal such that $\kappa \leq |X|$, then we define

$$F_\kappa = \{f \in \mathrm{Inj}(X) : |X \setminus \mathrm{im}(f)| = \kappa\}.$$

Let $\mathcal{S}_0$ be the subspace topology on $\mathrm{Inj}(X)$ induced by the topology $\mathcal{I}_4$ on I_X. A subbasis for $\mathcal{S}_0$ is given in (9). If $n \in \mathbb{N}$ and $n > 0$, then we define $\mathcal{S}_n$ to be the topology generated by $\mathcal{S}_{n-1}$ and F_{n-1}. It follows that $\mathcal{S}_n$ is the least topology containing $\mathcal{S}_0$ and every set F_m where $m < n$. The topology $\mathcal{J}$ is the topology generated by $\mathcal{S}_0$ and the collection of all F_κ where $\kappa \leq |X|$.

We begin by showing that $\mathcal{J}$ and every $\mathcal{S}_n$, $n \in \mathbb{N}$, is a semigroup topology on $\mathrm{Inj}(X)$. It follows from the definition of $\mathcal{S}_0$ that $\mathcal{S}_0$ is a semigroup topology. Suppose that $\mathcal{T}$ is $\mathcal{J}$ or $\mathcal{S}_n$ where $n \in \mathbb{N}$. To show that $\mathcal{T}$ is a semigroup topology, it suffices to show that if $F_\kappa \in \mathcal{T}$, then the preimage of F_κ under the multiplication function is open in $\mathrm{Inj}(X) \times \mathrm{Inj}(X)$ with the product topology induced by $\mathcal{T}$. Note that

$$|X \setminus \mathrm{im}(fg)| = |X \setminus \mathrm{im}(f)| + |X \setminus \mathrm{im}(g)|$$

for all $f, g \in \mathrm{Inj}(X)$. Suppose that $F_\kappa \in \mathcal{T}$ for some κ such that $\kappa \leq |X|$. If $f, g \in \mathrm{Inj}(X)$ are such that $fg \in F_\kappa$, then $|X \setminus \mathrm{im}(f)| = \alpha$ and $|X \setminus \mathrm{im}(g)| = \beta$ for some cardinals $\alpha, \beta \leq \kappa$ such that $\alpha + \beta = \kappa$. Thus $f \in F_\alpha$, $g \in F_\beta$, and $F_\alpha F_\beta \subseteq F_\kappa$, and so $\mathcal{T}$ is a semigroup topology.

Next, we will show that the topologies $\mathcal{S}_0, \mathcal{S}_1, \ldots$ and $\mathcal{J}$ are distinct. Suppose that $n \in \mathbb{N}$ is arbitrary. If $f \in F_n$ is arbitrary and B is any basic open neighbourhood of f in $\mathcal{S}_n$, then $f \notin F_m$ for every $m \neq n$, and so B is a basic open set for $\mathcal{S}_0$. In other words,

$$B = \{g \in \mathrm{Inj}(\mathbb{N}) : h \subsetneq g \text{ and } \mathrm{im}(g) \cap Y = \varnothing\}$$

for finite $h \in I_\mathbb{N}$ and finite $Y \subseteq \mathbb{N}$. It is routine to verify that there exists $g \in B$ such that $g \notin F_n$, and hence F_n is not open in $\mathcal{S}_n$ for every $n \in \mathbb{N}$. It follows that the topologies $\mathcal{S}_0, \mathcal{S}_1, \ldots$ are distinct. Clearly, $\mathcal{J}$ contains the strictly ascending chain of topologies $\mathcal{S}_0, \mathcal{S}_1, \ldots$ and hence $\mathcal{J}$ is distinct from every $\mathcal{S}_n$, $n \in \mathbb{N}$.

(ii). Let $s \in \mathrm{Inj}(X)$ be arbitrary. We show that there exist $f_s, g_s \in \mathrm{Inj}(X)$ and $a_s \in \mathrm{Sym}(X)$ such that $s = f_s a_s g_s$ and for every basic open neighbourhood B of a_s there exists an open neighbourhood U of s such that $U \subseteq f_s(B \cap \mathrm{Sym}(X))g_s$.

Let $f_s = s$, $a_s = 1_{\mathrm{Inj}(X)}$, and $g_s = 1_{\mathrm{Inj}(X)}$ and let B be a basic open neighbourhood of a_s. As $a_s = 1_{\mathrm{Inj}(X)}$ there is some finite $Y \subseteq X$ such that

$$B \cap \mathrm{Sym}(X) = \{h \in \mathrm{Sym}(X) : (y)h = y \text{ for all } y \in Y\}.$$

Then $h \in \mathrm{Inj}(X)$ is an element of $f_s(B \cap \mathrm{Sym}(X))g_s$ if and only if

 (1) $y \notin \mathrm{im}(h)$ for all $y \in Y \setminus \mathrm{im}(f_s)$;
 (2) $((y)s^{-1}, y) \in h$ for all $y \in Y \cap \mathrm{im}(f_s)$;
 (3) $|X \setminus \mathrm{im}(s)| = |X \setminus \mathrm{im}(h)|$.

So $f_s(B \cap \mathrm{Sym}(X))g_s$ is an open neighbourhood of s as required.

(iii). This follows by part (ii) together with Theorem 3.1(ii).

(iv). This part of the theorem follows from Lemma 5.3.

(v). This follows by part (ii) together with Theorem 3.1(iv) and the automatic continuity of $\mathrm{Sym}(X)$.

(vi). We will show that, when X is countable, the topologies $\mathcal{S}_0, \mathcal{S}_1, \ldots$ and $\mathcal{J}$ from the proof of part (i) are Polish. As already discussed at the start of this section, $\mathcal{S}_0$ is Polish. To show that $\mathcal{S}_n$ is Polish we proceed by induction. Given that $\mathcal{S}_{n-1}$ is Polish, for some $n > 0$, it suffices by [42, Lemma 13.2] to show that F_n is closed in $\mathcal{S}_n$. The complement of F_n in $\mathrm{Inj}(X)$ is given by

$$
\begin{aligned}
\mathrm{Inj}(X) \setminus F_n &= \{f \in \mathrm{Inj}(X) : |X \setminus \mathrm{im}(f)| \neq n\} \\
&= \{f \in \mathrm{Inj}(X) : |X \setminus \mathrm{im}(f)| > n\} \cup \{f \in \mathrm{Inj}(X) : |X \setminus \mathrm{im}(f)| < n\}.
\end{aligned}
$$

The set $\{f \in \mathrm{Inj}(X) : |X \setminus \mathrm{im}(f)| > n\}$ is open in $\mathcal{S}_0$ and hence is open in $\mathcal{S}_n$. The set

$$
\{f \in \mathrm{Inj}(X) : |X \setminus \mathrm{im}(f)| < n\} = \bigcup_{m=0}^{n-1} F_m
$$

is also open by assumption. Hence the complement of F_n is open, and so F_n is closed. Therefore $\mathcal{S}_n$ is Polish also. Hence the topology generated by $\bigcup_{n=0}^{\infty} \mathcal{S}_n$ is Polish by [42, Lemma 13.3]. Finally, by a similar argument, $F_{\aleph_0}$ is closed in the topology generated by $\bigcup_{n=0}^{\infty} \mathcal{S}_n$, so by applying [42, Lemma 13.2] again, $\mathcal{J}$ is Polish.

Although it is not strictly necessary, we note that a complete metric on $\mathrm{Inj}(\mathbb{N})$ that induces $\mathcal{J}$ is:

$$
d(f,g) = \begin{cases} 0 & \text{if } f = g; \\ 1 & \text{if } |\mathbb{N} \setminus \mathrm{im}(f)| \neq |\mathbb{N} \setminus \mathrm{im}(g)| \\ \frac{1}{m+1} & \text{otherwise}; \end{cases}
$$

where $m = \min\{n \in \mathbb{N} : (n \times n) \cap f \neq (n \times n) \cap g\}$.

(vii). This part of the theorem follows immediately from parts (iv) and (v), respectively. $\square$

Theorem 5.23. *Let X be an infinite set, let $\mathcal{S}_1$ denote the topology on $\mathrm{Surj}(X)$ generated by the pointwise topology and the set $\mathrm{Sym}(X)$, and let $\mathcal{S}_2$ be the topology generated by the pointwise topology together with the collection of sets*

$$
U_{\kappa,x} := \{f \in \mathrm{Surj}(X) : |(x)f^{-1}| = \kappa\}
$$

for all $x \in X$ and cardinals $\kappa \leq |X|$. Then the following hold:

   (i) *the pointwise topology and $\mathcal{S}_1$ are distinct semigroup topologies on $\mathrm{Surj}(X)$;*
  (ii) *if X is countable, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are Polish semigroup topologies;*
 (iii) *if X is countable, then there are infinitely many distinct Polish semigroup topologies on $\mathrm{Surj}(X)$ containing the pointwise topology and contained in $\mathcal{S}_2$;*
 (iv) *if X is countable, then $\mathrm{Surj}(X)$ with the topology $\mathcal{S}_1$ embeds into $\mathbb{N}^{\mathbb{N}}$ with the pointwise topology.*

Proof. **(i).** Since the pointwise topology is a semigroup topology for X^X, it is also a semigroup topology for $\mathrm{Surj}(X)$. If X is countable, then $\mathrm{Surj}(X)$ is G_δ in X^X, and so it follows that the pointwise topology is a Polish semigroup topology on $\mathrm{Surj}(X)$.

Since $\mathrm{Inj}(X)$ is closed in X^X, it follows that $\mathrm{Sym}(X) = \mathrm{Inj}(X) \cap \mathrm{Surj}(X)$ is closed in $\mathrm{Surj}(X)$ with respect to the pointwise topology. To show that $\mathcal{S}_1$ is compatible with the multiplication in $\mathrm{Surj}(X)$, it suffices to show that $\{(f,g) : fg \in \mathrm{Sym}(X)\}$ is open in $\mathrm{Surj}(X) \times \mathrm{Surj}(X)$. It is routine to verify that $\mathrm{Surj}(X) \setminus \mathrm{Sym}(X)$ is an ideal in $\mathrm{Surj}(X)$. Hence if $f, g \in \mathrm{Surj}(X)$ are such that $fg \in \mathrm{Sym}(X)$, then $f, g \in \mathrm{Sym}(X)$. Hence $\mathrm{Sym}(X) \times \mathrm{Sym}(X) = \{(f,g) : fg \in \mathrm{Sym}(X)\}$ is open, as required.

(ii). Since $\mathrm{Sym}(X)$ is closed in $\mathrm{Surj}(X)$ with respect to the pointwise topology, it follows by [42, Lemma 13.2], that $\mathcal{S}_1$ is Polish for countable X.

It remains to show that $\mathcal{S}_2$ is a Polish semigroup topology when $|X| = \aleph_0$. We may assume without loss of generality that $X = \mathbb{N}$. First note that for all $f, g \in \mathrm{Surj}(\mathbb{N})$ and $n \in \mathbb{N}$ we have that

$$
|(n)(fg)^{-1}| = \sum_{i \in (n)g^{-1}} |(i)f^{-1}|.
$$

If $fg \in U_{m,n}$, then there are three cases to consider: $m \in \mathbb{N}$; $m = \aleph_0 = |(n)g^{-1}|$; and $m = \aleph_0 \neq |(n)g^{-1}|$.

(1) If $m \in \mathbb{N}$, then

$$fg \in \left(\bigcap_{i \in (n)g^{-1}} U_{|(i)f^{-1}|, i} \right) \cdot \left(U_{|(n)g^{-1}|, n} \cap \bigcap_{i \in (n)g^{-1}} \{ h \in \mathrm{Surj}(\mathbb{N}) : (i)h = (i)g \} \right) \subseteq U_{m,n}.$$

(2) If $m = \aleph_0$ and $|(n)g^{-1}| = m$, then

$$fg \in \mathrm{Surj}(X) \cdot U_{m,n} \subseteq U_{m,n}.$$

(3) If $m = \aleph_0$ and $|(n)g^{-1}| \neq \aleph_0$, then there is some $i \in (n)g^{-1}$ such that $|(i)f^{-1}| = \aleph_0$ and so

$$fg \in U_{\aleph_0, i} \cdot \{ h \in \mathrm{Surj}(\mathbb{N}) : (i)h = n \} \subseteq U_{\aleph_0, n}.$$

To show that $\mathcal{S}_2$ is Polish, we define $\mathcal{T}_{m,n}$ to be the topology generated by the pointwise topology together with the sets $U_{i,n}$ for $i \leq m$ and for all $m, n \in \mathbb{N}$. We show by induction on m, that $\mathcal{T}_{m,n}$ is Polish for all $m, n \in \mathbb{N}$.

By the definition of $\mathrm{Surj}(\mathbb{N})$, $U_{0,n} = \varnothing$ for all $n \in \mathbb{N}$. It follows that, $\mathcal{T}_{0,n}$ is the pointwise topology, which is Polish, for all $n \in \mathbb{N}$. This establishes the base case of the induction.

For the inductive step, suppose that $m > 0$ and $\mathcal{T}_{m-1,n}$ is Polish for all $n \in \mathbb{N}$. The topology $\mathcal{T}_{m,n}$ is generated by $\mathcal{T}_{m-1,n}$ and $U_{m,n}$. To apply [42, Lemma 13.2] and conclude that $\mathcal{T}_{m,n}$ is Polish, it suffices to show that $U_{m,n}$ is closed in $\mathcal{T}_{m-1,n}$. By the definition of $U_{m,n}$,

$$\mathrm{Surj}(\mathbb{N}) \backslash U_{m,n} = \{ f \in \mathrm{Surj}(\mathbb{N}) : |(n)f^{-1}| < m \} \cup \{ f \in \mathrm{Surj}(\mathbb{N}) : |(n)f^{-1}| > m \}.$$

The set $V = \{ f \in \mathrm{Surj}(\mathbb{N}) : |(n)f^{-1}| < m \}$ is the union of the open sets $\{ f \in \mathrm{Surj}(\mathbb{N}) : |(n)f^{-1}| = i \} = U_{i,n}$, for all $i < m$, and so V is open in $\mathcal{T}_{m-1,n}$. The set $W = \{ f \in \mathrm{Surj}(\mathbb{N}) : |(n)f^{-1}| > m \}$ is the union of the open sets

$$\{ f \in \mathrm{Surj}(\mathbb{N}) : (y)f = n \text{ for all } y \in Y \}$$

in the pointwise topology, for all finite subsets Y of $\mathbb{N}$ with at least $m+1$ elements, and so W too is open. Therefore $\mathrm{Surj}(\mathbb{N}) \backslash U_{m,n}$ is open and so $U_{m,n}$ is closed.

