Monoids of injective maps closed under conjugation by permutations*

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Abstract

Let Ω be a countably infinite set, $\mathrm{Inj}(\Omega)$ the monoid of all injective endomaps of Ω, and $\mathrm{Sym}(\Omega)$ the group of all permutations of Ω. We classify all submonoids of $\mathrm{Inj}(\Omega)$ that are closed under conjugation by elements of $\mathrm{Sym}(\Omega)$.

1 Introduction

Let Ω be a countably infinite set and $\mathrm{Sym}(\Omega)$ the group of all permutations of Ω. In 1933 Schreier and Ulam [8] showed that $\mathrm{Sym}(\Omega)$ has precisely four normal subgroups. This result was then generalized by Baer [1] to arbitrary sets in place of Ω. (See also [2] and [4] for other related results.) In this paper we generalize the Schreier-Ulam Theorem in a different direction, by classifying all the normal submonoids (i.e., ones that are closed under conjugation by elements of $\mathrm{Sym}(\Omega)$) of $\mathrm{Inj}(\Omega)$, the monoid of all injective endomaps of Ω. Unlike normal subgroups of $\mathrm{Sym}(\Omega)$, there are infinitely many (in fact, $2^{\aleph_0}$) normal submonoids of $\mathrm{Inj}(\Omega)$. However, it is possible to describe them.

Given a normal submonoid $M \subseteq \mathrm{Inj}(\Omega)$, our strategy will be to split M into the smaller semigroups M_{gp} (consisting of the permutations in M), M_{fin} (consisting of the elements $f \in M$ satisfying $1 \leq |\Omega \setminus (\Omega)f| < \aleph_0$), and M_∞ (consisting of the elements $f \in M$ such that $|\Omega \setminus (\Omega)f| = \aleph_0$). We shall then describe these three subsemigroups (each of which is also normal) individually. Even though our definition does not explicitly say that M_{gp} is closed under taking inverses, it turns out that this must be the case, and hence that M_{gp} must always be one of the four groups mentioned in the Schreier-Ulam Theorem. Further, M_∞ must either be empty or contain every element $f \in \mathrm{Inj}(\Omega)$ satisfying $|\Omega \setminus (\Omega)f| = \aleph_0$. The semigroup M_{fin} is more difficult to describe, and its structure depends on that of M_{gp}. But, roughly speaking, M_{fin} must either be of the form

$$\{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus \{0\}\},$$

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where N is an additive submonoid of the natural numbers, or be a slightly smaller subsemigroup of such a semigroup.

The main tool used in proving the result described above is the following theorem from [7]: given any three maps $f, g, h \in \mathrm{Inj}(\Omega) \setminus \mathrm{Sym}(\Omega)$, there exist permutations $a, b \in \mathrm{Sym}(\Omega)$ such that $h = afa^{-1}bgb^{-1}$ if and only if

$$|\Omega \setminus (\Omega)f| + |\Omega \setminus (\Omega)g| = |\Omega \setminus (\Omega)h|.$$

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2 Conjugation basics

We begin with some basic definitions and facts about conjugation of injective set maps. The results in this section are all easy and are discussed in detail in [7], so we omit their proofs here.

Let Ω be an arbitrary infinite set, $\mathrm{Inj}(\Omega)$ the monoid of all injective endomaps of Ω, and $\mathrm{Sym}(\Omega)$ the group of all permutations of Ω. We shall write set maps on the right of their arguments. The set of integers will be denoted by $\mathbb{Z}$, the set of positive integers will be denoted by $\mathbb{Z}_+$, the set of nonnegative integers will be denoted by $\mathbb{N}$, and the cardinality of a set Σ will be denoted by $|\Sigma|$.

Definition 1. *Let $f \in \mathrm{Inj}(\Omega)$ be any element, and let $\Sigma \subseteq \Omega$ be a nonempty subset. We shall say that Σ is a* cycle *under f if the following two conditions are satisfied:*

(i) *for all $\alpha \in \Omega$, $(\alpha)f \in \Sigma$ if and only if $\alpha \in \Sigma$;*

(ii) *Σ has no proper nonempty subset satisfying* (i).

We shall say that Σ is a forward cycle *under f if Σ is an infinite cycle under f and $\Sigma \setminus (\Omega)f \neq \emptyset$. If Σ is an infinite cycle under f that is not a forward cycle, we shall refer to it as an* open cycle.

It is easy to see that for any $\alpha \in \Omega$, the set

$$\{(\alpha)f^n : n \in \mathbb{N}\} \cup \{\beta \in \Omega : \exists n \in \mathbb{Z}_+ \, ((\beta)f^n = \alpha)\}$$

is a cycle under f. By condition (ii) above, it follows that every cycle of f is of this form. This also implies that that every $\alpha \in \Omega$ falls into exactly one cycle under f. Thus, we can define a collection $\{\Sigma_i\}_{i \in I}$ of disjoint subsets of Ω to be a *cycle decomposition* of f if each Σ_i is a cycle under f and $\bigcup_{i \in I} \Sigma_i = \Omega$. We note that f can have only one cycle decomposition, up to reindexing the cycles. For convenience, we shall therefore at times refer to *the* cycle decomposition of f.

Definition 2. *Let* $f, g \in \mathrm{Inj}(\Omega)$ *be any two elements. We shall say that* f *and* g *have* equivalent *cycle decompositions if there exist an indexing set* I *and cycle decompositions* $\{\Sigma_i\}_{i\in I}$ *and* $\{\Gamma_i\}_{i\in I}$ *of* f *and* g*, respectively, that satisfy the following two conditions:*

(i) *for all* $i \in I$, $|\Sigma_i| = |\Gamma_i|$;

(ii) *if* $|\Sigma_i| = |\Gamma_i| = \aleph_0$ *for some* $i \in I$*, then* Σ_i *is a forward cycle under* f *if and only if* Γ_i *is a forward cycle under* g.

As with permutations, we have the following fact.

Proposition 3 ([7, Proposition 3])**.** *Let* $f, g \in \mathrm{Inj}(\Omega)$ *be any two maps. Then* $g = afa^{-1}$ *for some* $a \in \mathrm{Sym}(\Omega)$ *if and only if* f *and* g *have equivalent cycle decompositions.*

The next two observations will also be useful in the sequel.

Lemma 4 ([7, Lemma 4])**.** *Let* $f \in \mathrm{Inj}(\Omega)$ *be any map. Then there is a one-to-one correspondence between the elements of* $\Omega \setminus (\Omega)f$ *and forward cycles in the cycle decomposition of* f.

Lemma 5 ([7, Lemma 5])**.** *Let* $f, g \in \mathrm{Inj}(\Omega)$ *be any two maps. Then*

$$|\Omega \setminus (\Omega)f| + |\Omega \setminus (\Omega)g| = |\Omega \setminus (\Omega)fg|.$$

We conclude this section by recalling another generalization of a familiar concept from group theory.

Definition 6. *We shall say that a subset* $M \subseteq \mathrm{Inj}(\Omega)$ *is* normal *if it is closed under conjugation by elements of* $\mathrm{Sym}(\Omega)$.

