# Endomorphism rings generated using small numbers of elements

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**Abstract**

Let $R$ be a ring, $M$ a nonzero left $R$-module, and $\Omega$ an infinite set, and set $E = \mathrm{End}_R(\bigoplus_\Omega M)$. Given two subrings $S_1, S_2 \subseteq E$, write $S_1 \approx S_2$ if there exists a finite subset $U \subseteq E$ such that $\langle S_1 \cup U \rangle = \langle S_2 \cup U \rangle$. We show that if $M$ is simple and $\Omega$ is countable, then the subrings of $E$ that are closed in the function topology and contain the diagonal subring of $E$ (consisting of endomorphisms that take each copy of $M$ to itself) fall into exactly two equivalence classes, with respect to the equivalence relation above. We also show that every countable subset of $E$ is contained in a 2-generator subsemigroup of $E$.

## 1 Introduction

Let $R$ be a ring, $M$ a nonzero left $R$-module, $\Omega$ an infinite set, $N = \bigoplus_\Omega M$, and $E = \mathrm{End}_R(N)$. In this paper we will show that the ring $E$ has some unusual properties that are analogous to known properties of the symmetric group of an infinite set. In the next section we will demonstrate that every countable subset of $E$ is contained in a 2-generator subring of $E$. (The proof also works if $N$ is taken to be $\prod_\Omega M$.) This parallels Galvin's result that every countable subset of the symmetric group of an infinite set is contained in a 2-generator subgroup (cf. [3]). As an immediate corollary, we will show that every countable ring can be embedded in a ring generated by two elements, reproducing a result of Maltsev (cf. [9] and [10] for different proofs). The group-theoretic analog of this fact has also been known for a long time (cf. [4] and [11]).

Actually, our proof of the above result shows that a countable subset of $E$ is contained in a 2-generator *subsemigroup* of $E$. This is a generalization of the result of Magill in [6] that every countable set of endomorphisms of an infinite-dimensional vector space is contained in a 2-generator subsemigroup of the semigroup of all endomorphisms of that vector space (see also [1] for a shorter proof).

Given two subrings $S_1, S_2 \subseteq E$, we will say that $S_1 \approx S_2$ if there exists a finite subset $U \subseteq E$ such that $\langle S_1 \cup U \rangle = \langle S_2 \cup U \rangle$. We will devote the remainder of the paper to exploring properties of this equivalence relation. In particular, we will show that if $M$ is finitely generated, $\mathrm{End}_R(M)$ is a simple ring (e.g., if $M$ is a simple module), $\Omega$ is countable, and $S \subseteq E$ is a subring that is closed in the function topology and contains $D$, the diagonal subring of $E$ (consisting of endomorphisms that take each copy of $M$ to itself), then either $S \approx D$ or $S \approx E$ (but not both). This is in the spirit the result of Bergman and Shelah in [2] that the subgroups of the symmetric group of a countably infinite set that are closed in the function topology fall into exactly four equivalence classes. (There the equivalence relation is defined the same way as the relation above, with subgroups in place of subrings.)

Along the way, we will also note a natural way of associating to every preordering $\rho$ on $\Omega$ a subring $E(\rho)$ of $E$. We will then show that if $\Omega$ is countable and $M$ is finitely generated, then the subrings of the form $E(\rho)$ fall into exactly two equivalence classes (again, represented by $D$ and $E$). The result mentioned in the previous paragraph is actually a special case of this.

A curious example is that if we view $E$ as a ring of row-finite matrices over $\mathrm{End}_R(M)$, then the subring of upper-triangular matrices is equivalent to $E$, while the subring of lower-triangular matrices is equivalent to $D$.

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## 2 Countable sets of endomorphisms

Let $R$ denote a unital associative ring, $M$ a nonzero left $R$-module, and $\Omega$ an infinite set. $N$ will denote either $\bigoplus_{\alpha\in\Omega} M_\alpha$ or $\prod_{\alpha\in\Omega} M_\alpha$ (the arguments in this section work under either interpretation), where each $M_\alpha = M$. We will write $E$ to denote $\mathrm{End}_R(N)$. Endomorphisms will be written on the right of their arguments. Also, given a subset $\Sigma \subseteq \Omega$, we will write $M^\Sigma$ for the $R$-submodule of $N$ consisting of elements $(n_\alpha)_{\alpha\in\Omega}$ with $n_\alpha = 0$ for all $\alpha \notin \Sigma$, and $\pi_\Sigma$ for the projection from $N$ to $M^\Sigma$ along $M^{\Omega\setminus\Sigma}$, so in particular, $N = M^\Sigma \oplus M^{\Omega\setminus\Sigma}$. Finally, $\mathbb{Z}^+$ will denote the set of positive integers, and if $\Gamma$ is a set, $|\Gamma|$ will denote the cardinality of $\Gamma$.