We have shown that $\mathcal{T}_{m,n}$ is Polish for every $m, n \in \mathbb{N}$. For every $m \in \mathbb{N}$, we define $\mathcal{T}_m$ to be the least topology containing $\mathcal{T}_{m,n}$ for all $n \in \mathbb{N}$. By [42, Lemma 13.3], $\mathcal{T}_m$ is Polish for all $m \in \mathbb{N}$. By considering only case (1) in the proof that $\mathcal{S}_2$ is a semigroup topology, it follows that $\mathcal{T}_m$ is also a semigroup topology for all $m \in \mathbb{N}$.

If $\mathcal{T}$ is the least topology containing $\mathcal{T}_m$ for all $m \in \mathbb{N}$, then, by [42, Lemma 13.3] again, $\mathcal{T}$ is Polish. If $n \in \mathbb{N}$ is arbitrary, then

$$\mathrm{Surj}(\mathbb{N}) \setminus U_{\aleph_0, n} = \{ f \in \mathrm{Surj}(\mathbb{N}) : |(n)f^{-1}| < \aleph_0 \}$$

which is the union of all of the open sets $U_{i,n}$, $i \in \mathbb{N}$, as in the inductive step above, is open in $\mathcal{T}$. Hence the topology generated by $\mathcal{T}$ and any of the sets $U_{\aleph_0, n}$, $n \in \mathbb{N}$, is a Polish topology. The topology generated by the (countable) union of all such topologies is $\mathcal{S}_2$, and so, by applying [42, Lemma 13.3] one last time, $\mathcal{S}_2$ is Polish.

(iii). By the proof of part (ii), it suffices to show that $\mathcal{T}_i$ and $\mathcal{T}_j$ are distinct for all $i \neq j$. Furthermore, since $\mathcal{T}_i \subseteq \mathcal{T}_j$ whenever $i \leq j$, it suffices to show that $\mathcal{T}_i \neq \mathcal{T}_{i+1}$ for every $i \in \mathbb{N}$. We show that $U_{i+1,0}$ is not open in $\mathcal{T}_i$. Suppose that $f \in U_{i+1,0}$ is arbitrary. Every basic open neighbourhoods of f in $\mathcal{T}_i$ is of the form

$$\bigcap_{(m,n) \in Z} U_{m,n} \cap \{ g \in \mathrm{Surj}(\mathbb{N}) : h \subseteq g \} = \{ g \in \mathrm{Surj}(\mathbb{N}) : h \subseteq g \text{ and } m = |(n)g^{-1}| \text{ for all } (m,n) \in Z \},$$

where $h \in P_{\mathbb{N}}$ and $Z \subseteq \{ 1, 2 \ldots, i \} \times \mathbb{N}$ are finite. In particular, every basic open neighbourhood of f contains $g \in \mathrm{Surj}(\mathbb{N})$ such that $|(0)g^{-1}| > i + 1$ and hence no basic open neighbourhood of f is contained in $U_{i+1,0}$.

(iv). We will show that the topology $\mathcal{S}_1$ on $\mathrm{Surj}(\mathbb{N})$ satisfies Theorem 5.5(i), and so it is possible to topologically embed $\mathrm{Surj}(\mathbb{N})$ with the topology $\mathcal{S}_1$ into $\mathbb{N}^{\mathbb{N}}$ with the pointwise topology. Suppose that σ is the equivalence relation of $\mathrm{Surj}(\mathbb{N})$ with classes $\mathrm{Sym}(\mathbb{N})$ and $\mathrm{Surj}(\mathbb{N}) \backslash \mathrm{Sym}(\mathbb{N})$. Then σ is a right congruence on $\mathrm{Surj}(\mathbb{N})$ (even a two-sided congruence, but this is not important here). If $\{ \rho_i : i \in \mathbb{N} \}$ is a sequence of right congruences on $\mathrm{Surj}(\mathbb{N})$ such that $\{ m/\rho_i : m \in \mathrm{Surj}(\mathbb{N}) \}$ is a subbasis for the pointwise topology, then $\{ m/\rho_i : m \in \mathrm{Surj}(\mathbb{N}) \} \cup \{ \sigma \}$ satisfies Theorem 5.5(i). $\square$

5.6. **Diagram Monoids.** In this section we consider topologies on the infinite partition monoid; see [20, 21, 24] for previous literature on partition monoids. The so-called *diagram monoids*, including the partition monoid, arose as the multiplicative monoids of certain algebras in representation theory [28]. In some sense, diagram monoids are also a generalisation of transformation monoids, in a somewhat different direction than monoids of binary relations. The results of this section are rather negative, in the sense that, we will show that no second countable T_1 topology is semitopological for the partition monoid.

For a set X we define the partition monoid $\mathfrak{P}_X$ as follows. The underlying set of $\mathfrak{P}_X$ is the set of partitions of $X \times \{0, 1\}$. We will use partitions and the corresponding equivalence relations interchangeably. We define $\mathrm{Diag}(s, t)$ to be the least equivalence relation on $X \times \{0, 1, 2\}$ containing

$$\{((x, a), (y, b)) \in (X \times \{0, 1, 2\})^2 : ((x, a), (y, b)) \in s \text{ or } ((x, a - 1), (y, b - 1)) \in t\}.$$

The product of s and t is defined to be

$$\{((x, a), (y, b)) \in (X \times \{0, 1\})^2 : ((x, 2a), (y, 2b)) \in \mathrm{Diag}(s, t)\}$$

and is denoted st. It is possible to show that this multiplication is associative, and that the identity is

$$\{\{(x, 0), (x, 1)\} : x \in X\}.$$

To prove the main theorem in this section we require the following technical lemma.

Lemma 5.24. *Let X be a set, and let $SL_X := X \cup \{0\}$ where $0 \notin X$ be the semigroup with multiplication defined by*

$$st = \begin{cases} s & \text{if } s = t \\ 0 & \text{if } s \neq t \end{cases}$$

(SL_X is a meet semilattice where every pair of non-zero elements is incomparable). If SL_X is a T_1 semitopological semigroup, then the minimum size of a basis for SL_X is at least $|X|$.

Proof. Suppose that $\mathcal{T}$ is a T_1 topology that is semitopological for SL_X. Let $x \in X$. Then $\{0\}\rho_x^{-1} = SL_X \backslash \{x\}$. As $\mathcal{T}$ is T_1, it follows that $\{x\}$ is open. Thus the subspace topology on X is the discrete topology and so the minimum size of a basis is $|X|$. $\qquad\square$

Theorem 5.25. *Let X be an infinite set. Then the following hold:*

(i) *if X is countable, then $\mathfrak{P}_X$ admits a second countable T_0 semigroup topology;*
(ii) *no second countable T_1 topology is semitopological for $\mathfrak{P}_X$.*

Proof. **(i).** We define $\mathcal{T}$ to be the topology on $\mathfrak{P}_X$ with subbasis consisting of sets of the form:

$$\{f \in \mathfrak{P}_X : (a, b) \in f\}$$

for all $a, b \in X \times \{0, 1\}$.

This topology is clearly T_0. It is also second countable as it has been defined by a countable subbasis. Let $f, g \in \mathfrak{P}_X$ and let $U := \{h \in \mathfrak{P}_X : (a, b) \in h\}$ be an arbitrary subbasic open neighbourhood of fg. It suffices to find neighbourhoods U_f and U_g of f and g, respectively, such that $U_f U_g \subseteq U$. From the definition of the multiplication in $\mathfrak{P}_X$ there is a finite path in $\mathrm{Diag}(f, g)$ f and g connecting a to b. We can now choose U_f to be those elements of $\mathfrak{P}_X$ which share all pairs belonging to f that are involved in this path. We may define U_g analogously.

(ii). For any partition P of X, we define the partition P' of $X \times \{0, 1\}$ to consist of the blocks $B \times \{0, 1\}$ for every block B of P. If S is the subsemigroup of $\mathfrak{P}_X$ consisting of those partitions P' with at most two parts, then it is easy to verify that S is a semilattice isomorphic to $SL_{2^{|X|}}$. Thus, by Lemma 5.24, $\mathfrak{P}_X$ is not second countable. $\qquad\square$

It might be worth noting that Theorem 5.25(ii) holds for the dual symmetric inverse monoid and the factorisable dual symmetric inverse monoid, the definitions of which are even longer than their names, and hence are omitted.

6. Monoids of continuous functions

If X is a topological space, then we denote by $C(X)$ the monoid of continuous functions from X to X. The *Cantor set* is $2^{\mathbb{N}}$ and the *Hilbert cube* is $[0,1]^{\mathbb{N}}$ where $[0,1]$ is the closed unit interval in $\mathbb{R}$. In this section, we consider semigroup topologies on the monoids $C([0,1]^{\mathbb{N}})$ and $C(2^{\mathbb{N}})$.

One standard topology on $C(X)$ is the compact-open topology, which we now define. If X and Y are topological spaces, then the *compact-open topology* on $C(X, Y)$ (the space of all continuous functions from X to Y) is the topology generated by the subbasis consisting of the sets:

$$[K, U] = \{f \in C(X, Y) : (K)f \subseteq U\},$$

where $K \subseteq X$ is compact, and $U \subseteq Y$ is open.

If X is compact and metrizable with compatible metric d, then $C(X, X) = C(X)$ is separable with respect to the compact-open topology, and the topology is induced by

$$(10) \qquad d_{\infty}(f, g) = \sup\{d((x)f, (x)g) : x \in X\};$$

see [81, Proposition 1.3.3]. The space X is complete since it is compact, and d_{∞} is therefore complete also, meaning that $C(X)$ with the compact-open topology is Polish.

Theorem 6.1. *Let X and Y be compact metrizable spaces. If $C(X, Y)$ is first countable and Hausdorff with respect to some topology $\mathcal{T}$, and $a : X \times C(X, Y) \longrightarrow Y$ defined by*

$$(p, f)a = (p)f$$

is continuous, then $\mathcal{T}$ contains the compact-open topology.

Proof. It suffices to show that if K is compact in X and U is open in Y, then the complement $F = \{f \in C(X, Y) : \exists k \in K, \ f(k) \notin U\}$ of $[K, U]$ is closed in $\mathcal{T}$.

Suppose that $f_0, f_1, \ldots \in F$ converges to $f \in C(X, Y)$ with respect to $\mathcal{T}$. For every $i \in \mathbb{N}$, there exists $x_i \in K$ such that $(x_i)f_i \notin U$. Since K is compact, it follows that $(x_i)_{i \in \mathbb{N}}$ contains a convergent subsequence $(x_{(i)y})_{i \in \mathbb{N}}$. If x is the limit of this subsequence, then

$$(x)f = (x, f)a = \left(\lim_{i \longrightarrow \infty} x_{(i)y}, \lim_{i \longrightarrow \infty} f_{(i)y} \right) a = \lim_{i \longrightarrow \infty} (x_{(i)y}, f_{(i)y})a = \lim_{i \longrightarrow \infty} (x_{(i)y})f_{(i)y} \in Y \setminus U$$

and so $f \in F$, and F is closed in $\mathcal{T}$, as required. $\qquad \square$

Lemma 6.2. *Let X be topological space and suppose that $C(X)$ is a first countable Hausdorff topological semigroup with respect to a topology $\mathcal{T}$. Then the set F of constant functions in $C(X)$ is closed in $\mathcal{T}$.*

Proof. It is routine to verify that F is the set of right zeros in $C(X)$. If $f \in C(X)$ and $(k_i)_{i \in \mathbb{N}}$ is a sequence in F that converges to $k \in C(X)$, then $fk = \lim_{i \longrightarrow \infty} fk_i = \lim_{i \longrightarrow \infty} k_i = k$ and so $k \in F$. $\qquad \square$

6.1. **The Hilbert cube.** In this section, we show that the compact-open topology is the unique Polish semigroup topology on the monoid $C(Q)$ of continuous functions on the Hilbert cube $Q = [0,1]^{\mathbb{N}}$. Since the compact-open topology is Polish on the group of homeomorphisms of any compact metric space (see, for example, [81, Proposition 1.3.3]), it follows that the group of homeomorphisms $H(Q)$ of the Hilbert cube is a Polish subgroup of $C(Q)$ with the compact-open topology. In fact, it was shown by Kallman [40] that the compact-open topology is the unique Polish topology on $H(Q)$.

We will use the following metric on Q:

$$(11) \qquad d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{i+1}},$$

which is bounded above by 1, and so the corresponding d_{∞} is also bounded above by 1.

We require Theorem 5.2.4 from [81], which we state for the sake of completeness.

Proposition 6.3 (Theorem 5.2.4 in [81]). *Let K_1 and K_2 be compact subsets of $(0,1)^{\mathbb{N}}$ and let $k : K_1 \longrightarrow K_2$ be a homeomorphism with $d_{\infty}(k, \mathrm{id}_{K_1}) < \varepsilon$. Then there exists a homeomorphism $h_k \in H(Q)$ such that $d_{\infty}(h_k, \mathrm{id}_Q) < \varepsilon$ and $h_k|_{K_1} = k$.*

The next proposition provides a key step in the proof of the result of this section.

Proposition 6.4. *Suppose that $C(Q)$ is a Polish semigroup with respect to some topology. Then the function $a : Q \times C(Q) \longrightarrow Q$ defined by*

$$(p, f)a = (p)f$$

is continuous.

Proof. Let $\mathcal{T}$ denote the given Polish semigroup topology on $C(Q)$. Suppose that F denotes the set of constant functions in $C(Q)$ and that $\phi : F \longrightarrow Q$ is the function that sends $f \in F$ to the unique value in $(Q)f$.

We will show that ϕ is a homeomorphism, which will allow us to conclude the proof as follows. If $\gamma : Q \times C(Q) \longrightarrow C(Q) \times C(Q)$ is defined by $(p, f)\gamma = (p\phi^{-1}, f)$, then γ is continuous since it is continuous in each coordinate, and if $M : C(Q) \times C(Q) \longrightarrow C(Q)$ is the multiplication function on $C(Q)$, then M is continuous also. Thus $(p, f)\gamma M\phi = (p\phi^{-1}, f)M\phi = (p\phi^{-1} \circ f)\phi$ and $(p)\phi^{-1} \in F$ is constant with value p, and so $p\phi^{-1} \circ f$ is constant with value $(p)f$, and, finally, $(p\phi^{-1} \circ f)\phi = (p)f = (p, f)a$. In other words, $a : Q \times C(Q) \longrightarrow Q$ as a composite of the continuous functions γ, M, and ϕ, is itself continuous.