3 Some general considerations

From now on we shall assume that Ω is countable. Let $\mathrm{Inj}_{\mathrm{fin}}(\Omega) \subseteq \mathrm{Inj}(\Omega)$ denote the subset consisting of all elements f such that $1 \leq |\Omega \setminus (\Omega)f| < \aleph_0$, and let $\mathrm{Inj}_\infty(\Omega) \subseteq \mathrm{Inj}(\Omega)$ denote the subset consisting of all elements f such that $|\Omega \setminus (\Omega)f| = \aleph_0$. By Lemma 5, these two sets are subsemigroups of $\mathrm{Inj}(\Omega)$. Further, by Proposition 3 and Lemma 4, if $f, h \in \mathrm{Inj}(\Omega)$ are conjugate to each other, then $|\Omega \setminus (\Omega)f| = |\Omega \setminus (\Omega)h|$. Hence, $\mathrm{Inj}_{\mathrm{fin}}(\Omega)$ and $\mathrm{Inj}_\infty(\Omega)$ are normal. We shall not delve very deeply into its structure, but let us note that $\mathrm{Inj}_\infty(\Omega)$ is known as the *Baer-Levi semigroup* (of type $(\aleph_0, \aleph_0)$). See, for instance, [3] and [6] for more information about it.

Given a submonoid $M \subseteq \mathrm{Inj}(\Omega)$, we set $M_{\mathrm{gp}} = M \cap \mathrm{Sym}(\Omega)$, $M_{\mathrm{fin}} = M \cap \mathrm{Inj}_{\mathrm{fin}}(\Omega)$, and $M_\infty = M \cap \mathrm{Inj}_\infty(\Omega)$. Then $M = M_{\mathrm{gp}} \cup M_{\mathrm{fin}} \cup M_\infty$, and the union is disjoint. As intersections of semigroups, M_{gp}, M_{fin}, and M_∞ are semigroups.

Suppose that a submonoid $M \subseteq \mathrm{Inj}(\Omega)$ is normal. Then, by the above remarks, the same is true of M_{gp}, M_{fin}, and M_∞. In order to understand the structure of M, we shall try to understand the structures of these three "pieces" of M individually. In the case of M_{gp}, we can accomplish this task very quickly, by relying on the Schreier-Ulam Theorem. We require a little more notation in order to state it in full detail.

Definition 7. *Given a map* $g \in \mathrm{Sym}(\Omega)$, *the* support *of* g *is the set* $\{\alpha \in \Omega : (\alpha)g \neq \alpha\}$. *The subgroup of* $\mathrm{Sym}(\Omega)$ *consisting of all the elements having finite support will be denoted by* $\mathrm{Fin}(\Omega)$. *The elements of* $\mathrm{Fin}(\Omega)$ *are said to be* finitary. *Finally,* $\mathrm{Alt}(\Omega) \subseteq \mathrm{Fin}(\Omega)$ *will denote the alternating subgroup (consisting of even finitary permutations).*

Theorem 8 (Schreier and Ulam [8])**.** $\mathrm{Sym}(\Omega)$ *has precisely four normal subgroups, specifically,* $\{1\}$, $\mathrm{Alt}(\Omega)$, $\mathrm{Fin}(\Omega)$, *and* $\mathrm{Sym}(\Omega)$.

Lemma 9. *Let* $M \subseteq \mathrm{Inj}(\Omega)$ *be a normal submonoid. Then* $M_{\mathrm{gp}} = M \cap \mathrm{Sym}(\Omega)$ *is a normal subgroup of* $\mathrm{Sym}(\Omega)$.

Proof. As we have noted above, M_{gp} must be a normal submonoid of $\mathrm{Inj}(\Omega)$. Hence, it suffices to show that M_{gp} is closed under taking inverses. But, for any permutation $f \in \mathrm{Sym}(\Omega)$, it is easy to see that f and f^{-1} have equivalent cycle decompositions, and hence $f^{-1} = afa^{-1}$ for some $a \in \mathrm{Sym}(\Omega)$, by Proposition 3. Therefore, since M_{gp} is closed under conjugation, it is closed under taking inverses as well. □

By the Lemma 9 and Theorem 8, if $M \subseteq \mathrm{Inj}(\Omega)$ is a normal submonoid, then M_{gp} must be one of $\{1\}$, $\mathrm{Alt}(\Omega)$, $\mathrm{Fin}(\Omega)$, and $\mathrm{Sym}(\Omega)$.

Let us next recall a result mentioned in the Introduction that will play an important role throughout this note.

Theorem 10 ([7, Corollary 10])**.** *Let* $f, g, h \in \mathrm{Inj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ *be any three maps. Then there exist permutations* $a, b \in \mathrm{Sym}(\Omega)$ *such that* $h = afa^{-1}bgb^{-1}$ *if and only if*

$$|\Omega \setminus (\Omega)f| + |\Omega \setminus (\Omega)g| = |\Omega \setminus (\Omega)h|.$$

Corollary 11. *Let* $M \subseteq \mathrm{Inj}(\Omega)$ *be a normal submonoid. Then either* $M \cap \mathrm{Inj}_{\infty}(\Omega) = \emptyset$ *or* $\mathrm{Inj}_{\infty}(\Omega) \subseteq M$.

Proof. Suppose that there is an element $f \in M \cap \mathrm{Inj}_{\infty}(\Omega)$. Then, $afa^{-1}bfb^{-1} \in M$ for all $a, b \in \mathrm{Sym}(\Omega)$, since M is closed under conjugation and composition. Hence, $\mathrm{Inj}_{\infty}(\Omega) \subseteq M$, by the previous theorem. □

Let $M \subseteq \mathrm{Inj}(\Omega)$ be a normal submonoid. Theorem 8, Lemma 9, and Corollary 11 allow us to completely characterize M_{gp} and M_{∞}, and so it remains to explore M_{fin}. Unlike M_{∞}, the structure of M_{fin} depends on whether M_{gp} is $\{1\}$, $\mathrm{Alt}(\Omega)$, $\mathrm{Fin}(\Omega)$, or $\mathrm{Sym}(\Omega)$. We shall discuss these four cases individually in the four sections that follow. Then, in Section 8 we shall collect all those pieces for a complete description of the normal submonoids of $\mathrm{Inj}(\Omega)$. It is easiest to describe M_{fin} when $M_{\mathrm{gp}} = \mathrm{Sym}(\Omega)$, so we start there.

4 Monoids containing Sym(Ω)

We note that any submonoid of $\mathrm{Inj}(\Omega)$ that contains $\mathrm{Sym}(\Omega)$ is automatically normal. More generally, we have the following fact.

Lemma 12. *Let* $M \subseteq \mathrm{Inj}(\Omega)$ *be any submonoid that contains* $\mathrm{Sym}(\Omega)$, *and let* $f \in M$ *be any element. Suppose that* $g \in \mathrm{Inj}(\Omega)$ *satisfies* $|\Omega \setminus (\Omega)g| = |\Omega \setminus (\Omega)f|$. *Then* $g \in M$.