The following argument was obtained by tinkering with the proofs of Theorem 2.6 in [6] and Theorem 3.1 in [3].

**Theorem 1.** *Every countable subset of $E = \mathrm{End}_R(N)$ is contained in a 2-generator subsemigroup of $E$ (viewed as a multiplicative semigroup).*

*Proof.* We may assume that $\Omega = \mathbb{Z} \times \Gamma$, where $|\Gamma| = |\Omega|$. Let us set $\Sigma = \{0\} \times \Gamma$. Also, let $g_1 \in E$ be an endomorphism that takes $N$ isomorphically to $M^\Sigma$, and let $g_2 \in E$ be the right inverse of $g_1$ that takes $M^\Sigma$ isomorphically to $N$ and takes $M^{\Omega\setminus\Sigma}$ to zero. We note that $g_2 g_1 = \pi_\Sigma$.

Now, let $U \subseteq E$ be a countably infinite subset. We will show that $U$ is contained in a 2-generator subsemigroup of $E$. Since $E = g_1\pi_\Sigma E \pi_\Sigma g_2$, we can find a subset $\bar{U} \subseteq \pi_\Sigma E \pi_\Sigma$ such that $U = g_1 \bar{U} g_2$. Since $\bar{U}$ is countable, we can write $\bar{U} = \{u_i : i \in \mathbb{Z}\}$. For each $i \in \mathbb{Z}$ let us define $\hat{u}_i \in \mathrm{End}_R(M^{\{i\}\times\Gamma})$ so that $\hat{u}_i$ acts on $M^{\{i\}\times\Gamma}$ as $u_i$ acts on $M^\Sigma$ (upon identifying $M_{(i,\,\gamma)}$ with $M_{(0,\,\gamma)}$ for each $\gamma \in \Gamma$). Also, let $g_3 \in E$ be an endomorphism such that for each $i \in \mathbb{Z}$ the restriction of $g_3$ to $M^{\{i\}\times\Gamma}$ is $\hat{u}_i$. Finally, let $g_4 \in E$ be the automorphism that takes $M_{(i,\,\gamma)}$ identically to $M_{(i+1,\,\gamma)}$ for each $i \in \mathbb{Z}$ and $\gamma \in \Gamma$. Then for each $i \in \mathbb{Z}$, we have $u_i = \pi_\Sigma g_4^i g_3 g_4^{-i} = g_2 g_1 g_4^i g_3 g_4^{-i}$. Writing $g_5 = g_4^{-1}$ and recalling that $g_2$ is a right inverse of $g_1$, we conclude that $U = \{g_1 g_4^i g_3 g_5^i g_2 : i \in \mathbb{Z}\}$.

It remains to be shown that $\{g_1, g_2, g_3, g_4, g_5\}$ is contained in a 2-generator subsemigroup of $E$. Write $\Omega$ as $\bigcup_{i=1}^{7} \Sigma_i$, where the union is disjoint, and $|\Sigma_i| = |\Omega|$ for $i \in \{1, 2, \ldots, 7\}$. Also, write $\Delta = \bigcup_{i=2}^{7} \Sigma_i$. Now, let us choose an endomorphism $f_1 \in E$ such that

(1) $f_1$ takes $M^{\Sigma_i}$ isomorphically to $M^{\Sigma_{i+1}}$ for $i \in \{1, 2, \ldots, 5\}$, and takes $M^{\Sigma_6 \cup \Sigma_7}$ isomorphically to $M^{\Sigma_7}$.

Such an endomorphism will necessarily map $N$ isomorphically to $M^{\Delta}$. Let $f_2 \in E$ be an endomorphism such that

(2) $f_2$ takes $N$ isomorphically to $M^{\Sigma_1}$.