We start by showing that ϕ is continuous. If U is an open set in Q, it suffices to show that $(U)\phi^{-1} = \{f \in F : (Q)f = \{q\},\ q \in U\}$ is open in F. If $f : Q \longrightarrow Q$ is defined by

$$(q)f = (d(q, Q \setminus U), d(q, Q \setminus U), \ldots),$$

then since f is continuous in every coordinate, f is continuous. Since $C(Q)$ is T_1, F is T_1, and so the singleton set containing the constant function $g : Q \longrightarrow Q$ with value $(0, 0, \ldots)$ is closed. Hence $\{g\}\rho_f^{-1} \cap F$ is closed in F (recall that $\rho_f : C(Q) \longrightarrow C(Q)$ is defined by $(h)\rho_f = h \circ f$ for all $h \in C(Q)$), and so $F \setminus (\{g\}\rho_f^{-1})$ is open. If $h \in F \setminus (\{g\}\rho_f^{-1})$, then there exists $q \in Q$ such that

$$(q)hf \neq (0, 0, \ldots)$$

and so $(q)h \in U$ by the definition of f. On the other hand, if $h \in F$ is such that $(q)h \in U$, then $(q)hf \neq (0, 0, \ldots)$, and so $F \setminus (\{g\}\rho_f^{-1}) = (U)\phi^{-1} = \{f \in F : (Q)f = \{q\},\ q \in U\}$ is open, as required.

Next, we show that $\phi^{-1}|_{(0,1)^{\mathbb{N}}} : (0, 1)^{\mathbb{N}} \longrightarrow F$ is continuous also. Since F is closed in $C(Q)$ with respect to $\mathcal{T}$, by Lemma 6.2, it follows that F is Polish. Since ϕ is continuous, ϕ is Borel measurable, and so by Proposition 3.3, ϕ^{-1} is Borel measurable also. It suffices, by Proposition 3.2, to endow $(0, 1)^{\mathbb{N}}$ and $((0, 1)^{\mathbb{N}})\phi^{-1} \subseteq F$ with Polish semitopological group structures such that $\phi^{-1}|_{(0,1)^{\mathbb{N}}} : (0, 1)^{\mathbb{N}} \longrightarrow ((0, 1)^{\mathbb{N}})\phi^{-1}$ is a homomorphism. With the operation of component-wise addition, $\mathbb{R}^{\mathbb{N}}$ is an abelian topological group, and $\mathbb{R}^{\mathbb{N}}$ is homeomorphic to $(0, 1)^{\mathbb{N}}$. Therefore we may endow $(0, 1)^{\mathbb{N}}$ with the additive group structure of $\mathbb{R}^{\mathbb{N}}$ corresponding to an order-isomorphism between $\mathbb{R}$ and $(0, 1)$. We define $*$ on $((0, 1)^{\mathbb{N}})\phi^{-1}$ by $x * y = ((x)\phi + (y)\phi)\phi^{-1}$. It remains to show that $((0, 1)^{\mathbb{N}})\phi^{-1}$ with $*$ is right topological. That $((0, 1)^{\mathbb{N}})\phi^{-1}$ with $*$ is left topological will then follow immediately since $\mathbb{R}^{\mathbb{N}}$ is abelian. Suppose that $g = (g_0, g_1, \ldots) \in (0, 1)^{\mathbb{N}}$ is arbitrary. Then $\rho_g : (0, 1)^{\mathbb{N}} \longrightarrow (0, 1)^{\mathbb{N}}$ is continuous since $(0, 1)^{\mathbb{N}}$ is a topological group. We extend ρ_g to $\rho_g' : Q \longrightarrow Q$ so that the ith coordinate of $(x_0, x_1, \ldots)\rho_g'$ is $x_i + g_i$ if $x_i \in (0, 1)$ and x_i if $x_i = 0$ or $x_i = 1$. Since ρ_g' is an order-isomorphism onto $[0, 1]$ in every component, it follows that ρ_g' is continuous in every component, and is hence continuous, i.e. $\rho_g' \in C(Q)$. Thus $x \mapsto x \circ \rho_g'$, where $x \in C(Q)$ is continuous, and so, in particular, $x \mapsto x \circ \rho_g'$ restricted to $x \in ((0, 1)^{\mathbb{N}})\phi^{-1}$ is also continuous (on $((0, 1)^{\mathbb{N}})\phi^{-1}$ with the subspace topology). If $x \in ((0, 1)^{\mathbb{N}})\phi^{-1} \subseteq F$ is arbitrary, then x is a constant function with value $(x_0, x_1, \ldots) \in (0, 1)^{\mathbb{N}}$ and so

$$x * (g)\phi^{-1} = ((x)\phi + g)\phi^{-1} = (x)\phi\rho_g\phi^{-1}$$

by the definition of $*$ and ρ_g. Recall that $(x)\phi = (x_0, x_1, \ldots) \in (0, 1)^{\mathbb{N}}$ by the definition of ϕ. But

$$(x)\phi\rho_g\phi^{-1} = (x_0, x_1, \ldots)\rho_g\phi^{-1} = (x_0 + g_0, x_1 + g_1, \ldots)\phi^{-1}$$

and $(x_0 + g_0, x_1 + g_1, \ldots)\phi^{-1}$ is the constant function from Q to Q with value $(x_0 + g_0, x_1 + g_1, \ldots)$, which is equal to $(x) \circ \rho_g$. Therefore

$$x * (g)\phi^{-1} = ((x)\phi + g)\phi^{-1} = (x)\phi\rho_g\phi^{-1} = x \circ \rho_g = x \circ \rho_g'$$

and since $x \mapsto x \circ \rho_g'$ is continuous on $((0, 1)^{\mathbb{N}})\phi^{-1}$, it follows that $((0, 1)^{\mathbb{N}})\phi^{-1}$ with $*$ is a right topological group.

To conclude the proof, we must show that $\phi^{-1} : Q \longrightarrow F$ is continuous. We define $\theta : [0, 1] \longrightarrow [1/4, 3/4]$ by $(x)\theta = x/2 + 1/4$ and let $\bar{\theta} \in C(Q)$ be the function that applies θ in every coordinate of Q. If $\theta' : [0, 1] \longrightarrow [0, 1]$ is any continuous extension of θ^{-1}, and $\bar{\theta'} \in C(Q)$ applies θ' in every coordinate, then $\theta\theta' = \mathrm{id}_Q$ and $\phi^{-1} = \bar{\theta}\phi^{-1}|_{(0,1)^{\mathbb{N}}}\rho_{\bar{\theta'}}$. Hence ϕ^{-1} being the composite of the continuous functions $\bar{\theta}$, $\phi^{-1}|_{(0,1)^{\mathbb{N}}}$, and $\rho_{\bar{\theta'}}$ is itself continuous. $\square$

Theorem 6.5. *The monoid $C(Q)$ of continuous functions on the Hilbert cube $Q = [0, 1]^{\mathbb{N}}$ has the following properties:*

  (i) *$C(Q)$ has property $\boldsymbol{X}$ with respect to its group of units $H(Q)$;*
  (ii) *the compact-open topology is the unique Polish semigroup topology on $C(X)$;*

(iii) *if the group of homeomorphisms $H(Q)$ has automatic continuity with respect to a class of topological semigroups, then $C(Q)$ has automatic continuity with respect to the same class.*

Proof. **(i)**. If S is $C(Q)$ with the compact-open topology, $\mathscr{B}$ is the basis for the compact-open topology on $C(Q)$ consisting of open balls with respect to the d_∞ metric on $C(Q)$, and T is $H(Q)$, then, we show that there exist $f, g \in C(Q)$ and $t_s \in H(Q)$ such that $s = f t_s g$ and for every ball $B = B_{d_\infty}(t_s, \varepsilon) \in \mathscr{B}$ there exists $\delta > 0$ such that $B_{d_\infty}(s, \delta) \subseteq f(B \cap H(Q))g$.

Suppose that $s \in C(Q)$ is arbitrary. We define the required $f, g \in C(Q)$ via two continuous functions $f', g' : [0,1] \longrightarrow [0,1]$ that are defined by:

$$(x)f' = \frac{2x+1}{4}, \qquad (x)g' = \begin{cases} 0 & \text{if } x \leq \frac{1}{4} \\ \frac{4x-1}{2} & \text{if } \frac{1}{4} \leq x \leq \frac{3}{4} \\ 1 & \text{if } x \geq \frac{3}{4}. \end{cases}$$

Then both f' and g' are continuous, and, moreover, $f'g'$ is the identity function $\text{id}_{[0,1]}$. We define $f, g, h \in C(Q)$ by applying f' and g' in certain coordinates, as follows:

$$\begin{aligned} (x_0, x_1, \ldots)f &= ((x_0)f', (x_1)f', \ldots) \\ (x_0, x_1, \ldots)g &= ((x_1)g', (x_3)g', \ldots) \\ (x_0, x_1, \ldots)h &= ((x_0)g', (x_1)g', \ldots). \end{aligned}$$

Since $f'g' = \text{id}_{[0,1]}$, it follows that $fh = \text{id}_Q$. Let $A = [1/4, 3/4]^{\mathbb{N}}$. Of course, A is homeomorphic to Q, and A is a compact subset of $(0,1)^{\mathbb{N}}$. To find $t_s \in H(Q)$ such that $ft_s g = s$, we show that there exists a homeomorphism t'_s from A into a subspace of A, that can be extended to a homeomorphism of Q by Proposition 6.3. We denote by $\pi_i : Q \longrightarrow [0,1]$ the ith projection of Q, that is,

$$(x_0, x_1, \ldots)\pi_i = x_i$$

for any $(x_0, x_1, \ldots) \in Q$. If $\vec{x} = (x_0, x_1, \ldots) \in A$ and $k \in C(Q)$ is arbitrary, then we define $t'_k : A \longrightarrow A$ by

$$(12) \qquad (\vec{x})t'_k = (x_0, x_1, \ldots)t'_k = (x_0, (\vec{x})hk\pi_0 f', x_1, (\vec{x})hk\pi_1 f', \ldots).$$

From the definition of t'_k, if $\vec{x} \in Q$ is arbitrary, then

$$\begin{aligned} (\vec{x})ft'_k g &= (x_0, x_1, \ldots)ft'_s g \\ &= ((x_0)f', (x_1)f', \ldots)t'_s g \\ &= ((x_0)f', (\vec{x})fhk\pi_0 f', (x_1)f', (\vec{x})fhk\pi_1 f', \ldots)g \\ &= ((x_0)f', (\vec{x})k\pi_0 f', (x_1)f', (\vec{x})k\pi_1 f', \ldots)g \\ &= ((\vec{x})k\pi_0 f'g', (\vec{x})k\pi_1 f'g', \ldots) \\ &= ((\vec{x})k\pi_0, (\vec{x})k\pi_1, \ldots) \\ &= (\vec{x})k \end{aligned}$$

and so $k = ft'_k g$.

Since t'_k is continuous in every component, t'_k is continuous, and it is clearly a bijection onto its image. Since A is compact, and every bijective continuous function between compact Hausdorff spaces is a homeomorphism, it follows that t'_k is a homeomorphism for every $k \in C(Q)$. By our choice of metric for Q in (11), $d_\infty(t'_s, \text{id}_A) < 2$. Hence by Proposition 6.3, we may extend t'_s to $t_s \in H(Q)$.

If $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, is arbitrary, then we will show that $B_{d_\infty}(s, \varepsilon) \subseteq f(B_{d_\infty}(t_s, \varepsilon) \cap H(Q))g$.

If $a \in B_{d_\infty}(s, \varepsilon)$ and $t'_a : A \longrightarrow A$ is defined as in (12), then $ft'_a g = a$. Since $f' : [0,1] \longrightarrow [0,1]$ is a contraction and $d_\infty(a, s) < \varepsilon$, it follows that

$$
\begin{aligned}
d_\infty(t'_a, t'_s) &= \sup\{d(\vec{x}t'_s, \vec{x}t'_a) : \vec{x} \in A\} \\
&= \sup\left\{ \sum_{i \in \mathbb{N}} \frac{|\vec{x}hs\pi_i f' - \vec{x}ha\pi_i f'|}{2^{2i+2}} : \vec{x} \in A \right\} \\
&\leq \sup\left\{ \sum_{i \in \mathbb{N}} \frac{|\vec{x}hs\pi_i - \vec{x}ha\pi_i|}{2^{2i+2}} : \vec{x} \in A \right\} \\
&= \sup\left\{ \sum_{i \in \mathbb{N}} \frac{|\vec{x}s\pi_i - \vec{x}a\pi_i|}{2^{2i+2}} : \vec{x} \in (A)h = Q \right\} \\
&\leq \sup\left\{ \sum_{i \in \mathbb{N}} \frac{|\vec{x}s\pi_i - \vec{x}a\pi_i|}{2^{i+1}} : \vec{x} \in Q \right\} \\
&= d_\infty(a, s) < \varepsilon.
\end{aligned}
$$

Then, since $(t'_s)^{-1} : (A)t'_s \longrightarrow A$ is a surjective function,

$$d_\infty((t'_s)^{-1}t'_a, \mathrm{id}_{(A)t'_a}) = d_\infty((t'_s)^{-1}t'_a, (t'_s)^{-1}t'_s) = d_\infty(t'_a, t'_s) < \varepsilon,$$

and so, by Proposition 6.3, there exists $\phi \in H(Q)$ extending $(t'_s)^{-1}t'_a$ such that $d_\infty(\phi, \mathrm{id}_Q) < \varepsilon$. It follows that $t_s\phi \in B_{d_\infty}(t_s, \varepsilon)$ and $ft_s\phi g = ft'_s\phi g = ft'_a g = a$ and so $a \in f(B_{d_\infty}(t_s, \varepsilon) \cap H(Q))g$.

(ii). It follows immediately from Theorem 6.1 and Proposition 6.4, that every Polish semigroup topology on $C(Q)$ contains the compact-open topology. It therefore suffices to show that no Polish semigroup topology for $C(Q)$ contains the compact-open topology properly. This follows from part (i) together with Theorem 3.1(iii).

(iii). This follows immediately from part (i) and Theorem 3.1(iv). $\qquad\qquad\square$

6.2. **The Cantor space.** In this section, we show that the monoid of continuous functions $C(2^{\mathbb{N}})$ on the Cantor space $2^{\mathbb{N}}$ has a unique Polish semigroup topology – the compact-open topology. The proof is analogous to, but not the same as, the proof of the uniqueness of the Polish semigroup topology on the Hilbert cube given in the last section.