Proof. Since f and g are injective, the formula $((\alpha)f)h = (\alpha)g$ defines a bijection $h : (\Omega)f \to (\Omega)g$. Moreover, since $|\Omega \setminus (\Omega)f| = |\Omega \setminus (\Omega)g|$, we can extend h to a permutation of Ω, using any bijection $\Omega \setminus (\Omega)f \to \Omega \setminus (\Omega)g$. The desired conclusion now follows from the fact that $g = fh \in M$. □

Definition 13. *Given a subset* $M \subseteq \mathrm{Inj}(\Omega)$, *we define*

$$M_{\mathbb{N}} = \{|\Omega \setminus (\Omega)f| : f \in M\} \cap \mathbb{N}.$$

Also, given a subset $N \subseteq \mathbb{N}$, *we define*

$$S(N) = \{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus \{0\}\}.$$

By Lemma 5, $M_{\mathbb{N}}$ is a submonoid of the additive monoid $\mathbb{N}$ whenever M is a submonoid of $\mathrm{Inj}(\Omega)$, and $S(N)$ is a subsemigroup of $\mathrm{Inj}(\Omega)$ whenever N is a subsemigroup of $\mathbb{N}$. It is also easy to see that $S(N)$ is normal.

We can now quickly describe the submonoids of $\mathrm{Inj}(\Omega)$ that contain $\mathrm{Sym}(\Omega)$.

Proposition 14. *For any submonoid* $N \subseteq \mathbb{N}$, *both* $\mathrm{Sym}(\Omega) \cup S(N)$ *and* $\mathrm{Sym}(\Omega) \cup S(N) \cup \mathrm{Inj}_{\infty}(\Omega)$ *are submonoids of* $\mathrm{Inj}(\Omega)$.

Conversely, a submonoid $M \subseteq \mathrm{Inj}(\Omega)$ *that contains* $\mathrm{Sym}(\Omega)$ *must be either of the form* $\mathrm{Sym}(\Omega) \cup S(N)$ *or of the form* $\mathrm{Sym}(\Omega) \cup S(N) \cup \mathrm{Inj}_{\infty}(\Omega)$, *for some submonoid* $N \subseteq \mathbb{N}$. (*Specifically,* $N = M_{\mathbb{N}}$.)

Proof. By Lemma 5, for any submonoid $N \subseteq \mathbb{N}$, $S(N)$ is closed under multiplication by elements of $\mathrm{Sym}(\Omega)$, and $\mathrm{Inj}_{\infty}(\Omega)$ closed under multiplication by elements of $\mathrm{Sym}(\Omega) \cup S(N)$. The first claim now follows from the fact that $\mathrm{Sym}(\Omega)$, $S(N)$, and $\mathrm{Inj}_{\infty}(\Omega)$ are all subsemigroups of $\mathrm{Inj}(\Omega)$.

For the converse, let $M \subseteq \mathrm{Inj}(\Omega)$ be a submonoid containing $\mathrm{Sym}(\Omega)$. Then, by Lemma 12, an element $f \in \mathrm{Inj}_{\mathrm{fin}}(\Omega)$ is in M_{fin} if and only if $|\Omega \setminus (\Omega)f| \in M_{\mathbb{N}} \setminus \{0\}$. Hence, $M_{\mathrm{fin}} = S(M_{\mathbb{N}})$. By the same lemma (or, by Corollary 11), $M_{\infty} = M \cap \mathrm{Inj}_{\infty}(\Omega)$ must be either empty or all of $\mathrm{Inj}_{\infty}(\Omega)$. Thus, either $M = \mathrm{Sym}(\Omega) \cup S(M_{\mathbb{N}})$ or $M = \mathrm{Sym}(\Omega) \cup S(M_{\mathbb{N}}) \cup \mathrm{Inj}_{\infty}(\Omega)$. □

In the following section we shall discuss the next simplest case, of submonoids $M \subseteq \mathrm{Inj}(\Omega)$ such that $M \cap \mathrm{Sym}(\Omega) = \{1\}$.

5 Monoids with trivial groups of units

We shall require the following well-known observation about additive submonoids of $\mathbb{N}$.

Lemma 15. *Every additive submonoid* $N \subseteq \mathbb{N}$ *has a unique minimal generating set.*

Proof. This is clear if $N = \{0\}$, so we may assume that $N \neq \{0\}$. Let $\{G_i\}_{i \in I}$ be the set of all generating sets for N as a monoid. We shall show that $G = \bigcap_{i \in I} G_i$ is a generating set for N. (Note that G is necessarily nonempty, since it contains the least nonzero element of N.)

Suppose that G does not generate N. Let $n \in N \setminus \{0\}$ be the least element that is not in $\langle G \rangle$, the monoid generated by G. Then there must be some generating set G_i ($i \in I$) such that $n \notin G_i$. Hence $n = n_1 + \cdots + n_k$ for some $n_1, \ldots, n_k \in G_i$, since G_i is a generating set for N. We must necessarily have $n_1, \ldots, n_k < n$. By our choice of n, this implies that $n_1, \ldots, n_k \in \langle G \rangle$. Hence $n \in \langle G \rangle$; a contradiction. Therefore G generates all of N. □

We can thus make the following

Definition 16. *Given an additive submonoid $N \subseteq \mathbb{N}$, let $\mathrm{Gn}(N)$ denote the unique minimal generating set for N as a monoid.*

We note, in passing, that $\mathrm{Gn}(N)$ is always finite (e.g., see [5, Theorem 2.4(2)]). We are now ready to describe the normal submonoids of $\mathrm{Inj}(\Omega)$ that have no nontrivial units.

Proposition 17. *Let $M \subseteq \mathrm{Inj}(\Omega)$ be any submonoid such that $M \cap \mathrm{Sym}(\Omega) = \{1\}$. Then M is normal if and only if either $M = \{1\} \cup M_{\mathrm{fin}}$ or $M = \{1\} \cup M_{\mathrm{fin}} \cup \mathrm{Inj}_{\infty}(\Omega)$, where*

$$M_{\mathrm{fin}} = B \cup \{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\mathrm{Gn}(N) \cup \{0\})\},$$

for some additive submonoid $N \subseteq \mathbb{N}$, and normal subset $B \subseteq \mathrm{Inj}_{\mathrm{fin}}(\Omega)$ that satisfies $B_{\mathbb{N}} = \mathrm{Gn}(N)$.

Proof. Suppose that M is normal. Let $N = M_{\mathbb{N}}$, and set

$$B = \{f \in M : |\Omega \setminus (\Omega)f| \in \mathrm{Gn}(N)\}.$$

Then B is normal, since M is. Further, if $h \in \mathrm{Inj}(\Omega)$ is any element such that $|\Omega \setminus (\Omega)h| \in M_{\mathbb{N}} \setminus (\mathrm{Gn}(M_{\mathbb{N}}) \cup \{0\})$, then $h \in M$, by Theorem 10. Hence,

$$B \cup \{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\mathrm{Gn}(N) \cup \{0\})\} = M \cap \mathrm{Inj}_{\mathrm{fin}}(\Omega).$$

The desired conclusion then follows from Corollary 11.

For the converse, suppose that M_{fin} has the form specified in the statement. First, we note that this set is a subsemigroup of $\mathrm{Inj}(\Omega)$. For, given any two elements $f, g \in M_{\mathrm{fin}}$, we have

$$fg \in \{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\mathrm{Gn}(N) \cup \{0\})\},$$

by Lemma 5. As usual, this implies that M is a submonoid. Since B is normal, so is M_{fin}, considering that

$$\{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\mathrm{Gn}(N) \cup \{0\})\}$$

is always normal. It follows that M is normal as well. □

In the above statement we describe the structure of a normal submonoid of $\mathrm{Inj}(\Omega)$ in terms of a normal set B, which may, at first glance, seem not especially helpful. However, by Proposition 3, constructing such a set B simply amounts to picking any subset of $\mathrm{Inj}(\Omega)$ satisfying $B_{\mathbb{N}} = \mathrm{Gn}(N)$ and then adding to it all maps that have cycle decompositions equivalent to those of the maps already in B.