Also, let us define $t_i \in \mathrm{Hom}_R(M^{\Sigma_1}, M^{\Sigma_{i+1}})$ ($i \in \{1, 2, \ldots, 5\}$) so that

(3) $t_i$ is the restriction of $f_1^i$ to $M^{\Sigma_1}$.

Then $t_i^{-1} f_2^{-1} g_i \in \mathrm{Hom}_R(M^{\Sigma_{i+1}}, N)$ for $i \in \{1, 2, \ldots, 5\}$, where $t_i^{-1}, f_2^{-1} \in E$ are right inverses of $t_i$ and $f_2$, respectively. Writing

(4) $t_6 = f_1^6$,

we see that $t_6$ is an isomorphism from $N$ to $M^{\Sigma_7}$ and $t_6^{-1} f_2 \in \mathrm{Hom}_R(M^{\Sigma_7}, N)$. Finally, let $f_3 \in E$ be an endomorphism such that

(5) $f_3$ restricted to $M^{\Sigma_{i+1}}$ is $t_i^{-1} f_2^{-1} g_i$ if $i \in \{1, 2, \ldots, 5\}$, and $t_6^{-1} f_2$ if $i = 6$.

Then $g_i = f_1^6 f_3 f_1^i f_3$ for $i \in \{1, 2, \ldots, 5\}$, and therefore $\{g_1, g_2, g_3, g_4, g_5\}$ is contained in the subsemigroup generated by $f_1$ and $f_3$. □

The following was originally proved by Maltsev.

**Corollary 2.** *Every countable ring can be embedded in a ring generated by two elements, using an embedding that respects central elements.*

*Proof.* Let $S$ be a countable ring and $\Omega$ an infinite set. Then $S$ embeds diagonally in $\mathrm{End}_S(\bigoplus_\Omega S)$. Thus, by the previous theorem, the image of $S$ is contained in a 2-generator subring of $\mathrm{End}_S(\bigoplus_\Omega S)$. It is clear that the diagonal embedding maps the center of $S$ into the center of $\mathrm{End}_S(\bigoplus_\Omega S)$. □

# 3 Equivalence classes

In this section we will keep $R$, $M$, $\Omega$, and $E$ as above, but restrict our attention to the case $N = \bigoplus_\Omega M$. However, we begin with two definitions applicable to an arbitrary ring.

**Definition 3.** *Let $S$ be a ring, $\kappa$ an infinite cardinal, and $S_1$, $S_2$ subrings of $S$. We will write $S_1 \preccurlyeq_{\kappa,S} S_2$ if there exists a subset $U \subseteq S$ of cardinality $< \kappa$ such that $S_1 \subseteq \langle S_2 \cup U \rangle$, the subring of $S$ generated by $S_2 \cup U$. If $S_1 \preccurlyeq_{\kappa,S} S_2$ and $S_2 \preccurlyeq_{\kappa,S} S_1$, we will write $S_1 \approx_{\kappa,S} S_2$, while if $S_1 \preccurlyeq_{\kappa,S} S_2$ and $S_2 \not\preccurlyeq_{\kappa,S} S_1$, we will write $S_1 \prec_{\kappa,S} S_2$. The subscripts $S$ and $\kappa$ will be omitted when their values are clear from the context.*

It is easy to see that $\preccurlyeq_{\kappa,S}$ is a preorder on subrings of $S$, and hence $\approx_{\kappa,S}$ is an equivalence relation. This equivalence relation and the results of this section are modeled on those in [2], where Bergman and Shelah define an analogous relation for groups and classify into equivalence classes the subgroups of the group of permutations of a countably infinite set that are closed in the function topology. Properties of such an equivalence relation defined for submonoids of the monoid of self-maps of an infinite set are investigated in [8].

**Definition 4.** *Let $S$ be a ring. Then the* cofinality $c(S)$ *of $S$ is the least cardinal $\kappa$ such that $S$ can be expressed as the union of an increasing chain of $\kappa$ proper subrings.*

Cofinality can be defined analogously for any algebra (in the sense of universal algebra). It has received much attention in the literature in connection with permutation groups. In particular, Macpherson and Neumann show in [5] that $c(\mathrm{Sym}(\Omega)) > |\Omega|$, where $\mathrm{Sym}(\Omega)$ is the group of all permutations of an infinite set $\Omega$. It is shown in [7] that the ring $E$ likewise satisfies $c(E) > |\Omega|$.