To prove that the compact-open topology is also a maximal Polish semigroup topology on $2^{\mathbb{N}}$ we require an analogue of Proposition 6.3. We were not able to locate such an analogue in the literature, and so we provide our own, the proof of which is similar to the proof of Proposition 6.3 given in [81]. In the following proposition, we require the d_∞ metric on $C(2^{\mathbb{N}})$ for which we ought to fix a metric on $2^{\mathbb{N}}$. Since $2^{\mathbb{N}}$ is a subset of the Hilbert cube, we can define the metric d on $2^{\mathbb{N}}$ to be the restriction of the metric on the Hilbert cube defined in (11). Then

$$d((x_0, x_1, \ldots), (y_0, y_1, \ldots)) = \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{2^{i+1}},$$

where $(x_0, x_1, \ldots), (y_0, y_1, \ldots) \in 2^{\mathbb{N}}$. We also require a metric on $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ which we define by

$$\rho((a_0, a_1), (b_0, b_1)) = \max\{d(a_0, b_0), d(a_1, b_1)\}.$$

Let the associated metric ρ_∞ be defined as in (10). If $n > 0$, we denote the finite sequence of length n consisting solely of 0 by 0^n, and we denote the infinite sequence consisting solely of the value 0 by 0^∞.

Proposition 6.6. *Suppose that A and B are closed subsets of $2^{\mathbb{N}}$ such that there exists a homeomorphism $\phi : A \longrightarrow B$ such that $d_\infty(\phi, \mathrm{id}_A) < \varepsilon$. Then there exists a homeomorphism $\phi' : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that $(a, 0^\infty)\phi' = (a\phi, 0^\infty)$ for all $a \in A$, and $\rho_\infty(\phi', \mathrm{id}_{2^{\mathbb{N}} \times 2^{\mathbb{N}}}) < \varepsilon$.*

Proof. It is well-known that if F is a non-empty closed subset of the Cantor space, then there exists a continuous function $g_F : 2^{\mathbb{N}} \longrightarrow F$ such that $(x)g_F = x$ for all $x \in F$ (see, for example, the proof of [2, Lemma 3.59]).

Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, be fixed, and let $n \in \mathbb{N}$ be such that $\sum_{i=n}^{\infty} \frac{1}{2^{i+1}} < \varepsilon$. We will view $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ as $2^{\mathbb{N}} \times (2^n \times 2^{\mathbb{N}})$ in the obvious way, but continue using the metric ρ. Moreover, let $+$ and $-$ be componentwise addition

and subtraction modulo 2 on $2^{\mathbb{N}}$. Define $\phi_0, \phi_1, \phi_2 : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by

$$
\begin{aligned}
(a, (b, c))\phi_0 &= (a, (b, c + a)) \\
(a, (b, c))\phi_1 &= (a - (c)g_A + (c)g_A\phi, (b, c)) \\
(a, (b, c))\phi_2 &= (a, (b, c - (a)g_B\phi^{-1})).
\end{aligned}
$$

Then ϕ_0, ϕ_1, and ϕ_2 are continuous, since they are continuous in each component, and ϕ_0^{-1}, ϕ_1^{-1}, and ϕ_2^{-1} are well-defined and continuous. Hence ϕ_0, ϕ_1, and ϕ_2 are homeomorphisms. The required homeomorphism $\phi' : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is $\phi' = \phi_0\phi_1\phi_2$ since for $a \in A$

$$
\begin{aligned}
(a, 0^{\infty})\phi_0\phi_1\phi_2 &= (a, (0^n, a))\phi_1\phi_2 \\
&= (a - (a)g_A + (a)g_A\phi, (0^n, a))\phi_2 \\
&= ((a)\phi, (0^n, a))\phi_2 \\
&= ((a)\phi, (0^n, a - ((a)\phi g_B\phi^{-1}))) \\
&= ((a)\phi, 0^{\infty}).
\end{aligned}
$$

It remains to show that $\rho_{\infty}(\phi', \mathrm{id}_{2^{\mathbb{N}} \times 2^{\mathbb{N}}}) = \sup\{\rho((x, y)\phi', (x, y)) : x, y \in 2^{\mathbb{N}}\} < \varepsilon$. By definition,

$$
\rho((x, y)\phi', (x, y)) = \max\{d((x, y)\phi'\pi_0, x), d(((x, y)\phi'\pi_1, y)\},
$$

where $\pi_0, \pi_1 : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ are the projection functions. By the assumption on n, and the fact that in the definitions of ϕ_0, ϕ_1, and ϕ_2 the first n terms in the second component are not altered, it follows that

$$
d((x, y)\phi'\pi_1, y) < \sum_{i=n}^{\infty} \frac{1}{2^{i+1}} < \varepsilon.
$$

Since ϕ_0 and ϕ_2 do not change the first coordinate of their arguments, it suffices to check that $d((a, (b, c))\phi_1\pi_0, a) < \varepsilon$ for all $(a, (b, c)) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$. If $(a, (b, c)) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is arbitrary, then the map $x \mapsto (c)g_A - a + x$ is an isometry on $2^{\mathbb{N}}$, with respect to the metric d, and so

$$
\begin{aligned}
d(a, (a, (b, c))\phi_1\pi_0) &= d(a, a - (c)g_A + (c)g_A\phi) \\
&= d(((c)g_A - a) + a, ((c)g_A - a) + a - (c)g_A + (c)g_A\phi) \\
&= d((c)g_A, (c)g_A\phi) \\
&< \varepsilon,
\end{aligned}
$$

since $d_{\infty}(\phi, \mathrm{id}_A) < \varepsilon$. $\qquad\square$

Theorem 6.7. *The monoid $C(2^{\mathbb{N}})$ of continuous functions on the Cantor set together with the compact-open topology has property $\boldsymbol{X}$ with respect to its group of units $H(2^{\mathbb{N}})$.*

Proof. Similarly to the proof of Theorem 6.5(i), we will show that for every $s \in C(2^{\mathbb{N}})$, there exist $f, g \in C(2^{\mathbb{N}})$ and $t_s \in H(2^{\mathbb{N}})$ such that $s = ft_sg$ and for every ball $B = B_{d_{\infty}}(t_s, \varepsilon)$ there exists $\delta > 0$ such that $B_{d_{\infty}}(s, \delta) \subseteq f(B \cap H(2^{\mathbb{N}}))g$.

Let $\Phi : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ be a homeomorphism. If $\varepsilon > 0$, then since Φ is homeomorphism between compact metric spaces, Φ and Φ^{-1} are uniformly continuous, and so there exists $r(\varepsilon) \in \mathbb{R}$ such that $r(\varepsilon) > 0$ and the following conditions hold:

(i) $d(a, b) < r(\varepsilon)$ implies that $\rho((a)\Phi, (b)\Phi) < \varepsilon$ for any $a, b \in 2^{\mathbb{N}}$;
(ii) $\rho(a, b) < r(\varepsilon)$ implies that $d((a)\Phi^{-1}, (b)\Phi^{-1}) < \varepsilon$ for all $a, b \in 2^{\mathbb{N}} \times 2^{\mathbb{N}}$.

Let $s \in C(2^{\mathbb{N}})$ be arbitrary, and let $f, g \in C(2^{\mathbb{N}})$ be defined by:

$$
\begin{aligned}
(13) \qquad\qquad (x)f &= \left((x, x)\Phi^{-1}, 0^{\infty}\right)\Phi^{-1} \\
(14) \qquad\qquad (x)g &= (x)\Phi\pi_0\Phi\pi_0,
\end{aligned}
$$

where, as defined in the proof of Proposition 6.6, $\pi_0 : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ is the projection function. Note that fg is the identity on $2^{\mathbb{N}}$. Next, we will show that there exists $t_s \in H(2^{\mathbb{N}})$ such that $s = ft_sg$.

Let $s', \Delta : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ be defined by

$$
(15) \qquad\qquad\qquad (x)s' = ((x)s, x)
$$

and $(x)\Delta = (x, x)$. Clearly Δ is a homeomorphism from $2^{\mathbb{N}}$ to $(2^{\mathbb{N}})\Delta$, and, s' is injective and continuous, since it is continuous in each component, and the inverse of s' is also continuous because $((x)s, x) \mapsto x$ is a projection. Hence s' is a homeomorphism between $2^{\mathbb{N}}$ and $(2^{\mathbb{N}})s'$. If $\phi = \Phi\Delta^{-1}s'\Phi^{-1}$ and $x \in 2^{\mathbb{N}}$ is arbitrary, then

$$
(16) \qquad (x, x)\Phi^{-1}\phi = (x, x)\Phi^{-1}\Phi\Delta^{-1}s'\Phi^{-1} = (x)s'\Phi^{-1} = ((x)s, x)\Phi^{-1}
$$

and so ϕ is a homeomorphism from $A = (2^{\mathbb{N}})\Delta\Phi^{-1} = \{(x,x)\Phi^{-1} : x \in 2^{\mathbb{N}}\}$ to $B = (A)\phi$. By Proposition 6.6, there exists a homeomorphism $\zeta : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ such that

$$(17) \qquad\qquad ((x,x)\Phi^{-1}, 0^{\infty})\zeta = ((x,x)\Phi^{-1}\phi, 0^{\infty})$$

for all $x \in 2^{\mathbb{N}}$.

We will show that the required homeomorphism of $2^{\mathbb{N}}$ is

$$(18) \qquad\qquad t_s = \Phi\zeta\Phi^{-1}.$$

If $x \in 2^{\mathbb{N}}$ is arbitrary, then

$$
\begin{aligned}
(x)ft_sg &= ((x,x)\Phi^{-1}, 0^{\infty})\Phi^{-1}t_sg && \text{by (13)}\\
&= ((x,x)\Phi^{-1}, 0^{\infty})\zeta\Phi^{-1}g && \text{by (18)}\\
&= ((x,x)\Phi^{-1}\phi, 0^{\infty})\Phi^{-1}g && \text{by (17)}\\
&= ((x,x)\Phi^{-1}\phi, 0^{\infty})\Phi^{-1}\Phi\pi_0\Phi\pi_0 && \text{by (14)}\\
&= (x,x)\Phi^{-1}\phi\Phi\pi_0\\
&= ((x)s, x)\Phi^{-1}\Phi\pi_0 && \text{by (16)}\\
&= (x)s.
\end{aligned}
$$

Suppose that $\varepsilon > 0$ is arbitrary. It remains to show that there exists $\delta > 0$ such that $B_{d_\infty}(s,\delta) \subseteq f(B_{d_\infty}(t_s,\varepsilon) \cap H(2^{\mathbb{N}}))g$. We set $\delta = r(r(\varepsilon))$. Let $u \in B_{d_\infty}(s,\delta)$. We define $u' : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times 2^{\mathbb{N}}$ by $(x)u' = ((x)u, x)$ for all $x \in 2^{\mathbb{N}}$, and, similar to the proof above for s', u' is a homeomorphism from $2^{\mathbb{N}}$ to $(2^{\mathbb{N}})u'$. Then $\rho_\infty(u', s') = d_\infty(u,s) < \delta$. Hence

$$\rho_\infty(\mathrm{id}_{(2^{\mathbb{N}})s'}, (s')^{-1}u') = \rho_\infty((s')^{-1}s', (s')^{-1}u') = \rho_\infty(u', s') = d_\infty(u,s) < \delta$$

since left multiplication by $(s')^{-1}$ is an isometry. It follows that

$$d_\infty(\Phi\,\mathrm{id}_{(2^{\mathbb{N}})s'}\,\Phi^{-1}, \Phi(s')^{-1}u'\Phi^{-1}) < r(\varepsilon)$$

by (i) and (ii), from the second paragraph of this proof.

By Proposition 6.6 applied to the homeomorphism $\Phi(s')^{-1}u'\Phi^{-1} : (2^{\mathbb{N}})s'\Phi^{-1} \longrightarrow (2^{\mathbb{N}})u'\Phi^{-1}$, there exists $\gamma \in H(2^{\mathbb{N}} \times 2^{\mathbb{N}})$ such that

$$(19) \qquad\qquad (a, 0^{\infty})\gamma = (a\Phi(s')^{-1}u'\Phi^{-1}, 0^{\infty})$$

and $\rho_\infty(\gamma, \mathrm{id}_{2^{\mathbb{N}} \times 2^{\mathbb{N}}}) < r(\varepsilon)$. Since left multiplication by $\Phi^{-1}t_s\Phi$ is an isometry of $C(2^{\mathbb{N}} \times 2^{\mathbb{N}})$,

$$\rho_\infty(\Phi^{-1}t_s\Phi\gamma, \Phi^{-1}t_s\Phi) = \rho_\infty(\gamma, \mathrm{id}_{2^{\mathbb{N}} \times 2^{\mathbb{N}}}) < r(\varepsilon)$$

and so

$$d_\infty(t_s\Phi\gamma\Phi^{-1}, t_s) = d_\infty(\Phi\Phi^{-1}t_s\Phi\gamma\Phi^{-1}, \Phi\Phi^{-1}t_s\Phi\Phi^{-1}) < \varepsilon.$$

Hence $f(t_s\Phi\gamma\Phi^{-1})g \in f(B_{d_\infty}(t_s,\varepsilon) \cap H(2^{\mathbb{N}}))g$. We conclude the proof by showing that $f(t_s\Phi\gamma\Phi^{-1})g = u$. Suppose that $x \in 2^{\mathbb{N}}$ is arbitrary. Then

$$
\begin{aligned}
(x)f(t_s\Phi\gamma\Phi^{-1})g &= ((x,x)\Phi^{-1}, 0^{\infty})\Phi^{-1}t_s\Phi\gamma\Phi^{-1}g && \text{by (13)}\\
&= ((x,x)\Phi^{-1}, 0^{\infty})\zeta\gamma\Phi^{-1}g && \text{by (18)}\\
&= ((x,x)\Phi^{-1}\phi, 0^{\infty})\gamma\Phi^{-1}g && \text{by (17)}\\
&= ((x)s, x)\Phi^{-1}, 0^{\infty})\gamma\Phi^{-1}g && \text{by (16)}\\
&= (((x)s, x)\Phi^{-1}\Phi(s')^{-1}u'\Phi^{-1}, 0^{\infty})\Phi^{-1}g && \text{by (19)}\\
&= (((x)s, x)(s')^{-1}u'\Phi^{-1}, 0^{\infty})\Phi^{-1}g\\
&= ((x)u'\Phi^{-1}, 0^{\infty})\Phi^{-1}g && \text{by (15)}\\
&= (((x)u, x)\Phi^{-1}, 0^{\infty})\Phi^{-1}g\\
&= (((x)u, x)\Phi^{-1}, 0^{\infty})\Phi^{-1}\Phi\pi_0\Phi\pi_0 && \text{by (14)}\\
&= ((x)u, x)\Phi^{-1}\Phi\pi_0\\
&= (x)u,
\end{aligned}
$$

which concludes the proof. □

Theorem 6.8. *The monoid of continuous functions on the Cantor set $2^{\mathbb{N}}$ together with the compact-open topology has automatic continuity with respect to the class of second countable topological semigroups.*

Proof. This follows from Theorem 6.7, Theorem 3.1, and the automatic continuity of the group of homeomorphisms of the Cantor set. □

Proposition 6.9. *Suppose that $\mathcal{T}$ is a second countable Hausdorff semigroup topology on $C(2^{\mathbb{N}})$. Let F be the subspace of constant functions in $C(2^{\mathbb{N}})$ and let $\phi : F \longrightarrow 2^{\mathbb{N}}$ (with the usual topology) be the function mapping every $f \in F$ to the unique value in $(2^{\mathbb{N}})f$. Then ϕ is a homeomorphism.*

Proof. Suppose that $\mathscr{B}$ is any basis for $2^{\mathbb{N}}$ consisting of clopen sets and such that $2^{\mathbb{N}} \notin \mathscr{B}$.