6 Monoids containing $\mathrm{Fin}(\Omega)$

Our next goal is to describe the normal submonoids $M \subseteq \mathrm{Inj}(\Omega)$ satisfying $M \cap \mathrm{Sym}(\Omega) = \mathrm{Fin}(\Omega)$. To accomplish this we shall first describe how composition with elements of $\mathrm{Fin}(\Omega)$ affects the cycle decomposition of an arbitrary element of $\mathrm{Inj}(\Omega)$.

The following notation will be convenient in the sequel.

Definition 18. *For each $f \in \mathrm{Inj}(\Omega)$ and $n \in \mathbb{Z}_+$ let*

$$(f)\mathrm{C}_n = |\{\Sigma \subseteq \Omega : \Sigma \text{ is a cycle under } f \text{ of cardinality } n\}|.$$

Similarly, let

$$(f)\mathrm{C}_{\mathrm{open}} = |\{\Sigma \subseteq \Omega : \Sigma \text{ is an open cycle under } f\}|$$

and

$$(f)\mathrm{C}_{\mathrm{fwd}} = |\{\Sigma \subseteq \Omega : \Sigma \text{ is a forward cycle under } f\}|.$$

By Proposition 3, two elements $f, g \in \mathrm{Inj}(\Omega)$ are conjugates of one another if and only if $(f)\mathrm{C}_{\mathrm{open}} = (g)\mathrm{C}_{\mathrm{open}}$, $(f)\mathrm{C}_{\mathrm{fwd}} = (g)\mathrm{C}_{\mathrm{fwd}}$, and $(f)\mathrm{C}_n = (g)\mathrm{C}_n$ for all $n \in \mathbb{Z}_+$. We shall also require a more general equivalence relation on elements of $\mathrm{Inj}(\Omega)$.

Definition 19. *Given any two maps $f, g \in \mathrm{Inj}(\Omega)$, let us write $f \approx_{\mathrm{fin}} g$ if the following four conditions are satisfied:*

(i) $(f)\mathrm{C}_{\mathrm{open}} = (g)\mathrm{C}_{\mathrm{open}}$;

(ii) $(f)\mathrm{C}_{\mathrm{fwd}} = (g)\mathrm{C}_{\mathrm{fwd}}$;

(iii) $(f)\mathrm{C}_n \neq (g)\mathrm{C}_n$ *for only finitely many* $n \in \mathbb{Z}_+$;

(iv) *if* $(f)\mathrm{C}_n \neq (g)\mathrm{C}_n$ *for some* $n \in \mathbb{Z}_+$, *then* $(f)\mathrm{C}_n$ *and* $(g)\mathrm{C}_n$ *are both finite.*

Further, we shall say that a subset $B \subseteq \mathrm{Inj}(\Omega)$ is $\approx_{\mathrm{fin}}$-closed if for all $f, g \in \mathrm{Inj}(\Omega)$ such that $f \approx_{\mathrm{fin}} g$, $f \in B$ if and only if $g \in B$.

Clearly, $\approx_{\mathrm{fin}}$ is an equivalence relation on $\mathrm{Inj}(\Omega)$. We shall prove that for $f, g \in \mathrm{Inj}(\Omega)$, each having at least one infinite cycle in its cycle decomposition, $f \approx_{\mathrm{fin}} g$ if and only if f is conjugate to $h_1 g h_2$ for some $h_1, h_2 \in \mathrm{Fin}(\Omega)$. The argument is divided into several steps.

Lemma 20. *Let $f \in \mathrm{Inj}(\Omega)$ and $h \in \mathrm{Fin}(\Omega)$ be any two maps. Then*

(i) $(f)\mathrm{C}_{\mathrm{open}} = (hf)\mathrm{C}_{\mathrm{open}}$, *and*

(ii) $(f)\mathrm{C}_{\mathrm{open}} = (fh)\mathrm{C}_{\mathrm{open}}$.

Proof. (i) Since h can be written as a product of transpositions, it is enough to show this in the case where h is a transposition. Further, since under this assumption $f = hhf$, it is enough to show that $(f)\mathrm{C}_{\mathrm{open}} \leq (hf)\mathrm{C}_{\mathrm{open}}$. This is clear if h fixes all the elements in the open cycles of f, so suppose that $(\alpha)h = \beta \neq \alpha$ for some $\alpha \in \Sigma$, where Σ is an open cycle of f. We consider several different cases.

Suppose that $\beta \in \Sigma$. Without loss of generality, we may assume that $\beta = (\alpha)f^n$ for some $n \in \mathbb{Z}_+$. Then $\Sigma \setminus \{(\alpha)f, (\alpha)f^2, \dots, (\alpha)f^n\}$ is an open cycle of hf. Since h is a transposition, all open cycles of f other than Σ are open cycles of hf. Hence $(f)\mathrm{C}_{\mathrm{open}} \leq (hf)\mathrm{C}_{\mathrm{open}}$.

Suppose instead that $\beta \notin \Sigma$, and let Γ be the cycle of f that contains β. If Γ is finite, then $\Sigma \cup \Gamma$ is an open cycle under hf. If Γ is a forward cycle, then

$$\{(\beta)f^n : n \in \mathbb{Z}_+\} \cup \{\gamma \in \Omega : \exists n \in \mathbb{N}\,((\gamma)f^n = \alpha)\}$$

is an open cycle under hf, in place of Σ. If Γ is an open cycle of f, then

$$\{(\beta)f^n : n \in \mathbb{Z}_+\} \cup \{\gamma \in \Omega : \exists n \in \mathbb{N}\,((\gamma)f^n = \alpha)\}$$

and

$$\{(\alpha)f^n : n \in \mathbb{Z}_+\} \cup \{\gamma \in \Omega : \exists n \in \mathbb{N}\,((\gamma)f^n = \beta)\}$$

are open cycles of hf, in place of Σ and Γ. Again, in each of these three cases, $(f)\mathrm{C}_{\mathrm{open}} \leq (hf)\mathrm{C}_{\mathrm{open}}$.

(ii) By part (i), we have $(f)\mathrm{C}_{\mathrm{open}} = (hf)\mathrm{C}_{\mathrm{open}}$. Also, Proposition 3 implies that $(hf)\mathrm{C}_{\mathrm{open}} = (h^{-1}(hf)h)\mathrm{C}_{\mathrm{open}}$. But, the latter is just $(fh)\mathrm{C}_{\mathrm{open}}$, which completes the proof. □

Corollary 21. *Let $f, g \in \mathrm{Inj}(\Omega)$ be any two maps, and suppose that $f = hh_1gh_2h^{-1}$ for some $h \in \mathrm{Sym}(\Omega)$ and $h_1, h_2 \in \mathrm{Fin}(\Omega)$. Then $f \approx_{\mathrm{fin}} g$.*

Proof. By Proposition 3, $f \approx_{\mathrm{fin}} h_1gh_2$. Thus, it suffices to show that $h_1gh_2 \approx_{\mathrm{fin}} g$.