**Proposition 5.** *Let $S, S' \subseteq E$ be subrings.*
(i) *$S \preccurlyeq_{\aleph_0} S'$ if and only if $S \preccurlyeq_{\aleph_1} S'$ (and hence $S \approx_{\aleph_0} S'$ if and only if $S \approx_{\aleph_1} S'$).*
(ii) *$S \approx_{\aleph_0} E$ if and only if $S \approx_{|\Omega|^+} E$ (where $|\Omega|^+$ is the successor cardinal of $|\Omega|$).*

*Proof.* (i) follows from Theorem 1. (ii) follows from the fact that $c(E) > |\Omega|$. For, if $S \approx_{|\Omega|^+} E$, then among subsets $U \subseteq E$ of cardinality $\leq |\Omega|$ such that $\langle S \cup U \rangle = E$, we can choose one of least cardinality. Let us write $U = \{f_i : i \in |U|\}$. Then the subrings $S_i = \langle S \cup \{f_j : j < i\} \rangle$ $(i \in |U|)$ form a chain of $\leq |\Omega|$ proper subrings of $E$. If $|U|$ were infinite, this chain would have union $E$, contradicting $c(E) > |\Omega|$. Hence, $U$ is finite, and $S \approx_{\aleph_0} E$. □

We will devote the rest of this section to showing that a large natural class of subrings of $E$ consists of elements that are $\prec_\kappa E$. First, we need a few definitions and a lemma.

**Definition 6.** *Let $S$ be a ring and $U$ a subset of $S$. We will say that $s \in S$ is represented by a* ring word of length 1 *in elements of $U$ if $r \in U \cup \{0, 1, -1\}$, and, recursively, that $s \in S$ is represented by a* ring word of length $n$ *in elements of $U$ if $s = p + q$ or $s = pq$ for some elements $p, q \in S$ which can be represented by ring words of lengths $m_1$ and $m_2$ respectively, with $n = m_1 + m_2$.*

**Definition 7.** *Let $U \subseteq E$ be a subset and $x_1, x_2 \in N = \bigoplus_{\alpha\in\Omega} M_\alpha$. We will write $p_U(x_1, x_2) = r$ if $x_2 = x_1 f$ for some $f \in E$ that is represented by a ring word of length $r$ in elements of $U$, and $r$ is the smallest such integer. If no such integer exists, we will write $p_U(x_1, x_2) = \infty$. Also, given $x \in N$ and $r \in \mathbb{Z}^+$, let $B_U(x, r) = \{y \in N : p_U(x, y) \leq r\}$. (Here $p$ stands for "proximity," and $B$ stands for "ball.")*

**Definition 8.** *Given a nonzero subset $X \subseteq N$, we will say that $\Sigma \subseteq \Omega$ is the* support *of $X$ if $X \subseteq M^\Sigma$ and $\Sigma$ is the least such subset of $\Omega$. Also, if $\kappa$ is a regular infinite cardinal $\leq |\Omega|$, we will say that a subring $S \subseteq E$ is* $\kappa$-fearing *if for every $\alpha \in \Omega$, $(M_\alpha)S$ has support of cardinality $< \kappa$.*

**Lemma 9.** *Suppose that $\kappa$ is a regular infinite cardinal $\leq |\Omega|$, that $M$ can be generated by $< \kappa$ elements as an $R$-module, that $S \subseteq E$ is a $\kappa$-fearing subring, and that $U \subseteq E$ is a subset of cardinality $< \kappa$. Then for any $x \in N$ and any $r \in \mathbb{Z}^+$, $B_{S\cup U}(x,r)$ has support of cardinality $< \kappa$.*

*Proof.* Let $x \in N$ be any element. Then $|\{xf : f \in U\}| < \kappa$, since $|U| < \kappa$. Hence, the support of $\{xf : f \in U\}$ is contained the union of $< \kappa$ finite sets and therefore has cardinality $< \kappa$. Also, $\{xf : f \in S\}$ has support of cardinality $< \kappa$. (There is a finite subset $\Gamma \subseteq \Omega$ such that $x \in M^\Gamma$. So, since $S$ is $\kappa$-fearing, $(M^\Gamma)S$ has support of cardinality $< \kappa$.) Therefore, $B_{S\cup U}(x,1) = \{xf : f \in S \cup U\}$ has support of cardinality $< \kappa$.