To show that ϕ is continuous, suppose that $U \in \mathscr{B}$. It suffices to show that $(U)\phi^{-1} = \{f \in F : (2^{\mathbb{N}})f = \{q\},\ q \in U\}$ is open in F. If $y \notin U$, then we define $f : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ such that

$$(x)f = \begin{cases} x & \text{if } x \in U \\ y & \text{if } x \notin U. \end{cases}$$

Since U and $2^{\mathbb{N}} \setminus U$ are open, and f is continuous when restricted to either of the sets, it follows that f is continuous. Since $C(2^{\mathbb{N}})$ is T_1, F is T_1, and so the singleton set containing the constant function $g : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ with value y is closed. Hence $\{g\}\rho_f^{-1} \cap F$ is closed in F, and so $F \setminus (\{g\}\rho_f^{-1})$ is open in F. If $h \in F \setminus (\{g\}\rho_f^{-1})$, then there exists $x \in 2^{\mathbb{N}}$ such that

$$(x)hf \neq y$$

and so $(x)h \in U$ by the definition of f. On the other hand, if $h \in F$ is such that $(x)h \in U$, then $(x)hf \neq y$, and so $F \setminus (\{g\}\rho_f^{-1}) = (U)\phi^{-1} = \{f \in F : (2^{\mathbb{N}})f = \{q\},\ q \in U\}$ is open, as required.

Next, we show that $\phi^{-1} : 2^{\mathbb{N}} \longrightarrow F$ is continuous also. Note that the topology induced on F by the compact-open topology is precisely the standard topology on $2^{\mathbb{N}}$ (when viewing an element of F as its image under ϕ). It therefore suffices to show that the topology induced on F by $\mathcal{T}$ is contained in the topology induced on F by the compact-open topology. This follows immediately from Theorem 6.8. $\qquad\square$

Proposition 6.10. *Suppose that $C(2^{\mathbb{N}})$ is a second countable Hausdorff semigroup with respect to some topology. Then the function $a : 2^{\mathbb{N}} \times C(2^{\mathbb{N}}) \longrightarrow 2^{\mathbb{N}}$ defined by*

$$(p, f)a = (p)f$$

is continuous.

Proof. Let $\mathcal{T}$ denote the given second countable Hausdorff semigroup topology on $C(2^{\mathbb{N}})$; F denote the set of constant functions in $C(2^{\mathbb{N}})$; and $\phi : F \longrightarrow 2^{\mathbb{N}}$ be the function that sends $f \in F$ to the unique value in $(2^{\mathbb{N}})f$. By the same argument as given in the second paragraph of the proof of Proposition 6.4, it suffices to show that ϕ is a homeomorphism, which follows by Proposition 6.9. $\qquad\square$

As immediate corollaries of the last proposition together with Theorem 6.1, we obtain the following.

Corollary 6.11. *Every second countable Hausdorff semigroup topology on the monoid of continuous functions on the Cantor space $2^{\mathbb{N}}$ contains the compact-open topology.*

Corollary 6.12. *The compact-open topology is the unique second countable Hausdorff semigroup topology on the monoid of continuous functions on the Cantor set.*

Proof. By Corollary 6.11 every such topology contains the compact-open topology and by Theorem 6.8 every such topology is contained in the compact-open topology. $\qquad\square$

If S is a semigroup, then we denote by $S^{\dagger}$ the *dual semigroup* of S, that is, $S^{\dagger}$ is the set S together with the operation $*$ where $s * t$ is defined to be the product of t and s in S for all $s, t \in S$. If $\mathcal{T}$ is topological for the semigroup S, then, by Proposition 2.3, $\mathcal{T}$ is also topological for $S^{\dagger}$.

Proposition 6.13. *Let $\mathcal{C}$ be a class of topological semigroups, and let $\mathcal{C}^{\dagger}$ denote the class of dual semigroups of semigroups in $\mathcal{C}$. If S is a topological semigroup, then the following are equivalent:*

(i) *S has automatic continuity with respect to $\mathcal{C}$;*
(ii) *the dual semigroup $S^{\dagger}$ has automatic continuity with respect to $\mathcal{C}^{\dagger}$;*
(iii) *every anti-homomorphism from S to a semigroup in $\mathcal{C}^{\dagger}$ is continuous.*

Proof. The identity function $\iota : T \longrightarrow T^{\dagger}$ is an anti-isomorphism from T to its dual semigroup $T^{\dagger}$. The proof can be completed using Proposition 2.3 and the fact that the composite of continuous functions is continuous. $\qquad\square$

A *Stone space* (also called a *Boolean space*) is a compact Hausdorff topological space with a basis of clopen sets. A classical result of Stone [75] gives a correspondence between Stone spaces and Boolean algebras. In modern terminology, this is an equivalence, called *Stone duality*, between the category of Boolean algebras and homomorphisms, and the dual category of Stone spaces and continuous functions; see, for example, [27, Theorem 34] for more details. Up to isomorphism, there is only one countably infinite atomless Boolean algebra which we denote by B_{∞}. This object is the Fraïssé limit of the class of non-trivial finite Boolean algebras. The countably infinite atomless Boolean algebra B_{∞} is also the Stone dual object of the Cantor space $2^{\mathbb{N}}$ and thus $\text{End}(B_{\infty})$ is anti-isomorphic to the monoid $C(2^{\mathbb{N}})$.

Theorem 6.14. *If B_∞ is the countably infinite atomless Boolean algebra, then the following hold:*

  (i) *the pointwise topology is the unique second countable Hausdorff semigroup topology on* $\mathrm{End}(B_\infty)$;

  (ii) $\mathrm{End}(B_\infty)$ *has automatic continuity with respect to the pointwise topology and the class of second countable topological semigroups.*

Proof. **(i).** Since $\mathrm{End}(B_\infty)$ is anti-isomorphic to $C(2^\mathbb{N})$, from Proposition 2.3 and Corollary 6.12, it follows that $\mathrm{End}(B_\infty)$ has a unique second countable Hausdorff semigroup topology. The pointwise topology is a second countable and Hausdorff semigroup topology on $\mathrm{End}(B_\infty)$, and hence the only one.

(ii). The automatic continuity then follows from Proposition 6.13 and Theorem 6.8 as any homomorphism from $\mathrm{End}(B_\infty)$ can be factored into an anti-isomorphism to $C(2^\mathbb{N})$ followed by an anti-homomorphism from $C(2^\mathbb{N})$. □

7. A LITTLE FORAY INTO THE LAND OF CLONES

Further examples of function clones which have automatic homeomorphicity with respect to close subclones of $\mathscr{O}_\mathbb{N}$ can be found :

  • the countable generic posets $(P, \leq)$ and $(P, <)$;
  • the rational Urysohn space;
  • the rational Urysohn sphere;
  • the countable random graph with all the loops added.

In this section we will extend some of the results in the preceding sections from monoids to clones, which are introduced in the following subsections.

7.1. Function clones. If C is any set, $g : C^m \longrightarrow C$ for some $m \in \mathbb{N} \setminus \{0\}$ and $f_1, \ldots, f_m : C^n \longrightarrow C$ for some $n \in \mathbb{N} \setminus \{0\}$, then we define $(f_1, \ldots, f_m) \circ_{m,n} g : C^n \longrightarrow C$ by

$$(x_1, \ldots, x_n) \mapsto ((x_1, \ldots, x_n)f_1, \ldots, (x_1, \ldots, x_n)f_m) \, g.$$

If the values of m and n are clear from the context, then we write $\circ$ instead of $\circ_{m,n}$. We denote by $\pi_i^n : C^n \longrightarrow C$ the *i-th projection of* C^n which is defined by

$$(x_1, \ldots, x_n) \mapsto x_i.$$

A *(function) clone* $\mathscr{C}$ with domain C is a set of functions of finite arity from C to C which is closed under composition and also contains all projections. More precisely, the following hold:

  (i) if $g \in \mathscr{C}$ is m-ary and $f_1, \ldots, f_m \in \mathscr{C}$ are n-ary, then $(f_1, \ldots, f_m) \circ g \in \mathscr{C}$;
  (ii) $\pi_i^n \in \mathscr{C}$ for every $n \in \mathbb{N}$ and every $i \in \{1, \ldots, n\}$.

If $\mathscr{C}$ is a function clone with domain C and $\mathcal{T}$ is a topology on C, then we will say that $\mathcal{T}$ is *topological* for $\mathscr{C}$, or $\mathcal{T}$ is a *clone topology*, on $\mathscr{C}$ if the composition function $\circ_{m,n}$ is continuous for every $m, n \in \mathbb{N}$; see [9] for more information.

Let $\mathscr{O}_X$ denote *the full function clone on a set X*, that is, the set of all finite arity functions from X to itself. In the definition of $\mathscr{O}_X$ given here, following [9, 44, 82], nullary functions are not permitted. Some authors, see, for example, [5, 51, 76], allow nullary operations in the definition of $\mathscr{O}_X$.

We define the *topology of pointwise convergence* on $\mathscr{O}_X$ in a similar way to its namesake for the full transformation monoid X^X, with the subbasic opens sets of the form

$$U_{(a_1, \ldots, a_n), b} = \{f \in \mathscr{O}_X : (a_1, \ldots, a_n)f = b\}$$

for some fixed $a_1, \ldots, a_n, b \in X$. Similarly to the monoid case, if X is countably infinite, this topology is Polish and topological for the full function clone $\mathscr{O}_X$.

A *polymorphism* of a structure $\mathbb{X}$ is a homomorphism from some finite and positive power of $\mathbb{X}$ into $\mathbb{X}$. The set of all polymorphisms of $\mathbb{X}$ forms a function clone on the set X which is closed in $\mathscr{O}_X$. This clone is called the *polymorphism clone of* $\mathbb{X}$ and denoted by $\mathrm{Pol}(\mathbb{X})$.

By restricting to the morphisms in a fixed category, we can extend this definition to obtain a clone of polymorphisms, for any object, in any category of sets and functions which allows finite products. For example we consider the clone $\mathrm{Pol}(2^\mathbb{N})$ of continuous maps from finite, positive powers of the Cantor space to itself.

A *clone homomorphism* is a map between function clones which preserves arities, maps projections to corresponding projections, and preserves the composition maps $\circ_{m,n}$ for every n and m. A topological clone $\mathscr{C}$ is said to have *automatic continuity* with respect to a class of topological clones, if every homomorphism from $\mathscr{C}$ to a member of that class is continuous. The following observation and lemma associate a semigroup

to each clone. They will be used to show that several clones have automatic continuity by using the automatic continuity of the associated semigroups.

Let $\mathscr{C}$ be a function clone which is topological with respect to the topology $\mathcal{T}$. Define a semigroup $S_\mathscr{C}$ in the following way: the elements of $S_\mathscr{C}$ are the functions in $\mathscr{C}$; and the binary operation $*$ on $S_\mathscr{C}$ is given by

$$(x_1, \ldots, x_n)f * g = ((x_1, \ldots, x_n)f, \ldots, (x_1, \ldots, x_n)f)g$$

for all $x_1, \ldots, x_n \in C$, whenever f is an n-ary function in $\mathscr{C}$. It is routine to verify that $*$, as defined above, is associative, and so $S_\mathscr{C}$ is indeed a semigroup. Moreover, all clone homomorphisms from a function clone to a function clone are also homomorphisms of the corresponding semigroups. Observe also, that if both f and g are unary functions, then $*$ is simply the usual composition of functions. Finally, it follows from the fact that composition of functions in $\mathscr{C}$ is continuous under $\mathcal{T}$, that $\mathcal{T}$ makes $S_\mathscr{C}$ into a topological semigroup.

Lemma 7.1. *Let $\mathscr{C}$ be a topological clone, and suppose that $S_\mathscr{C}$ has automatic continuity with respect to the topology on $\mathscr{C}$ and the class of second countable topological semigroups. Then $\mathscr{C}$ has automatic continuity with respect to the class of second countable topological clones.*

Proof. Suppose that $\mathscr{A}$ is a second countable topological clone. Let $f : \mathscr{C} \longrightarrow \mathscr{A}$ be a clone homomorphism. It follows from the definitions of $S_\mathscr{C}$ and $S_\mathscr{A}$ that f is also a homomorphism between these semigroups. The semigroups $S_\mathscr{C}$ and $S_\mathscr{A}$ are topological with respect to the topologies on $\mathscr{C}$ and $\mathscr{A}$, respectively, as discussed before the lemma. Thus by the assumption that $S_\mathscr{C}$ has automatic continuity, the map f is continuous. $\square$

7.2. **The full function clone on a countably infinite set.** Let X be a countably infinite set. In Section 5.1 it was shown that the pointwise topology is the unique Polish semigroup topology on the full transformation monoid X^X and that X^X under the pointwise topology has automatic continuity with respect to the class of second countable topological semigroups. We will now show that the full function clone $\mathscr{O}_X$ has the analogous properties for clones.

Theorem 7.2. *Let X be a countably infinite set. The topological clone $\mathscr{O}_X$ with the pointwise topology has automatic continuity with respect to the class of second countable topological clones.*

Proof. As discussed above, the topology of pointwise convergence $\mathcal{P}$ makes the semigroup $S_{\mathscr{O}_X}$ into a topological semigroup. Moreover, $\mathrm{Sym}(X) \subseteq \mathscr{O}_X$ and it has automatic continuity under the subspace topology of $\mathcal{P}$. Therefore, by Theorem 3.1(iv), it suffices to show that $S_{\mathscr{O}_X}$ satisfies property **X** with respect to $\mathrm{Sym}(X)$. In order to do so fix $s \in \mathscr{O}_X$ of arity $n \geq 1$.

Let f be any injective n-ary function such that the set $X \setminus \mathrm{im}(f)$ is infinite, and let g be a unary surjective function such that $(x)g^{-1}$ is infinite for every $x \in X$. It is now routine to define a unary t_s such that $s = ft_sg$. In particular, choose t', an injective partial transformation on X, such that $((\mathbf{x})f)\, t' \in ((\mathbf{x})s)\, g^{-1}$ for ever $\mathbf{x} \in X^n$. Since t' could have been chosen so that the sets $X \setminus \mathrm{dom}(t')$ and $X \setminus \mathrm{im}(t')$ are both countably infinite, t' can be extended to $t_s \in \mathrm{Sym}(X)$, so that $s = ft_sg$.