By the previous lemma, $(g)\mathrm{C}_{\mathrm{open}} = (h_1g)\mathrm{C}_{\mathrm{open}} = (h_1gh_2)\mathrm{C}_{\mathrm{open}}$. Since $|\Omega \setminus (\Omega)g| = |\Omega \setminus (\Omega)h_1gh_2|$, Lemma 4 implies that $(g)\mathrm{C}_{\mathrm{fwd}} = (h_1gh_2)\mathrm{C}_{\mathrm{fwd}}$. The desired conclusion then follows from the fact that g and h_1gh_2 can disagree on only finitely many elements of Ω, which implies that these two maps must have the same finite cycles, except for possibly finitely many. □

Lemma 22. *Let $f \in \mathrm{Inj}(\Omega)$ be a map that has at least one infinite cycle in its cycle decomposition, and let $n \in \mathbb{Z}_+$. Then there exists a transposition $h \in \mathrm{Fin}(\Omega)$ such that $(fh)\mathrm{C}_n = (f)\mathrm{C}_n + 1$ and $(fh)\mathrm{C}_m = (f)\mathrm{C}_m$ for all $m \in \mathbb{Z}_+ \setminus \{n\}$.*

Proof. Let $\Sigma \subseteq \Omega$ be an infinite cycle of f. Then we can write $\Sigma = \{\alpha_i : i \in I\}$, where either $I = \mathbb{Z}$ or $I = \mathbb{Z}_+$, and $(\alpha_i)f = \alpha_{i+1}$ for all $i \in I$. Let us fix an element $\alpha_i \in \Sigma$ and define $h \in \mathrm{Fin}(\Omega)$ by $(\alpha_i)h = \alpha_{i+n}$, $(\alpha_{i+n})h = \alpha_i$, and $(\alpha)h = \alpha$ for all $\alpha \in \Omega \setminus \{\alpha_i, \alpha_{i+n}\}$. Then $\{\alpha_i, \dots, \alpha_{i+n-1}\}$ and $\Sigma \setminus \{\alpha_i, \dots, \alpha_{i+n-1}\}$ become cycles under fh, in place of Σ, and otherwise f and fh have the same cycle decomposition. □

Lemma 23. *Let $f \in \mathrm{Inj}(\Omega)$ be a map that has at least one infinite cycle in its cycle decomposition, and let $n \in \mathbb{Z}_+$ be such that $(f)\mathrm{C}_n > 0$. Then there exists a transposition $h \in \mathrm{Fin}(\Omega)$ such that $(fh)\mathrm{C}_n = (f)\mathrm{C}_n - 1$ and $(fh)\mathrm{C}_m = (f)\mathrm{C}_m$ for all $m \in \mathbb{Z}_+ \setminus \{n\}$.*

Proof. Let $\Sigma \subseteq \Omega$ be an infinite cycle and $\{\beta_1, \dots, \beta_n\}$ an n-cycle in the cycle decomposition of f. Let us fix an element $\alpha \in \Sigma$ and define $h \in \mathrm{Fin}(\Omega)$ by $(\alpha)h = \beta_1$, $(\beta_1)h = \alpha$, and $(\gamma)h = \gamma$ for all $\gamma \in \Omega \setminus \{\alpha, \beta_1\}$. Then $\Sigma \cup \{\beta_1, \dots, \beta_n\}$ becomes a cycle under fh, and otherwise f and fh have the same cycle decomposition. Thus, fh has one fewer n-cycle than f but the same finite cycles of other cardinalities. □

Putting together the last four results, we obtain our description of $\approx_{\text{fin}}$ in terms of composition with elements of $\text{Fin}(\Omega)$, for maps having infinite cycles.

Proposition 24. *Let $f, g \in \text{Inj}(\Omega)$ be any two elements, each having at least one infinite cycle in its cycle decomposition. Then $f \approx_{\text{fin}} g$ if and only if $f = hh_1gh_2h^{-1}$ for some $h \in \text{Sym}(\Omega)$ and $h_1, h_2 \in \text{Fin}(\Omega)$.*

Proof. By Corollary 21, we only need to show the forward implication, so let us assume that $f \approx_{\text{fin}} g$. Repeatedly applying the previous two lemmas, we can find a finite sequence of transpositions $h_1, \dots, h_n \in \text{Fin}(\Omega)$ such that $gh_1 \dots h_n$ has a cycle decomposition equivalent to that of f. (For any $h_1, \dots, h_n \in \text{Fin}(\Omega)$, we have $(gh_1 \dots h_n)\text{C}_{\text{open}} = (f)\text{C}_{\text{open}}$, by Lemma 20, and $(gh_1 \dots h_n)\text{C}_{\text{fwd}} = (f)\text{C}_{\text{fwd}}$, by Lemma 4.) The result then follows from Proposition 3. □

In the above proposition, the assumption that f and g both have an infinite cycle is necessary. For instance, let $f \in \text{Sym}(\Omega)$ be an element such that $(f)\text{C}_{\text{open}} = 0$ and $(f)\text{C}_n = 1$ for all $n \in \mathbb{Z}_+$, and let $g \in \text{Sym}(\Omega)$ be an element such that $(g)\text{C}_{\text{open}} = 0$, $(g)\text{C}_1 = 2$, and $(g)\text{C}_n = 1$ for all $n > 1$. Then, clearly, $f \approx_{\text{fin}} g$. But, $f \neq hh_1gh_2h^{-1}$ for all $h \in \text{Sym}(\Omega)$ and $h_1, h_2 \in \text{Fin}(\Omega)$. For, supposing otherwise, there exist $h_1, h_2 \in \text{Fin}(\Omega)$ such that h_1gh_2 has a cycle decomposition equivalent to that of f, by Proposition 3. Let us list the cycles of g as $\{\alpha_1\}, \{\alpha_2\}, \{\alpha_3, \alpha_4\}, \{\alpha_5, \alpha_6, \alpha_7\}, \{\alpha_8, \alpha_9, \alpha_{10}, \alpha_{11}\}, \dots$, where $\Omega = \{\alpha_i : i \in \mathbb{Z}_+\}$. Since g and h_1gh_2 can disagree on only finitely many elements of Ω, there is a positive integer n such that for all $i > n$ we have $(\alpha_i)g = (\alpha_i)h_1gh_2$, and such that α_{n+1} is the element with the least index in some cycle of g. Let us denote the cardinality of the cycle to which α_{n+1} belongs by m. Then, by our definition of f, $\{\alpha_1, \dots, \alpha_n\}$ must contain exactly one cycle of h_1gh_2 of each cardinality less than m and no other cycles. Comparing this with our cycle decomposition for g yields a contradiction (since $\{\alpha_1, \dots, \alpha_n\}$ contains two cycles of g of cardinality 1, in addition to a cycle of each cardinality less than m but greater than 1).

We are now ready to describe the normal submonoids $M \subseteq \text{Inj}(\Omega)$ having the property that $M \cap \text{Sym}(\Omega) = \text{Fin}(\Omega)$.