Now, let $X \subseteq N$ be a subset that has support of cardinality $< \kappa$. Since $M$ can be generated by $< \kappa$ elements, $X$ is contained in a submodule of $N$ that can be generated by $< \kappa$ elements. Let $\{x_\varphi : \varphi \in \Phi\}$ be a generating set for such a submodule, where $|\Phi| < \kappa$. Then the submodule of $N$ generated by $\{Xf : f \in S\cup U\}$ is contained in the submodule of $N$ generated by $\bigcup_{\varphi\in\Phi}\{x_\varphi f : f \in S \cup U\}$, which has support of cardinality $< \kappa$, by the previous paragraph and the regularity of $\kappa$. Thus, $\{Xf : f \in S \cup U\}$ has support of cardinality $< \kappa$ as well.

Hence, letting $x \in N$ be any element and taking $X = \{xg : g \in S \cup U\}$, we see that $\{xgf : g, f \in S \cup U\}$ has support of cardinality $< \kappa$. Also, $\{x(g + f) : g, f \in S \cup U\} = \{xg : g \in S \cup U\} + \{xf : f \in S \cup U\}$, as subsets of $N$, and so $\{x(g + f) : g, f \in S \cup U\}$ has support of cardinality $< \kappa$. Therefore, by induction, for all $x \in N$ and $r \in \mathbb{Z}^+$, $B_{S\cup U}(x,r)$ has support of cardinality $< \kappa$. □

**Theorem 10.** *Suppose that $\kappa$ is a regular infinite cardinal $\leq |\Omega|$, that $M$ can be generated by $< \kappa$ elements as an $R$-module, and that $S \subseteq E$ is a $\kappa$-fearing subring. Then $S \not\approx_\kappa E$.*

*Proof.* Let $U \subseteq E$ be a subset such that $|U| < \kappa$. We will show that $E \not\subseteq \langle S \cup U\rangle$.

Let $f \in \langle S \cup U\rangle$ be any element. Then $f$ is represented by a word of length $r$ in elements of $S \cup U$ for some $r \in \mathbb{Z}^+$, which implies that $p_{S\cup U}(x, xf) \leq r$ for every $x \in N$. Hence, if we find an endomorphism $g \in E$ such that $\{p_{S\cup U}(x, xg) : x \in N\}$ has no finite upper bound, then $g \notin \langle S \cup U\rangle$.

In order to construct such a $g$, let us first define two sequences of elements of $N$. We can pick $x_1 = y_1 \in N$ arbitrarily, and then, assuming that elements with subscripts $i < j$ have been chosen, we can find a finite subset $\Gamma \subseteq \Omega$ such that $x_1, \dots, x_{j-1}, y_1, \dots, y_{j-1} \in M^\Gamma$. Now, let $x_j$ be any nonzero element in $M^{\Omega\setminus\Gamma}$. Since $x_j \neq 0$ and $|\Omega \setminus \Gamma| \geq \kappa$, $M^{\Omega\setminus\Gamma} \cap x_jE$ has support of cardinality $\geq \kappa$. (For every $\alpha \in \Omega \setminus \Gamma$ there is an endomorphism $f \in E$ such that $y = x_jf$ has the property that $y_\alpha \neq 0$.) Thus, $(M^{\Omega\setminus\Gamma} \cap x_jE) \setminus B_{S\cup U}(x_j, j)$ is nonempty, since, by the previous lemma, $B_{S\cup U}(x_j, j)$ has support of cardinality $< \kappa$; let $y_j$ be any element thereof. Now, let $\{\Delta_j : j \in \mathbb{Z}^+\}$ be a collection of disjoint subsets of $\Omega$ such that $x_j, y_j \in M^{\Delta_j}$ for each $j \in \mathbb{Z}^+$. Let $g_j \in \mathrm{End}_R(M^{\Delta_j})$ be an endomorphism such that $y_j = x_jg_j$. Finally, let $g \in E$ be an endomorphism such that the restriction of $g$ to each $M^{\Delta_j}$ is $g_j$. Such an endomorphism will have the desired property. □