Let B be a basic open ball containing t_s, that is,

$$B = \{h \in X^X : (x)h = (x)t_s \text{ for all } x \in F\}$$

for some fixed finite set $F \subseteq X$. Next, we will show that the open neighbourhood $U = \{u \in X^{X^n} : (\mathbf{x})u = (\mathbf{x})s \text{ for all } \mathbf{x} \in (F)f^{-1}\}$ of s is contained in $f(B \cap \mathrm{Sym}(X))g$. Let $u \in U$. Similarly to before, let b' be an injective partial transformation on X such that $((\mathbf{x})f)\, b' \in ((\mathbf{x})u)\, g^{-1}$ for all $\mathbf{x} \in X^n$. Moreover, this can be done in such a way that $(x)b' = (x)t_s$ for all $x \in F$. Since F is finite, we may assume that b' was chosen so that the sets $X \setminus \mathrm{dom}(b')$ and $X \setminus \mathrm{im}(b')$ are countably infinite. Hence b' can be extended to $b \in \mathrm{Sym}(X)$, and therefore $u = fbg \in f(B \cap \mathrm{Sym}(X))g$, as required. $\square$

The proof of the next lemma is very similar to the proof of Lemma 5.1(ii) $\Rightarrow$ (iii) and omitted.

Lemma 7.3. *Let $\mathscr{C}$ be a function clone on an infinite set X containing all of the constant functions and such that for every $x \in X$ there exists a unary $f_x \in \mathscr{C}$ such that $(x)f_x^{-1} = \{x\}$ and $(X)f_x$ is finite. If $\mathcal{T}$ is a T_1 clone topology of $\mathscr{C}$, then $\{f \in \mathscr{C} : (a_1, \ldots, a_n)f = b\}$ is open in $\mathcal{T}$ for all $n \geq 1$ and all $a, b \in X$.*

Combining the lemmas above we obtain the following result.

Corollary 7.4. *Let X be a countably infinite set. The topology of pointwise convergence is the unique Polish topology which makes $\mathscr{O}_X$ into a topological clone.*

7.3. The Horn clone. Having considered $\mathscr{O}_X$, the analogue of the semigroup of all transformations on X, we now consider the analogue of the monoid $\mathrm{Inj}(X)$ of injective functions from X to X. The function clone $\mathscr{H}_X$ is defined to be the subclone of $\mathscr{O}_X$ generated by the injective n-ary function for every arity $n \in \mathbb{N} \setminus \{0\}$. In the literature, this object is referred to as the *Horn clone*; see [7, 9].

In this section, we show that the pointwise topology can be extended to infinitely many Polish topologies for $\mathscr{H}_X$ when X is countably infinite. Among these Polish topologies on $\mathscr{H}_X$ there is a maximum one, and $\mathscr{H}_X$ has automatic continuity with respect to this topology and the class of second countable topological clones. The main theorem, and its proof, in this section are similar to Theorem 5.22.

For n-ary $f \in \mathscr{H}_X$, there exists a non-empty subset I_f of $\{1, \ldots, n\}$ such that

$$(\mathbf{x})f = (\mathbf{y})f \quad \text{if and only if} \quad (\mathbf{x})\pi_i^n = (\mathbf{y})\pi_i^n \text{ for all } i \in I_f$$

for all $\mathbf{x}, \mathbf{y} \in X^n$. The elements of $\mathscr{H}_X$ are exactly characterised by the existence of such a non-empty set.

Since nullary operations are vacuously injective, if they were permitted in the definition of $\mathscr{O}_X$, then the sets I_f in the definition of $\mathscr{H}_X$ could be empty. Although we have opted to disallow nullary functions in the definition of $\mathscr{O}_X$, and consequently require the sets I_f to be non-empty in the definition of $\mathscr{H}_X$, the results in this paper hold regardless for the other versions of $\mathscr{H}_X$.

The following is the main result of this section.

Theorem 7.5. *Let X be a countably infinite set and let $\mathcal{H}$ be the topology on the Horn clone $\mathscr{H}_X$ generated by the pointwise topology together with the sets*

$$V_y := \{f \in \mathscr{H}_X : y \notin \mathrm{im}(f)\} \quad \text{and} \quad F_n := \{f \in \mathscr{H}_X : |X \setminus \mathrm{im}(f)| = n\}$$

for $y \in X$ and $n \in \mathbb{N} \cup \{\aleph_0\}$. Then the following hold:

(i) *$\mathscr{H}_X$ admits infinitely many distinct Polish clone topologies, including the pointwise topology and $\mathcal{H}$;*

(ii) *$S_{\mathscr{H}_X}$ has property $\mathbf{X}$ with respect to $\mathcal{H}$ and $\mathrm{Sym}(X)$;*

(iii) *$\mathscr{H}_X$ has automatic continuity with respect to the topology $\mathcal{H}$ and the class of second countable topological clones;*

(iv) *$\mathcal{H}$ is the maximum Polish clone topology on $\mathscr{H}_X$;*

(v) *if $\mathcal{T}$ is any clone topology on $\mathscr{H}_X$ and $\mathcal{T}$ induces the pointwise topology on $\mathrm{Sym}(X)$, then $\mathcal{T}$ is contained in $\mathcal{H}$.*

We require the following three lemmas in the proof of Theorem 7.5.

Lemma 7.6. *If X is countably infinite, $f \in \mathscr{H}_X^{(n)}$, and $x \in X$ is such that $(x)f^{-1}$ is non-empty, then $(x)f^{-1} = T_1 \times \cdots \times T_n$ where $|T_i| = 1$ if $i \in I_f$ and $T_i = X$ otherwise.*

Proof. Let $T = (x)f^{-1}$ be non-empty and let $\mathbf{x} \in T$. If $\mathbf{y} \in X^n$, then, by the definition of I_f, $(\mathbf{y})f = (\mathbf{x})f$ if and only if $(\mathbf{x})\pi_i = (\mathbf{y})\pi_i$ for all $i \in I_f$. We define T_i to be X if $i \notin I_f$ and $\{(\mathbf{x})\pi_i\}$ otherwise. Hence

$$(x)f^{-1} = \{\mathbf{y} \in X^n : (\mathbf{y})f = (\mathbf{x})f\} = \{\mathbf{y} \in X^n : (\mathbf{y})\pi_i = (\mathbf{x})\pi_i \text{ for } i \in I_f\} = T_1 \times \cdots \times T_n,$$

as required. $\square$

Lemma 7.7. *If X is countably infinite, then the sets*

$$\mathscr{H}_X^{(n)} := \{f \in \mathscr{H}_X : \mathrm{arity}(f) = n\} \quad \text{and} \quad W_{S,n} := \{f \in \mathscr{H}_X^{(n)} : I_f = S\}$$

are open in the pointwise topology on $\mathscr{H}_X$ for all $n \in \mathbb{N}$, and all finite $S \subseteq \mathbb{N}$.

Proof. For $n \in \mathbb{N}$, let $\mathbf{x} \in X^n$ be fixed. Then the set $\mathscr{H}_X^{(n)}$ is the union of the subbasic open sets $U_{\mathbf{x},y} = \{f \in \mathscr{H}_X : (\mathbf{x})f = y\}$ (for the pointwise topology) for all $y \in X$. Hence $\mathscr{H}_X^{(n)}$ is open in the pointwise topology.

Let $n \in \mathbb{N}$ be arbitrary and $S \subseteq \{1, \ldots n\}$ be finite. For every $i \in \{1, \ldots n\}$, we let $\mathbf{x}_i, \mathbf{y}_i \in X^n$ be such that $(\mathbf{x}_i)\pi_j^n \neq (\mathbf{y}_i)\pi_j^n$ if and only if $i = j$ ($\mathbf{x}_i$ and $\mathbf{y}_i$ disagree only in their ith component). We will now show that

$$W_{S,n} = \left(\bigcap_{i \in S} \{g \in \mathscr{H}_X^{(n)} : (\mathbf{x}_i)g \neq (\mathbf{y}_i)g\} \right) \cap \left(\bigcap_{i \in \{1, \ldots, n\} \setminus S} \{g \in \mathscr{H}_X^{(n)} : (\mathbf{x}_i)g = (\mathbf{y}_i)g\} \right).$$

We denote the intersection in the previous displayed equations by J. Note that if $f \in \mathscr{H}_X^{(n)}$, then

(20) $(\mathbf{x}_i)f \neq (\mathbf{y}_i)f$ if and only if $i \in I_f$.

If $f \in W_{S,n}$, then $f \in \mathscr{H}_X^{(n)}$ and $I_f = S$. Hence $(\mathbf{x}_i)f \neq (\mathbf{y}_i)f$ if and only if $i \in S$. Hence $f \in J$. Conversely, if $f \in J$, then $f \in \mathscr{H}_X^{(n)}$ and so I_f is well-defined. If $i \in S$, then, since $f \in J$, $(\mathbf{x}_i)f \neq (\mathbf{y}_i)f$ and so $i \in I_f$ by (20). Similarly, if $i \notin S$, then $(\mathbf{x}_i)f = (\mathbf{y}_i)f$ and so $i \notin I_f$. It follows that $I_f = S$ and we have shown that $W_{S,n} = J$. $\qquad \square$

Lemma 7.8. *Let X be countably infinite, and let $\mathcal{H}_1$ be the topology on $\mathscr{H}_X$ generated by the pointwise topology and the sets $V_y = \{f \in \mathscr{H}_X : y \notin \mathrm{im}(f)\}$ for all $y \in X$. Then the set*

$$P_{T,y,n} := \{f \in \mathscr{H}_X^{(n)} : (y)f^{-1} = T\}$$

is open in $\mathcal{H}_1$ for every $y \in X$, $n \in \mathbb{N}$, and $T \subseteq X^n$.

Proof. If $T = \varnothing$, then $P_{T,y,n} = V_y$ which is open in $\mathcal{H}_1$. If $T \neq \varnothing$ and $f \in P_{T,y,n}$, then, by Lemma 7.6, $T = T_1 \times \cdots \times T_n$ where T_i is a singleton for $i \in I_f$ and $T_i = X$ for $i \notin I_f$. Note that T does not depend on the choice of $f \in P_{T,y,n}$, and so, in particular, $I_f = I_g$ for all $f, g \in P_{T,y,n}$. We denote this set by Y and fix any $\mathbf{x} \in T$. Therefore $P_{T,y,n} = U_{\mathbf{x},y} \cap W_{Y,n}$, where $U_{\mathbf{x},y}$ is one of the sub-basic open sets in the definition of the pointwise topology, and $W_{Y,n}$ is open in the pointwise topology by Lemma 7.7. $\qquad \square$

Proof of Theorem 7.5. **(i).** First, we demonstrate that $\mathscr{H}_X$ is a closed subset of $\mathscr{O}_X$ in the pointwise topology. Let $f \in \mathscr{O}_X \setminus \mathscr{H}_X$ be n-ary. Then for every non-empty $S \subseteq \{1, \ldots, n\}$ there are $\mathbf{x}_S, \mathbf{y}_S \in X^n$ such that $(\mathbf{x}_S)f = (\mathbf{y}_S)f$ if and only if there exists $i \in S$ such that $(\mathbf{x}_S)\pi_i \neq (\mathbf{y}_S)\pi_i$. Hence

$$\bigcap_{\varnothing \neq S \subseteq \{1,\ldots,n\}} U_{\mathbf{x}_S,(\mathbf{x}_S)f} \cap U_{\mathbf{y}_S,(\mathbf{y}_S)f}$$

is an open neighbourhood of f contained in $\mathscr{O}_X \setminus \mathscr{H}_X$. Thus $\mathscr{O}_X \setminus \mathscr{H}_X$ is open in the pointwise topology, and so $\mathscr{H}_X$ is closed in $\mathscr{O}_X$. Hence $\mathscr{H}_X$ with the pointwise topology is a Polish clone.

Next, we show that $\mathcal{H}$ is a clone topology by showing that $\circ_{n,m} : (\mathscr{H}_X^{(n)})^m \times \mathscr{H}_X^{(m)} \to \mathscr{H}_X$ is continuous for every $n, m \in \mathbb{N}$. Since $\mathscr{H}_X$ is a Polish clone with respect to the pointwise topology, the set $(U_{\mathbf{x},y})\circ_{n,m}^{-1}$ is open in the pointwise topology, and hence in $\mathcal{H}$. It suffices to consider $f_1, \ldots, f_m \in \mathscr{H}_X^{(n)}$ and $g \in \mathscr{H}_X^{(m)}$ such that $(f_1, f_2, \ldots, f_m) \circ g$ belongs to a subbasic open set B and either $B = V_y$ for some $y \in X$ or $B = F_n$ for some $n \in \mathbb{N} \cup \{\aleph_0\}$. We will find a neighbourhood of $((f_1, \ldots, f_m), g)$ which is mapped into B by $\circ_{n,m}$.

Suppose that $y \in X$ is such that $(f_1, \ldots, f_m) \circ_{n,m} g \in V_y$. Then either: $y \notin \mathrm{im}(g)$ or

$$((\mathbf{z})f_1, \ldots, (\mathbf{z})f_m) \notin (y)g^{-1}$$

for all $\mathbf{z} \in X^n$. In the former case, $(\mathscr{H}_X^{(n)})^m \times (\mathscr{H}_X^{(m)} \cap V_y)$ is the required neighbourhood.

In the latter case, it follows from Lemma 7.6, that $(y)g^{-1} = T_1 \times \cdots \times T_m$ where $|T_i| = 1$ if $i \in I_g$ and $T_i = X$ if $i \notin I_g$. Hence if $\mathbf{z} \in X^n$ is arbitrary, then $((\mathbf{z})f_1, \ldots, (\mathbf{z})f_m) \notin (y)g^{-1}$ and so there exists $i \in I_g$ such that $\{(\mathbf{z})f_i\} \neq T_i$. In other words, $\bigcap_{i \in I_g} (T_i)f_i^{-1} = \varnothing$. If $i \in I_g$ and $h_i \in P_{(T_i)f_i^{-1}, T_i, n}$, which is an open set by Lemma 7.8, then $(T_i)h_i^{-1} = (T_i)f_i^{-1}$, and so $\bigcap_{i \in I_g} (T_i)h_i^{-1} = \bigcap_{i \in I_g} (T_i)f_i^{-1} = \varnothing$. Hence

$$\left(\prod_{i=1}^{m} A_i\right) \times P_{(y)g^{-1}, y, m},$$

where $A_i = P_{(T_i)f_i^{-1}, T_i, n}$ if $i \in I_g$ and $A_i = \mathscr{H}_X^{(n)}$ if $i \notin I_g$, is the required open neighbourhood of $((f_1, \ldots, f_m), g)$.