Proposition 25. *Let $M \subseteq \text{Inj}(\Omega)$ be any submonoid such that $M \cap \text{Sym}(\Omega) = \text{Fin}(\Omega)$. Then M is normal if and only if either $M = \text{Fin}(\Omega) \cup M_{\text{fin}}$ or $M = \text{Fin}(\Omega) \cup M_{\text{fin}} \cup \text{Inj}_\infty(\Omega)$, where*

$$M_{\text{fin}} = B \cup \{f \in \text{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\text{Gn}(N) \cup \{0\})\},$$

for some additive submonoid $N \subseteq \mathbb{N}$ and some $\approx_{\text{fin}}$-closed subset $B \subseteq \text{Inj}_{\text{fin}}(\Omega)$ that satisfies $B_{\mathbb{N}} = \text{Gn}(N)$.

Proof. This proof is very similar to that of Proposition 17.

Suppose that M is normal. Let $N = M_{\mathbb{N}}$, and set

$$B = \{f \in M : |\Omega \setminus (\Omega)f| \in \text{Gn}(N)\}.$$

Since M is normal and contains $\text{Fin}(\Omega)$, B is $\approx_{\text{fin}}$-closed, by Proposition 24, and it clearly satisfies $B_{\mathbb{N}} = \text{Gn}(N)$. Further, if $h \in \text{Inj}(\Omega)$ is any element such that $|\Omega \setminus (\Omega)h| \in M_{\mathbb{N}} \setminus (\text{Gn}(M_{\mathbb{N}}) \cup \{0\})$, then $h \in M$, by Theorem 10. Hence,

$$B \cup \{f \in \text{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\text{Gn}(N) \cup \{0\})\} = M \cap \text{Inj}_{\text{fin}}(\Omega).$$

The desired conclusion then follows from Corollary 11.

For the converse, suppose that M_{fin} has the form specified in the statement. First, we note that this set is a subsemigroup of $\text{Inj}(\Omega)$. For, given any two elements $f, g \in M_{\text{fin}}$, we have

$$fg \in \{f \in \text{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\text{Gn}(N) \cup \{0\})\},$$

by Lemma 5. Further, by the same lemma and Proposition 24, M_{fin} is closed under multiplication by elements of $\text{Fin}(\Omega)$, and, as always, $\text{Inj}_\infty(\Omega)$ is closed under multiplication by elements of $\text{Fin}(\Omega) \cup M_{\text{fin}}$. Therefore M is indeed a submonoid. Since $\text{Fin}(\Omega)$, B,

$$\{f \in \text{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\text{Gn}(N) \cup \{0\})\},$$

and $\text{Inj}_\infty(\Omega)$ are normal, it follows that M is as well. □

7 Monoids containing Alt(Ω)

This section is devoted to submonoids $M \subseteq \text{Inj}(\Omega)$ satisfying $M \cap \text{Sym}(\Omega) = \text{Alt}(\Omega)$. As in the previous section, we shall first describe how composition with elements of $\text{Alt}(\Omega)$ affects the cycle decomposition of an arbitrary element of $\text{Inj}(\Omega)$.

Lemma 26. *Let $h \in \text{Fin}(\Omega)$ be any element, and let $f \in \text{Inj}(\Omega)$ be a map that satisfies either of the following conditions:*

(i) $(f)\text{C}_{\text{open}} + (f)\text{C}_{\text{fwd}} \geq 2$;

(ii) $(f)\text{C}_{\text{open}} + (f)\text{C}_{\text{fwd}} \geq 1$ *and* $(f)\text{C}_n = \aleph_0$ *for some* $n \in \mathbb{Z}_+$.

Then there exists a map $g \in \text{Alt}(\Omega)$ such that fh and fg have equivalent cycle decompositions (and hence so do hf and gf).

Proof. If $h \in \text{Alt}(\Omega)$, then there is nothing to prove. Let us therefore assume that $h \in \text{Fin}(\Omega) \setminus \text{Alt}(\Omega)$. Then for any transposition $a \in \text{Fin}(\Omega)$, we have $ha \in \text{Alt}(\Omega)$. In both cases, we shall define $g = ha$, using an appropriate transposition a.

Now, assume that f satisfies (i). Then, by Corollary 21, fh must have at least two infinite cycles in its cycle decomposition. Let $\Sigma, \Gamma \subseteq \Omega$ be such (distinct) cycles, and let us pick $\sigma \in \Sigma$ and $\gamma \in \Gamma$ arbitrarily. Let $a \in \text{Fin}(\Omega)$ be the transposition that interchanges σ and γ, and fixes all other elements of Ω. Then $ha \in \text{Alt}(\Omega)$, and fha has a cycle decomposition equivalent to that of fh (by the same argument as in the proof of Lemma 20).

Next, assume that f satisfies (ii). Again, by Corollary 21, fh must have at least one infinite cycle in its cycle decomposition and satisfy $(fh)\text{C}_n = \aleph_0$. Thus, by Lemma 22, there exists a transposition $a \in \text{Fin}(\Omega)$ such that $(fha)\text{C}_n = (fh)\text{C}_n + 1$ and $(fha)\text{C}_m = (fh)\text{C}_m$ for all $m \in \mathbb{Z}_+ \setminus \{n\}$. Since $(fh)\text{C}_n = \aleph_0$, it follows (by Corollary 21, once more) that fha has a cycle decomposition equivalent to that of fh, as before.

The parenthetical statement follows from the fact that for any $f \in \text{Inj}(\Omega)$ and $h \in \text{Sym}(\Omega)$, hf and $fh = h^{-1}(hf)h$ have equivalent cycle decompositions, by Proposition 3. □

With the above lemma and the results of the previous section in mind, to describe the effect of composing elements of $\mathrm{Alt}(\Omega)$ with elements of $\mathrm{Inj}(\Omega) \setminus \mathrm{Sym}(\Omega)$ we only need to consider maps having exactly one infinite cycle and finitely many n-cycles for each $n \in \mathbb{Z}_+$. The following equivalence relation will help us accomplish the task.

Definition 27. *Given any two maps $f, g \in \mathrm{Inj}(\Omega)$, let us write $f \approx_{\text{even}} g$ if $f \approx_{\text{fin}} g$ and*

$$\sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (g)\mathrm{C}_n)$$

is an even integer. (Here $(f)\mathrm{C}_n - (g)\mathrm{C}_n$ is understood to be 0 whenever $(f)\mathrm{C}_n = (g)\mathrm{C}_n$, even if both cardinals are infinite.)

Further, we shall say that a subset $B \subseteq \mathrm{Inj}(\Omega)$ is $\approx_{\text{even}}$-closed if for all $f, g \in \mathrm{Inj}(\Omega)$ such that $f \approx_{\text{even}} g$, $f \in B$ if and only if $g \in B$.