We note that in the above lemma and theorem the restriction on the size of $M$ is necessary. For example, suppose that $M = \bigoplus_{\aleph_0} L$, for some nonzero left $R$-module $L$, and also that $\Omega = \aleph_0$. Let $D$ be the diagonal subring of $\mathrm{End}_R(N)$, consisting of all elements $f \in \mathrm{End}_R(N)$

such that for each $\alpha \in \Omega$, $M_\alpha f \subseteq M_\alpha$. $D$ is clearly $\aleph_0$-fearing, but $D \approx_{\aleph_0} \mathrm{End}_R(N)$, since if we take any $f \in \mathrm{End}_R(N)$ that restricts to an isomorphism $M_\alpha \cong_R N$ for some $\alpha \in \Omega$ and let $g \in \mathrm{End}_R(N)$ be the inverse of that isomorphism composed with the inclusion of $M_\alpha$ in $N$, then $\mathrm{End}_R(N) \subseteq gDf$. In particular, for any nonzero element $x \in N$, $B_{D\cup\{f,g\}}(x, 3)$ has support of cardinality $\aleph_0$.

In subsequent sections we will focus on the case where $\Omega$ is countable. However, if $\Omega$ is assumed to be uncountable, then Lemma 9 can be used to obtain a conclusion stronger than the one in Theorem 10.

**Proposition 11.** *Suppose that $\kappa$ is a regular uncountable cardinal $\leq |\Omega|$, that $M$ can be generated by $< \kappa$ elements as an $R$-module, that $S \subseteq E$ is a $\kappa$-fearing subring, and that $U \subseteq E$ is a subset of cardinality $< \kappa$. Then $\langle S \cup U\rangle$ is also $\kappa$-fearing.*

*Proof.* Let $x \in N$ be any element. Then, by Lemma 9, $B_{S\cup U}(x, r)$ has support of cardinality $< \kappa$ for all $r \in \mathbb{Z}^+$. As a regular uncountable cardinal, $\kappa$ has uncountable cofinality, so this implies that $x\langle S \cup U\rangle (= \bigcup_{r\in\mathbb{Z}^+} B_{S\cup U}(x, r))$ has support of cardinality $< \kappa$. Also, the support of $R(x\langle S \cup U\rangle)$ is contained in the support of $x\langle S \cup U\rangle$, so $(Rx)\langle S \cup U\rangle$ has support of cardinality $< \kappa$. Now, pick any element $\alpha \in \Omega$. Letting $x$ range over a generating set of cardinality $< \kappa$ for $M_\alpha$ as an $R$-module, we conclude that $M_\alpha\langle S \cup U\rangle$ has support of cardinality $< \kappa$. Hence, $\langle S \cup U\rangle$ is a $\kappa$-fearing subring. □

# 4 Weakly $\aleph_0$-fearing subrings

In this section we will keep $R$, $N = \bigoplus_\Omega M$, and $E$ as before, but will now focus on the case when $\Omega$ is countable and $M$ is finitely generated. For simplicity, we will assume that $\Omega = \mathbb{Z}^+$. From now on $\prec_{\aleph_0,E}$, $\preccurlyeq_{\aleph_0,E}$, and $\approx_{\aleph_0,E}$ will be written simply as $\prec$, $\preccurlyeq$, and $\approx$, respectively. Also, we will view elements of $E$ as row-finite matrices over $\mathrm{End}_R(M)$, whenever convenient.

As in the paragraph following Theorem 10, we define $D \subseteq E$ to be the subring consisting of all elements $f \in \mathrm{End}_R(N)$ such that for each $\alpha \in \Omega$, $M_\alpha f \subseteq M_\alpha$. Let $T \subseteq E$ denote the subring of lower-triangular matrices, consisting of all elements $f \in E$ such that for each $\alpha \in \Omega\ (= \mathbb{Z}^+)$, $M_\alpha f \subseteq M^\Sigma$, where $\Sigma = \{\gamma \in \Omega : \gamma \leq \alpha\}$. Also, let $\bar{T} \subseteq E$ denote the subring of upper-triangular matrices, consisting of all elements $f \in E$ such that for each $\alpha \in \Omega$, $M_\alpha f \subseteq M^\Gamma$, where $\Gamma = \{\gamma \in \Omega : \gamma \geq \alpha\}$.