Suppose that $\kappa \leq \aleph_0$ and $(f_1, \ldots, f_m) \circ_{n,m} g \in F_\kappa$. Note that

$$\kappa = |X \setminus \mathrm{im}((f_1, \ldots, f_m) \circ_{n,m} g)|$$
$$= |X \setminus \mathrm{im}(g)| + |\{x \in \mathrm{im}(g) : ((\mathbf{y})f_1, \ldots, (\mathbf{y})f_m) \notin (x)g^{-1} \text{ for all } \mathbf{y} \in X^n\}|.$$

Denote the second cardinal in the sum above by λ. Then, by Lemma 7.6,

$$\lambda = |\{x \in \mathrm{im}(g) : ((\mathbf{y})f_1, \ldots, (\mathbf{y})f_m) \notin (x)g^{-1} \text{ for all } \mathbf{y} \in X^n\}|$$
$$= |\{x \in \mathrm{im}(g) : \text{ for each } \mathbf{y} \in X^n \text{ there is } i \in I_g \text{ such that } (\mathbf{y})f_i \notin (x)g^{-1}\pi_i\}|$$
$$= |\{x \in \mathrm{im}(g) : \text{ there is no } \mathbf{y} \in X^n \text{ with } (\mathbf{y})f_i \in (x)g^{-1}\pi_i \text{ for all } i \in I_g\}|.$$

For every $x \in \operatorname{im}(g)$, define $\alpha_x \in X^{I_g}$ to be such that $(i)\alpha_x$ is the unique element in $(x)g^{-1}\pi_i$ for every $i \in I_g$. Then, by definition of I_g, the map $\Phi \colon \operatorname{im}(g) \longrightarrow X^{I_g}$, given by $x \mapsto \alpha_x$, is an injection. Therefore

$$\lambda = |\{\alpha_x \in X^{I_g} : x \in \operatorname{im}(g) \text{ and there is no } \mathbf{y} \in X^n \text{ with } (\mathbf{y})f_i \in (x)g^{-1}\pi_i \text{ for all } i \in I_g\}|$$

$$= |\{\alpha_x \in X^{I_g} : x \in \operatorname{im}(g) \text{ and there is no } \mathbf{y} \in X^n \text{ with } (\mathbf{y})f_i = (i)\alpha_x \text{ for all } i \in I_g\}|$$

Moreover, suppose $\alpha \in X^{I_g}$ is such that there is no $\mathbf{y} \in X^n$ with $(\mathbf{y})f_i = (i)\alpha$ for all $i \in I_g$. Then we can choose $\mathbf{z} \in X^n$ such that $(\mathbf{z})\pi_i = (i)\alpha$ for all $i \in I_g$. Hence $\alpha = \alpha_x$ where $x = (\mathbf{z})g$ and there is no $\mathbf{y} \in X^n$ with $(\mathbf{y})f_i = (i)\alpha_x$ for all $i \in I_g$, and so the sets $\{\alpha_x \in X^{I_g} : x \in \operatorname{im}(g) \text{ and there is no } \mathbf{y} \in X^n \text{ with } (\mathbf{y})f_i = (i)\alpha_x \text{ for all } i \in I_g\}$ and $\{\alpha \in X^{I_g} : \text{there is no } \mathbf{y} \in X^n \text{ with } (\mathbf{y})f_i = (i)\alpha \text{ for all } i \in I_g\}$ are equal. Therefore,

$$(21) \qquad \kappa = |X \setminus \operatorname{im}(g)| + |\{\alpha \in X^{I_g} : \text{there is no } \mathbf{y} \in X^n \text{ with } (\mathbf{y})f_i = (i)\alpha \text{ for all } i \in I_g\}|.$$

There are 4 cases to consider. Since $g \in \mathscr{H}_X$, $I_g \neq \varnothing$.

Case α. Suppose that $I_g = \{i\}$ for some $i \in \{1, \ldots, m\}$. Then $\kappa = |X \setminus \operatorname{im}(g)| + |X \setminus \operatorname{im}(f_i)|$ by (21). Hence

$$\left(\prod_{k=1}^n A_k\right) \times \left(W_{I_g,m} \cap F_{|X \setminus \operatorname{im}(g)|}\right),$$

where $A_i = F_{|X \setminus \operatorname{im}(f_i)|} \cap \mathscr{H}_X^{(n)}$ and $A_k = \mathscr{H}_X^{(n)}$ if $k \neq i$, is the required neighbourhood.

Case β. Suppose there are distinct $i, j \in I_g$ with $I_{f_i} \cap I_{f_j} \neq \varnothing$. We will show that $\kappa = \aleph_0$.

Suppose that $k \in I_{f_i} \cap I_{f_j}$. Let $(\mathbf{x}_r)_{r \in \mathbb{N}}$ be an infinite sequence of pairwise distinct tuples in X^n such that $(\mathbf{x}_a)\pi_l = (\mathbf{x}_b)\pi_l$ for all $a, b \in \mathbb{N}$ and all $l \neq k$. There is at most one $a \in \mathbb{N}$ with $(\mathbf{x}_1)f_i = (\mathbf{x}_a)f_j$ as $(\mathbf{x}_a)f_j \neq (\mathbf{x}_b)f_j$ whenever $a \neq b$. For each a with $(\mathbf{x}_1)f_i \neq (\mathbf{x}_a)f_j$ we choose an m-tuple $\mathbf{y}_a$ whose ith entry is $(\mathbf{x}_1)f_i$ and whose jth entry is $(\mathbf{x}_a)f_j$. Suppose that there is $\mathbf{z} \in X^n$ such that $(\mathbf{y}_a)g = ((\mathbf{z})f_1, \ldots, (\mathbf{z})f_m) \circ g$. Then $(\mathbf{x}_1)f_i = (\mathbf{y}_a)\pi_i = (\mathbf{z})f_i$ and $(\mathbf{x}_a)f_j = (\mathbf{y}_a)\pi_j = (\mathbf{z})f_j$ since $i, j \in I_g$. Moreover, it follows that $(\mathbf{x}_1)\pi_k = (\mathbf{z})\pi_k = (\mathbf{x}_a)\pi_k$ since $k \in I_{f_i} \cap I_{f_j}$, which is a contradiction if $a > 1$. Therefore, the points $(\mathbf{y}_a)g$ are distinct and not in the image of $(f_1, \ldots, f_m) \circ g$ for $a > 1$, and thus $\kappa = \aleph_0$.

Then

$$\left(\prod_{k=1}^m W_{I_{f_k},n}\right) \times W_{I_g,m}$$

is the required neighbourhood.

Case γ. Suppose that $|I_g| \geq 2$ and $I_{f_i} \cap I_{f_j} = \varnothing$ for all distinct $i, j \in I_g$, and suppose further that f_i is surjective for all $i \in I_g$. Then for every $\alpha \in X^{I_g}$ we may find $\mathbf{y} \in X^n$ such that $(\mathbf{y})f_i = (i)\alpha$ for all $i \in I_g$, by simply choosing the appropriate values on each set of coordinates I_{f_i} separately. Hence $\kappa = |X \setminus \operatorname{im}(g)|$ by (21). Let $A_k = W_{I_{f_k},n} \cap F_0$ for $k \in I_g$ and let $A_k = \mathscr{H}_X^{(n)}$ for $k \notin I_g$. Then

$$\left(\prod_{k=1}^m A_k\right) \times (W_{I_g,m} \cap F_{|X \setminus \operatorname{im}(g)|})$$

is the required neighbourhood.

Case δ. Suppose that $|I_g| \geq 2$ and $I_{f_i} \cap I_{f_j} = \varnothing$ for all distinct $i, j \in I_g$, and suppose further that f_i is not surjective for some $i \in I_g$. Then $\aleph_0 \geq \kappa \geq |X \setminus \operatorname{im}(g)| + |(X \setminus \operatorname{im}(f_i)) \times X^{|I_g|-1}| = \aleph_0$ by (21). Hence $\kappa = \aleph_0$, and so

$$\left(\prod_{k=1}^m W_{I_{f_k},n} \cap F_{|X \setminus \operatorname{im}(f_k)|}\right) \times (W_{I_g,m} \cap F_{|X \setminus \operatorname{im}(g)|})$$

is the required neighbourhood. Therefore, concluding the argument that $\mathcal{H}$ is a clone topology.

For each $y \in X$, the set V_y is closed in the pointwise topology. It follows by [42, Lemma 13.2] that for each $y \in X$ the topology generated by the pointwise topology and V_y is Polish, hence by [42, Lemma 13.3] the topology $\mathcal{H}_1$ generated by the pointwise topology together with all of them is Polish.

As in the proof of Theorem 5.22, we can now define a sequence of topologies $\mathcal{S}_0 = \mathcal{H}_1, \mathcal{S}_1, \ldots$ where $\mathcal{S}_{n+1}$ is generated by $\mathcal{S}_n$ and F_n. By an argument similar to the one in the proof of Theorem 5.22, it can be shown that F_n is closed in $\mathcal{S}_n$ and $F_{\aleph_0}$ is closed in the topology generated by $\bigcup_{n=0}^\infty \mathcal{S}_n$. Hence all of these topologies $\mathcal{S}_1, \mathcal{S}_2, \ldots$ as well as $\mathcal{H}$ are Polish. Moreover, it follows from the proof that $\mathcal{H}$ is a clone topology that $\mathcal{S}_0, \mathcal{S}_1, \ldots$ are also clone topologies. Finally, when restricted to $\operatorname{Inj}(X)$, the topologies in this

proof agree with the corresponding topologies in Theorem 5.22, which are all distinct. It follows that the clone topologies in this proof are distinct also.

(ii). Let $s \in S_{\mathscr{H}_X}$ be n-ary for some $n \in \mathbb{N}$. We show that there exist $f_s, g_s \in S_{\mathscr{H}_X}$ and $a_s \in \mathrm{Sym}(X)$ such that $s = f_s a_s g_s$ and for every basic open neighbourhood B of a_s there exists an open neighbourhood U of s such that $U \subseteq f_s(B \cap \mathrm{Sym}(X))g_s$.

Let $f_s = s$, let a_s and g_s both be the identity map on X, and let B be a basic open neighbourhood of a_s. Then there is some finite $Y \subseteq X$ such that

$$B \cap \mathrm{Sym}(X) = \{h \in \mathrm{Sym}(X) : (y)h = y \text{ for all } y \in Y\}.$$

It can be shown that $h \in \mathscr{H}_X^{(n)}$ is an element of $f_s(B \cap \mathrm{Sym}(X))g_s = s(B \cap \mathrm{Sym}(X))$ if and only if the following hold

(1) $y \notin \mathrm{im}(h)$ for all $y \in Y \setminus \mathrm{im}(s)$;
(2) $(y)s^{-1} = (y)h^{-1}$ for all $y \in Y \cap \mathrm{im}(s)$;
(3) $|X \setminus \mathrm{im}(s)| = |X \setminus \mathrm{im}(h)|$;
(4) $I_h = I_s$.

In the reverse implication, the required element of $B \cap \mathrm{Sym}(X)$ can be chosen to be any bijection extending the partial bijection mapping $x \in \mathrm{im}(s)$ to the single point in the set $(x)s^{-1}h$.

It follows that

$$f_s(B \cap \mathrm{Sym}(X))g_s = \bigcap_{y \in Y \setminus \mathrm{im}(f_s)} V_y \cap \bigcap_{y \in Y \cap \mathrm{im}(f_s)} P_{(y)s^{-1},y,n} \cap F_{|X \setminus \mathrm{im}(s)|} \cap W_{I_s,n},$$

and so $f_s(B \cap \mathrm{Sym}(X))g_s$ is an open neighbourhood of s in $\mathcal{H}$, as required.

(iii). From part (ii) the monoid $S_{\mathscr{H}_X}$ has property **X** with respect to $\mathrm{Sym}(X)$ and the topology $\mathcal{H}$. Property **X** implies automatic continuity by Theorem 3.1, and so $S_{\mathscr{H}_X}$ has automatic continuity with respect to the class of second countable topological semigroups. It follows from Lemma 7.3 that $\mathscr{H}_X$ has automatic continuity with respect to the class of second countable topological clones.

(iv). This follows immediately from part (iii).

(v). Let $\mathcal{T}$ be a clone topology for $\mathscr{H}_X$ that induces the pointwise topology on $\mathrm{Sym}(X)$. Every clone topology for $\mathscr{H}_X$ is also a semigroup topology for $S_{\mathscr{H}_X}$. By part (ii) and Theorem 3.1, every semigroup topology on $S_{\mathscr{H}_X}$ which induces the pointwise topology on $\mathrm{Sym}(X)$, such as $\mathcal{T}$, is contained in $\mathcal{H}$. □

7.4. **The clones of the Cantor space and the countably infinite atomless Boolean algebra.** In this section, we consider the polymorphism clone of the Cantor space $2^{\mathbb{N}}$ which is defined as

$$\mathrm{Pol}(2^{\mathbb{N}}) := \bigcup_{n \in \mathbb{N} \setminus \{0\}} C((2^{\mathbb{N}})^n, 2^{\mathbb{N}}),$$

where $C(X, Y)$ denotes the set of continuous functions from a topological space X to a space Y. The polymorphism clone of the countably infinite atomless Boolean algebra, which is

$$\mathrm{Pol}(B_\infty) := \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathrm{Hom}((B_\infty)^n, B_\infty)$$

where $\mathrm{Hom}((B_\infty)^n, B_\infty)$ denotes the set of homomorphisms from $(B_\infty)^n$ to B_∞. For more information on this Boolean algebra see [27].

The compact-open topology on $\mathrm{Pol}(2^{\mathbb{N}})$ is defined so that for each $n \in \mathbb{N} \setminus \{0\}$ we consider the function space $C((2^{\mathbb{N}})^n, 2^{\mathbb{N}})$ with the compact-open topology (as defined previously) and view $\mathrm{Pol}(2^{\mathbb{N}})$ as the disjoint union of these spaces with the union topology. One can verify that this topology makes $\mathrm{Pol}(2^{\mathbb{N}})$ a topological clone.

The clone $\mathrm{Pol}(2^{\mathbb{N}})$ is an analogue of the monoid $C(2^{\mathbb{N}})$ of continuous functions on the Cantor space $2^{\mathbb{N}}$. In Theorem 6.8, it is shown that $C(2^{\mathbb{N}})$ with the compact-open topology has automatic continuity with respect to the class of second countable topological semigroups and that the compact-open topology is the unique second countable Hausdorff semigroup topology on $C(2^{\mathbb{N}})$. The Stone duality theorem gives an anti-isomorphism (see, for example, [27, Theorem 34]) between the (topological) semigroups $C(2^{\mathbb{N}})$ and $\mathrm{End}(B_\infty)$, which was used in Theorem 6.14 to prove analogues of the results about $C(2^{\mathbb{N}})$ for $\mathrm{End}(B_\infty)$.