Lemma 28. *The binary relation $\approx_{\text{even}}$ on elements of $\mathrm{Inj}(\Omega)$ is an equivalence relation.*

Proof. It is clear that $\approx_{\text{even}}$ is reflexive and symmetric. Let us then suppose that $f \approx_{\text{even}} g$ and $g \approx_{\text{even}} h$ for some $f, g, h \in \mathrm{Inj}(\Omega)$, and show that $f \approx_{\text{even}} h$. Since $\approx_{\text{fin}}$ is an equivalence relation, we only need to show that the integer

$$\sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (h)\mathrm{C}_n)$$

is even. Let $I \subseteq \mathbb{Z}_+$ be a finite set such that for all $n \in \mathbb{Z}_+ \setminus I$, $(f)\mathrm{C}_n = (g)\mathrm{C}_n = (h)\mathrm{C}_n$. (Such a set exists because $f \approx_{\text{fin}} g \approx_{\text{fin}} h$.) Computing modulo 2, we have

$$0 \equiv \sum_{n\in\mathbb{Z}_+} ((g)\mathrm{C}_n - (h)\mathrm{C}_n) = \sum_{n\in I} ((g)\mathrm{C}_n - (h)\mathrm{C}_n) = \sum_{n\in I} (g)\mathrm{C}_n - \sum_{n\in I} (h)\mathrm{C}_n.$$

Hence

$$\sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (h)\mathrm{C}_n) = \sum_{n\in I} (f)\mathrm{C}_n - \sum_{n\in I} (h)\mathrm{C}_n \equiv \sum_{n\in I} (f)\mathrm{C}_n - \sum_{n\in I} (g)\mathrm{C}_n$$
$$= \sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (g)\mathrm{C}_n) \equiv 0.$$

□

We shall prove that for $f, g \in \mathrm{Inj}(\Omega)$, each having exactly one infinite cycle and finitely many n-cycles for each $n \in \mathbb{Z}_+$, $f \approx_{\text{even}} g$ if and only if f is conjugate to $h_1 g h_2$ for some $h_1, h_2 \in \mathrm{Alt}(\Omega)$. The argument proceeds through three lemmas.

Lemma 29. *Let $f, g \in \mathrm{Inj}(\Omega)$ be any two maps, each having at least one infinite cycle in its cycle decomposition. If $f \approx_{\text{even}} g$, then $f = hgh'h^{-1}$ for some $h \in \mathrm{Sym}(\Omega)$ and $h' \in \mathrm{Alt}(\Omega)$.*

Proof. Suppose that $f \approx_{\text{even}} g$. Repeatedly applying Lemmas 22 and 23, we can find a finite sequence $h_1, \dots, h_m \in \mathrm{Fin}(\Omega)$ of transpositions such that $gh_1 \dots h_m$ has a cycle decomposition equivalent to that of f. Since

$$\sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (g)\mathrm{C}_n)$$

is an even integer, we can pick $h_1, \dots, h_m$ so that m is even as well, and hence $h' = h_1 \dots h_m \in \mathrm{Alt}(\Omega)$. The statement then follows from Proposition 3. □

By Lemma 26, the converse of the above lemma is generally false. However, we shall prove (in Proposition 32) that it holds for maps having exactly one infinite cycle and finitely many n-cycles for each $n \in \mathbb{Z}_+$.

Lemma 30. *Let $f \in \mathrm{Inj}(\Omega)$ be a map that satisfies $(f)\mathrm{C}_{\mathrm{open}} + (f)\mathrm{C}_{\mathrm{fwd}} = 1$ and $(f)\mathrm{C}_n < \aleph_0$ for all $n \in \mathbb{Z}_+$, and let $h \in \mathrm{Fin}(\Omega) \setminus \{1\}$ be a transposition. Then*

$$\sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (fh)\mathrm{C}_n) = \sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (hf)\mathrm{C}_n) \in \{-1, 1\}.$$

Proof. As noted before, fh and hf must have equivalent cycle decompositions, and hence the two sums above must always be equal. Thus it suffices to prove that the former, which we shall denote by A from now on, is either -1 or 1. Let $\Sigma \subseteq \Omega$ be the infinite cycle in the cycle decomposition of f. Then we can write $\Sigma = \{\alpha_i\}_{i\in I}$, where I is either $\mathbb{Z}$ or $\mathbb{Z}_+$, and $(\alpha_i)f = \alpha_{i+1}$ for all $i \in I$. We shall consider a number of different cases.

Suppose that h interchanges some α_i and α_{i+n} $(n > 0)$. Then $\{\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+n-1}\}$ is a finite cycle and $\Sigma \setminus \{\alpha_i, \alpha_{i+1}, \ldots, \alpha_{i+n-1}\}$ is an infinite cycle under fh. Thus f and fh have equal numbers of all types of cycles, except fh has one additional cycle of cardinality n (given that $(f)\mathrm{C}_n < \aleph_0$). Therefore $A = -1$.

Next, suppose that $\Gamma = \{\beta_0, \ldots, \beta_{m-1}\}$ is a finite cycle under f (with $m > 1$), where $(\beta_i)f = \beta_{i+1}$ (mod m), and that h interchanges β_0 and β_j $(0 < j \leq m-1)$. Then f and fh have the same cycles, except in place of $\{\beta_0, \ldots, \beta_{m-1}\}$, fh has the two cycles $\{\beta_0, \ldots, \beta_{j-1}\}$ and $\{\beta_j, \ldots, \beta_{m-1}\}$. Hence, compared to f, fh has one fewer cycle of cardinality m, one more cycle of cardinality j, and one more cycle of cardinality $m - j$. Therefore $A = -1$.

Now, let Σ and Γ be as before (though now m is allowed to be 1), and suppose that h interchanges β_0 and some α_i. Then $\Sigma \cup \Gamma$ is a cycle under fh, but otherwise fh has the same cycles as f. Hence fh has one fewer cycle of cardinality m than f, and therefore $A = 1$.

Finally, suppose that $\Gamma = \{\beta_0, \ldots, \beta_{m-1}\}$ and $\Delta = \{\delta_0, \ldots, \delta_{n-1}\}$ are distinct finite cycles under f (where $m, n \geq 1$, $(\beta_i)f = \beta_{i+1}$ (mod m), and $(\delta_i)f = \delta_{i+1}$ (mod n)), and that h interchanges β_0 and δ_0. Then $\Gamma \cup \Delta$ is a cycle under fh, but otherwise fh has the same cycles as f. Hence, compared to f, fh has one fewer cycle of cardinality m, one fewer cycle of cardinality n, and one more cycle of cardinality $m + n$. Therefore $A = 1$.

In all cases the sum A is either 1 or -1, as claimed. □

Lemma 31. *Let $f \in \mathrm{Inj}(\Omega)$ be a map that satisfies $(f)\mathrm{C}_{\mathrm{open}} + (f)\mathrm{C}_{\mathrm{fwd}} = 1$ and $(f)\mathrm{C}_n < \aleph_0$ for all $n \in \mathbb{Z}_+$, and let $h \in \mathrm{Alt}(\Omega)$ be any map. Then $fh \approx_{\mathrm{even}} f \approx_{\mathrm{even}} hf$.*

Proof. We shall only prove the first equivalence. By Lemma 28, it is enough to show this in the case where $h = h_1h_2$ for some transpositions h_1 and h_2. By Proposition 24, $f \approx_{\mathrm{fin}} fh$. Let $I \subseteq \mathbb{Z}_+$ be a finite set such that for all $n \in \mathbb{Z}_+ \setminus I$, $(f)\mathrm{C}_n = (fh_1)\mathrm{C}_n = (fh)\mathrm{C}_n$. Modulo 2, we have

$$\sum_{n\in I}((f)\mathrm{C}_n - (fh_1)\mathrm{C}_n) \equiv 1 \equiv \sum_{n\in I}((fh_1)\mathrm{C}_n - (fh_1h_2)\mathrm{C}_n),$$

by Lemma 30. Hence,

$$\sum_{n\in\mathbb{Z}_+} ((f)\mathrm{C}_n - (fh)\mathrm{C}_n) = \sum_{n\in I}((f)\mathrm{C}_n - (fh)\mathrm{C}_n) \equiv \sum_{n\in I}(f)\mathrm{C}_n - (\sum_{n\in I}(fh_1)\mathrm{C}_n - 1) \equiv 0.$$

Thus, $fh \approx_{\mathrm{even}} f$, by Definition 27. □

Combining the last three lemmas, we obtain a description of $\approx_{\text{even}}$ in terms of composition with members of $\text{Alt}(\Omega)$, for elements of $\text{Inj}(\Omega)$ that do not satisfy the hypotheses of Lemma 26.