**Proposition 12.** *There exist $g, h \in E$ such that $T \subseteq gDh$. In particular, $D \approx T$.*

*Proof.* Consider the following two matrices in $E$:

$$A = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \boxed{1 \quad 1} & & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \boxed{1 \quad 1 \quad 1} & & & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $Y \in T$ be any element. Then we can write

$$Y = \begin{pmatrix} a_{11} & 0 & 0 & 0 & \dots \\ a_{21} & a_{22} & 0 & 0 & \dots \\ a_{31} & a_{32} & a_{33} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

for some $a_{ij} \in \mathrm{End}_R(M)$. Let $X \in D$ be the matrix

$$\begin{pmatrix} \boxed{a_{11}} & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & a_{21} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_{22} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & a_{31} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & a_{32} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & a_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $AXB = Y$, and so $T \subseteq ADB$. The final assertion follows from the fact that $D \subseteq T$. □

We note that our definition of $T$ and the above proof make sense even without assuming that $M$ is finitely generated, since the elements of $T$ can be viewed as row-finite matrices regardless of the size of $M$. However, since we are assuming that $M$ is finitely generated, we have $T \prec E$, by Theorem 10. On the other hand, we can obtain a result of the opposite sort for $\bar{T}$.

**Corollary 13.** *$\bar{T} \approx E$.*

*Proof.* By the previous proposition, $T \preccurlyeq \bar{T}$, since $D \subseteq \bar{T}$. Hence, $E = T + \bar{T}$ implies that $\bar{T} \approx E$. □

Returning to the subring $D$, we can show that all $\aleph_0$-fearing subrings of $E$ are $\preccurlyeq D$. In fact, the same can be said of a larger class of subrings of $E$. We will devote the rest of the section to proving this.

**Definition 14.** *We will say that a subring $S \subseteq E$ is* weakly $\aleph_0$-fearing *if $|\{\alpha \in \Omega : (M_\alpha)S$ has infinite support$\}| < \aleph_0$.*

**Lemma 15.** *Let $S \subseteq E$ be a weakly $\aleph_0$-fearing subring. Then there exists an $\aleph_0$-fearing subring $S' \subseteq E$ such that $S \preccurlyeq S'$. In particular, $S \not\approx E$.*

*Proof.* Upon enlarging $S$, if necessary, we may assume that $D \subseteq S$. Let $\Sigma \subseteq \Omega$ be the finite subset consisting of the elements $\alpha \in \Omega$ such that $(M_\alpha)S$ has infinite support. Let $S' \subseteq S$ be the subring consisting of all elements $f \in S$ such that $(M_\alpha)f \subseteq (M_\alpha)$ for all $\alpha \in \Sigma$, and let $S'' \subseteq S$ be the subring consisting of all elements $f \in S$ such that $(M_\alpha)f \subseteq (M_\alpha)$ for all $\alpha \notin \Sigma$. In particular, $S = \pi_{\Omega \setminus \Sigma} S + \pi_\Sigma S \subseteq S' + S''$. Now, $S'$ is $\aleph_0$-fearing; let us show that $S' \approx S$. To this end, we will demonstrate that $S'' \subseteq \langle S' \cup U \rangle$ for some finite subset $U \subseteq E$.

Let $E_{\text{fin}} \subseteq E$ denote the subring consisting of all elements that have only finitely many off-diagonal entries. Also, let us write $e_{ij}$ to denote the standard matrix units. There are countably many such elements, so $\{e_{ij} : i, j \in \mathbb{Z}^+\} \subseteq U$, for some finite set $U \subseteq E$, by Theorem 1. Then $S'' \subseteq E_{\text{fin}} \subseteq \langle D \cup U\rangle$. But, we assumed that $D \subseteq S'$, so $S'' \subseteq \langle S' \cup U\rangle$, as desired.