Stone duality associates products of Boolean algebras with coproducts (disjoint unions) of Stone spaces. Although the Stone dual of $\mathrm{End}(B_\infty)$ is $C(2^{\mathbb{N}})$, the Stone dual of $\mathrm{Pol}(B_\infty)$ is not $\mathrm{Pol}(2^{\mathbb{N}})$. Instead, the Stone dual $\mathrm{Pol}(B_\infty)^\dagger$ of $\mathrm{Pol}(B_\infty)$ is the disjoint union

$$(22) \qquad \bigcup_{n \in \mathbb{N} \setminus \{0\}} C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\}),$$

and operation $\bullet_{n,m}$ defined by

$$(x)\big(f \bullet_{n,m} (g_1, \ldots, g_n)\big) = ((x)f\pi_1)g_{(x)f\pi_2}$$

for every $f \in C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$, $g_1, \ldots, g_n \in C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, m\})$ and $x \in 2^{\mathbb{N}}$ such that $(x)f = ((x)f\pi_1, (x)f\pi_2) \in 2^{\mathbb{N}} \times \{1, \ldots, n\}$. In this context, for every $n \in \mathbb{N}$, the Stone dual of $\mathrm{Hom}(B_\infty^n, B_\infty)$ is $C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$. Moreover, the Stone duality translates the pointwise topology to the compact-open topology and vice-versa. While the results in this section are stated in terms of $\mathrm{Pol}(2^{\mathbb{N}})$ and $\mathrm{Pol}(B_\infty)$, one of the proofs related to $\mathrm{Pol}(B_\infty)$ is given in terms of $\mathrm{Pol}(B_\infty)^\dagger$, which is the reason for introducing $\mathrm{Pol}(B_\infty)^\dagger$. It seems to the authors of this paper that the two clones $\mathrm{Pol}(2^{\mathbb{N}})$ and $\mathrm{Pol}(B_\infty)$ are in fact meaningfully distinct.

To prove the main theorem of this section, we require the following proposition, which is similar to Proposition 6.10.

Proposition 7.9. *Suppose that* $\mathrm{Pol}(2^{\mathbb{N}})$ *is a second countable Hausdorff topological clone with respect to some topology* $\mathcal{T}$*. Then for all* $n \in \mathbb{N}$ *the function* $a_n : (2^{\mathbb{N}})^n \times C((2^{\mathbb{N}})^n, 2^{\mathbb{N}}) \longrightarrow 2^{\mathbb{N}}$ *defined by*

$$(\mathbf{x}, f)a_n = (\mathbf{x})f$$

is continuous where $(2^{\mathbb{N}})^n \times C((2^{\mathbb{N}})^n, 2^{\mathbb{N}})$ *has the product topology given by the subspace topology on* $C((2^{\mathbb{N}})^n, 2^{\mathbb{N}})$ *induced by* $\mathcal{T}$ *and the usual topology on* $2^{\mathbb{N}}$*.*

Proof. If we restrict $\mathrm{Pol}(2^{\mathbb{N}})$ to its elements of arity 1, then we obtain $C(2^{\mathbb{N}})$. By Proposition 6.9, if F denotes the set of constant maps of arity 1, and $\phi : F \longrightarrow 2^{\mathbb{N}}$ is the function that sends $g \in F$ to the unique value in $(2^{\mathbb{N}})g$, then ϕ is a homeomorphism.

Let $\alpha : (2^{\mathbb{N}})^n \longrightarrow F^n$ be defined by $(x_1, x_2, \ldots, x_n)\alpha = ((x_1)\phi^{-1}, (x_2)\phi^{-1}, \ldots, (x_n)\phi^{-1})$. Then α is continuous since it is continuous in every component. If $\beta : (2^{\mathbb{N}})^n \times C((2^{\mathbb{N}})^n, 2^{\mathbb{N}}) \longrightarrow F^n \times C((2^{\mathbb{N}})^n, 2^{\mathbb{N}})$ is defined by $(\mathbf{x}, f)\beta = ((\mathbf{x})\alpha, f)$ for all $\mathbf{x} \in (2^{\mathbb{N}})^n$, $f \in C((2^{\mathbb{N}})^n, 2^{\mathbb{N}})$, then β is also continuous. Then a_n is the composite of the continuous functions β, the function clone $\circ_{1,n}$, and ϕ, and hence, a_n is continuous. $\square$

Theorem 7.10. *Let* $\mathcal{K}$ *be the compact-open topology on* $\mathrm{Pol}(2^{\mathbb{N}})$*. Then the following hold:*

(i) $S_{\mathrm{Pol}(2^{\mathbb{N}})}$ *has property* $\boldsymbol{X}$ *with respect to* $\mathcal{K}$ *and* $C(2^{\mathbb{N}})$*;*

(ii) *the clone* $\mathrm{Pol}(2^{\mathbb{N}})$ *has automatic continuity with respect to the class of second countable topological clones;*

(iii) $\mathcal{K}$ *is the only second countable Hausdorff clone topology on* $\mathrm{Pol}(2^{\mathbb{N}})$*.*

Proof. **(i).** Let $s \in \mathrm{Pol}(2^{\mathbb{N}})$ be arbitrary and of arity n. Let $f_s : (2^{\mathbb{N}})^n \longrightarrow 2^{\mathbb{N}}$ be a homeomorphism, $t_s := f_s^{-1}s$ and $g_s := 1_{2^{\mathbb{N}}}$ (the identity function on $2^{\mathbb{N}}$). Then $f_s t_s g_s = f_s f_s^{-1} s g_s = s$, $f_s, g_s \in \mathrm{Pol}(2^{\mathbb{N}})$ and $t_s \in C(2^{\mathbb{N}})$. Let B be a neighbourhood of t_s in $C(2^{\mathbb{N}})$. Then

$$t_s \in \bigcap_{i=1}^{k} [K_i, U_i] \subseteq B,$$

for some compact K_i, open U_i, and $k \in \mathbb{N}$. Hence

$$s = f_s t_s \in f_s \bigcap_{i=1}^{k} [K_i, U_i] \quad \text{and so} \quad \bigcap_{i=1}^{k} [(K_i)f_s^{-1}, U_i] = f_s \bigcap_{i=1}^{k} [K_i, U_i] \subseteq f_s B = f_s B g_s.$$

Since each of $[(K_i)f_s^{-1}, U_i]$ is a subbasic open set, $f_s B g_s$ is a neighbourhood of s, as required.

(ii). By part **(i)**, the semigroup $S_{\mathrm{Pol}(2^{\mathbb{N}})}$ has property $\boldsymbol{X}$ with respect to $C(2^{\mathbb{N}})$ and the compact-open topology. It is shown in Theorem 6.8 that $C(2^{\mathbb{N}})$ has automatic continuity with respect to the class of second countable topological semigroups and the compact-open topology. Hence, by Theorem 3.1(iv), and the fact $S_{\mathrm{Pol}(2^{\mathbb{N}})}$ has automatic continuity with respect to class of second countable topological semigroups. Therefore Lemma 7.1 implies that $\mathrm{Pol}(2^{\mathbb{N}})$ has automatic continuity, as required.

(iii). Suppose that $\mathrm{Pol}(2^{\mathbb{N}})$ is equipped with a second countable Hausdorff clone topology $\mathcal{T}$. By part **(ii)**, this topology must be contained in the compact-open topology. It is shown in Theorem 6.1 that if $C(X, Y)$

is first countable and Hausdorff with respect to some topology $\mathcal{S}$, for arbitrary compact metrizable spaces X and Y, and $a : X \times C(X,Y) \longrightarrow Y$ defined by

$$(p, f)a = (p)f$$

is continuous, then $\mathcal{S}$ contains the compact-open topology. Hence, by Proposition 7.9, $\mathcal{T}$ contains the compact-open topology. $\qquad\square$

Theorem 7.11. *The following hold:*

(i) $S_{\mathrm{Pol}(B_\infty)}$ *has property **X** with respect to* $\mathrm{End}(B_\infty)$ *and the pointwise topology;*

(ii) $\mathrm{Pol}(B_\infty)$ *has automatic continuity with respect to the pointwise topology and the class of second countable topological clones;*

(iii) *the pointwise topology is the only second countable Hausdorff clone topology on* $\mathrm{Pol}(B_\infty)$.

Proof. The proofs of parts (i) and (ii) of this theorem are similar to the corresponding parts of the proof of Theorem 7.10.

(i). Let $s \in \mathrm{Pol}(B_\infty)$ be arbitrary and of arity n. Note that all finite non-zero powers of B_∞ are isomorphic to B_∞. This can be seen by noting that they are all atomless, or by noting that every finite non-empty disjoint union of $2^{\mathbb{N}}$ with itself is homeomorphic to $2^{\mathbb{N}}$.

Let $f_s : (B_\infty)^n \longrightarrow B_\infty$ be an isomorphism, $t_s := f_s^{-1}s$, and $g_s := 1_{B_\infty}$ (the identity function on B_∞). Then $f_s t_s g_s = f_s f_s^{-1} s g_s = s$, $f_s, g_s \in \mathrm{Pol}(B_\infty)$ and $t_s \in \mathrm{End}(B_\infty)$. Let B be a neighbourhood of t_s in $\mathrm{End}(B_\infty)$. Then

$$t_s \in \bigcap_{i=1}^{k}\{h \in \mathrm{Hom}(B_\infty^n, B_\infty) : (a_i)h = b_i\} \subseteq B,$$

for some $a_i, b_i \in B_\infty$ and finite k. Since

$$s \in \bigcap_{i=1}^{k}\{h \in \mathrm{Hom}(B_\infty^n, B_\infty) : ((a_i)f_s^{-1})h = b_i\} \subseteq f_s B g_s,$$

it follows that $f_s B g_s$ is a neighbourhood of s.

(ii). The monoid $\mathrm{End}(B_\infty)$ has automatic continuity with respect to the pointwise topology and the class of second countable topological semigroups; see Theorem 6.14. The proof of this part of the theorem then follows by a similar argument to Theorem 7.10(ii).

(iii). Suppose that $\mathrm{Pol}(B_\infty)$ is equipped with a second countable Hausdorff clone topology $\mathcal{T}$. By part (ii), $\mathcal{T}$ is contained in the usual pointwise topology.

It remains to show that $\mathcal{T}$ contains the pointwise topology. Stone duality translates: the pointwise topology on $\mathrm{Pol}(B_\infty)$ to the compact-open topology on $\mathrm{Pol}(B_\infty)^\dagger$ (defined in (22)); and the topology $\mathcal{T}$ to a second countable Hausdorff topology $\mathcal{T}^\dagger$ on $\mathrm{Pol}(B_\infty)^\dagger$ such that $\bullet_{m,n}$ is continuous with respect to $\mathcal{T}^\dagger$ for all $m, n \in \mathbb{N}$. Hence it suffices to show that the topology $\mathcal{T}^\dagger$ on $\mathrm{Pol}(B_\infty)^\dagger$ obtained by translating $\mathcal{T}$ via Stone duality contains the compact-open topology. In other words, it is enough to fix $n \in \mathbb{N}$ and show that the subspace topology on $C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$ induced by $\mathcal{T}^\dagger$ contains the compact-open topology. As in the proof of Theorem 7.10(iii), by Theorem 6.1, it suffices to show that $a_n : 2^{\mathbb{N}} \times C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\}) \longrightarrow 2^{\mathbb{N}} \times \{1, \ldots, n\}$ defined by

$$(p, f)a_n = (p)f$$

is continuous. Recall that $\pi_i : 2^{\mathbb{N}} \times \{1, \ldots, n\} \longrightarrow 2^{\mathbb{N}}$ denotes the ith projection and note that $(p, f)a_n = ((p)a_n\pi_1, (p)a_n\pi_2)$ for all $p \in 2^{\mathbb{N}}$ and $f \in C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$. We conclude the proof by showing that $a_n\pi_1$ and $a_n\pi_2$ are continuous.

By Proposition 6.9, if F denotes the set of constant maps of arity 1 in $C(2^{\mathbb{N}})$, and $\phi : F \longrightarrow 2^{\mathbb{N}}$ is the function that sends $g \in F$ to the unique value in $(2^{\mathbb{N}})g$, then ϕ is a homeomorphism. Let $1_{2^{\mathbb{N}}}$ be the identity function on $2^{\mathbb{N}}$. If $p \in 2^{\mathbb{N}}$ and $f \in C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$, then $(p)\phi^{-1}f\pi_1$ is the constant map on $2^{\mathbb{N}}$ with value $(p)f\pi_1$. Hence

$$(p, f)a_n\pi_1 = (p)f\pi_1 = (p)\phi^{-1}f\pi_1\phi = (p)\phi^{-1}f\pi_1 1_{2^{\mathbb{N}}}\phi = \left(((p)\phi^{-1}f) \bullet_{n,1} (1_{2^{\mathbb{N}}}, \ldots, 1_{2^{\mathbb{N}}})\right)\phi$$

for all $p \in 2^{\mathbb{N}}$ and $f \in C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$.

Suppose that $n \in \mathbb{N}$ and that

$$M_n : C(2^{\mathbb{N}}) \times C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\}) \longrightarrow C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$$

is defined by $(f, g)M_n = f \circ g$. Then $(f, g)M_n = f \circ g = \overline{f} \bullet_{1,n} (g)$ where $\overline{f} : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}} \times \{1\}$ is defined by $(p)\overline{f} = ((p)f, 1)$, and so the functions M_n are continuous with respect to $\mathcal{T}^{\dagger}$. Since ϕ^{-1} and M_n are continuous, the maps $(p, f) \mapsto (p)\phi^{-1}f$ and $(p, f) \mapsto (1_{2^{\mathbb{N}}}, \ldots, 1_{2^{\mathbb{N}}})$ are continuous. Moreover, $\bullet_{n,1}$ and ϕ are continuous, and so $a_n\pi_1$ is continuous also.

Let $q : 2^{\mathbb{N}} \longrightarrow \{1, \ldots, n\}$ be continuous and surjective, and let $c_m : 2^{\mathbb{N}} \longrightarrow 2^{\mathbb{N}}$ be any constant map with value $x_m \in (m)q^{-1}$ for all $m \in \{1, \ldots, n\}$. Then

$$(p, f)a_n\pi_2 = (p)f\pi_2 = x_{(p)f\pi_2}q = c_{(p)f\pi_2}\phi q = \left((p)\phi^{-1}f \bullet_{n,1} (c_1, \ldots, c_n)\right)\phi q$$

for all $p \in 2^{\mathbb{N}}$ and $f \in C(2^{\mathbb{N}}, 2^{\mathbb{N}} \times \{1, \ldots, n\})$. The proof that $a_n\pi_2$ is continuous is then similar to the proof that $a_n\pi_1$ is continuous. $\qquad\square$

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