Proposition 32. *Let $f, g \in \text{Inj}(\Omega)$ be any two maps, each having exactly one infinite cycle in its cycle decomposition, and satisfying $(f)\text{C}_n, (g)\text{C}_n < \aleph_0$ for all $n \in \mathbb{Z}_+$. Then $f \approx_{\text{even}} g$ if and only if $f = hh_1gh_2h^{-1}$ for some $h \in \text{Sym}(\Omega)$ and $h_1, h_2 \in \text{Alt}(\Omega)$.*

Proof. The forward implication was proved in Lemma 29. For the converse, let us suppose that $f = hh_1gh_2h^{-1}$ for some $h \in \text{Sym}(\Omega)$ and $h_1, h_2 \in \text{Alt}(\Omega)$. By Proposition 3, it suffices to show that $h_1gh_2 \approx_{\text{even}} g$. But, this follows from the previous lemma (and Lemma 28). □

We are, at last, in a position to describe the normal submonoids of $\text{Inj}(\Omega)$ that have $\text{Alt}(\Omega)$ as the group of units.

Proposition 33. *Let $M \subseteq \text{Inj}(\Omega)$ be any submonoid such that $M \cap \text{Sym}(\Omega) = \text{Alt}(\Omega)$. Then M is normal if and only if either $M = \text{Alt}(\Omega) \cup M_{\text{fin}}$ or $M = \text{Alt}(\Omega) \cup M_{\text{fin}} \cup \text{Inj}_\infty(\Omega)$, where*

$$M_{\text{fin}} = B \cup \{f \in \text{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\text{Gn}(N) \cup \{0\})\},$$

for some additive submonoid $N \subseteq \mathbb{N}$ and subset $B \subseteq \text{Inj}_{\text{fin}}(\Omega)$ that satisfies the following conditions:

(i) $B_{\mathbb{N}} = \text{Gn}(N)$;

(ii) $\{f \in B : (f)\text{C}_{\text{open}} + (f)\text{C}_{\text{fwd}} \geq 2\}$ *and* $\{f \in B : \exists n \in \mathbb{Z}_+ ((f)\text{C}_n = \aleph_0)\}$ *are* $\approx_{\text{fin}}$*-closed;*

(iii) $\{f \in B : (f)\text{C}_{\text{open}} + (f)\text{C}_{\text{fwd}} = 1$ *and* $\forall n \in \mathbb{Z}_+ ((f)\text{C}_n < \aleph_0)\}$ *is* $\approx_{\text{even}}$*-closed.*

Proof. Again, this proof is very similar to those of Propositions 17 and 25.

Suppose that M is normal. Let $N = M_{\mathbb{N}}$, and set

$$B = \{f \in M : |\Omega \setminus (\Omega)f| \in \text{Gn}(N)\}.$$

Then B clearly satisfies (i); it satisfies (ii), by Proposition 24 and Lemma 26; and it satisfies (iii), by Proposition 32. Further, if $h \in \text{Inj}(\Omega)$ is any element such that $|\Omega \setminus (\Omega)h| \in M_{\mathbb{N}} \setminus (\text{Gn}(M_{\mathbb{N}}) \cup \{0\})$, then $h \in M$, by Theorem 10. Hence,

$$B \cup \{f \in \text{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\text{Gn}(N) \cup \{0\})\} = M \cap \text{Inj}_{\text{fin}}(\Omega).$$

The desired conclusion then follows from Corollary 11.

For the converse, suppose that M_{fin} has the form specified in the statement. By Propositions 24 and 32, B is normal and is closed under multiplication by elements of $\text{Alt}(\Omega)$. Since $\text{Alt}(\Omega)$,

$$\{f \in \text{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\text{Gn}(N) \cup \{0\})\},$$

and $\text{Inj}_\infty(\Omega)$ are also normal, it follows that M is as well. By the usual argument, it is easy to see that M must be a submonoid. □

8 Main result

Putting together the remarks made in Section 3 with Propositions 14, 17, 25 and 33, we obtain a classification of all the normal submonoids of $\mathrm{Inj}(\Omega)$. Some of the conditions are phrased differently in the theorem below than in the aforemetioned propositions, in order to make the statement more self-contained.

Theorem 34. *Let $M \subseteq \mathrm{Inj}(\Omega)$ be a normal submonoid. Then $M = M_{\mathrm{gp}} \cup M_{\mathrm{fin}} \cup M_{\infty}$, where*

(1) $M_{\mathrm{gp}} \in \{\{1\}, \mathrm{Alt}(\Omega), \mathrm{Fin}(\Omega), \mathrm{Sym}(\Omega)\}$;

(2)
$$M_{\mathrm{fin}} = B \cup \{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in N \setminus (\mathrm{Gn}(N) \cup \{0\})\}$$

for some additive submonoid $N \subseteq \mathbb{N}$, with minimal generating set $\mathrm{Gn}(N)$, and normal subset $B \subseteq \mathrm{Inj}(\Omega)$ that satisfies

$$\{|\Omega \setminus (\Omega)f| : f \in B\} = \mathrm{Gn}(N);$$

(3) $M_{\infty} \in \{\emptyset, \mathrm{Inj}_{\infty}(\Omega)\}$.

If $M_{\mathrm{gp}} \neq \{1\}$, then B must satisfy additional hypotheses, as follows.

If $M_{\mathrm{gp}} = \mathrm{Alt}(\Omega)$, then

(i) *the subset of B consisting of maps having at least two infinite cycles or infinitely many cycles of a particular finite cardinality in their cycle decompositions is $\approx_{\mathrm{fin}}$-closed (see Definition 19 for the notation $\approx_{\mathrm{fin}}$);*

(ii) *the subset of B consisting of maps having exactly one infinite cycle and finitely many cycles of each finite cardinality in their cycle decompositions is $\approx_{\mathrm{even}}$-closed (see Definition 27 for the notation $\approx_{\mathrm{even}}$).*

If $M_{\mathrm{gp}} = \mathrm{Fin}(\Omega)$, then B is $\approx_{\mathrm{fin}}$-closed.

If $M_{\mathrm{gp}} = \mathrm{Sym}(\Omega)$, then

$$B = \{f \in \mathrm{Inj}(\Omega) : |\Omega \setminus (\Omega)f| \in \mathrm{Gn}(N)\}.$$

Conversely, if $M = M_{\mathrm{gp}} \cup M_{\mathrm{fin}} \cup M_{\infty}$, where M_{gp} satisfies (1), M_{fin} satisfies (2), M_{∞} satisfies (3), and B satisfies the appropriate conditions above (depending on the form of M_{gp}), then M is a normal submonoid of $\mathrm{Inj}(\Omega)$.

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