The final assertion follows from Theorem 10. □

**Lemma 16.** *Let $S \subseteq E$ be an $\aleph_0$-fearing subring. Then $S \subseteq gT$ for some $g \in E$.*

*Proof.* For each $k \in \Omega\,(= \mathbb{Z}^+)$, let $l_k$ be the largest element in the union of the supports of $(M_j)S$ for all $j \leq k$. Let $f \in E$ be the endomorphism that takes $M_{l_k}$ identically to $M_k$ for each positive integer $k$, and takes $M_\alpha$ to zero if $\alpha \neq l_k$ for all $k \in \mathbb{Z}^+$. Also, let $g \in E$ be the endomorphism that takes $M_k$ identically to $M_{l_k}$ for each positive integer $k$. Now, let $h \in S$ be any element. Then $fh \in T$, and $g(fh) = (gf)h = 1 \cdot h = h$. Hence, $S \subseteq gT$. □

As in Proposition 12, the above proof works even without assuming that $M$ is finitely generated.

The following theorem summarizes the results of this section.

**Theorem 17.** *Let $S \subseteq E$ be a weakly $\aleph_0$-fearing subring. Then $S \preccurlyeq D$.*

*Proof.* This follows from Proposition 12, Lemma 15, and Lemma 16. □

# 5 Subrings arising from preorders

Keeping the notation from the previous section, we will now turn our attention to subrings $S \subseteq E$ such that $D \preccurlyeq S$. For this, we will need a new concept.

**Definition 18.** *Let $\rho$ be a preordering of $\Omega$. Define $E(\rho) \subseteq E$ to be the subset consisting of those elements $f \in E$ such that for all $\alpha, \beta \in \Omega$, $\pi_\alpha f \pi_\beta \neq 0$ implies $(\alpha, \beta) \in \rho$.*

It is clear that the subsets $E(\rho)$ are subrings. For example, $D$, $T$, $\bar{T}$, and $E$ are of this form. Indeed, recalling that $\Omega = \mathbb{Z}^+$, and setting $\rho_1 = \{(\alpha, \alpha) \in \Omega \times \Omega\}$, $\rho_2 = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha \geq \beta\}$, and $\rho_3 = \{(\alpha, \beta) \in \Omega \times \Omega : \alpha \leq \beta\}$, we have $D = E(\rho_1)$, $T = E(\rho_2)$, and $\bar{T} = E(\rho_3)$. We also note that every subring of $E$ of the form $E(\rho)$ contains $D$ and is closed in the function topology (i.e., the topology inherited from the set $N^N$ of all functions $\bigoplus_\Omega M \to \bigoplus_\Omega M$, where a subbasis of open sets is given by the sets $\{f \in N^N : mf = n\}$, for all $m, n \in \bigoplus_\Omega M$). In fact, if $\mathrm{End}_R(M)$ is a simple ring, then this characterizes such subrings of $E$.

**Proposition 19.** *Suppose that $\mathrm{End}_R(M)$ is a simple ring, and let $S \subseteq E$ be a subring. Then $S = E(\rho)$ for some preordering $\rho$ of $\Omega$ if and only if $S$ is closed in the function topology and $D \subseteq S$.*

*Proof.* Suppose that $S$ is closed in the function topology and $D \subseteq S$. Let $\rho = \{(\alpha, \beta) : \pi_\alpha S \pi_\beta \neq 0\} \subseteq \Omega \times \Omega$. Since $S$ contains the identity element, $\rho$ is reflexive.

Next, we note that for all $\alpha, \beta \in \Omega$ there is an obvious bijection between $\pi_\alpha E \pi_\beta$ and $\mathrm{End}_R(M)$, under which $\pi_\alpha S \pi_\beta$ corresponds to a 2-sided ideal (since $D \subseteq S$). Hence, $\mathrm{End}_R(M)$ being simple implies that either $\pi_\alpha S \pi_\beta = 0$ or $\pi_\alpha S \pi_\beta = \pi_\alpha E \pi_\beta$. In particular, if

$\pi_\alpha S\pi_\beta \neq 0$ and $\pi_\beta S\pi_\gamma \neq 0$ for some $\alpha, \beta, \gamma \in \Omega$, then $\pi_\alpha S\pi_\beta\pi_\beta S\pi_\gamma = \pi_\alpha E\pi_\beta\pi_\beta E\pi_\gamma = \pi_\alpha E\pi_\gamma$. Since $\pi_\beta \in D \subseteq S$, we have $0 \neq \pi_\alpha S\pi_\beta\pi_\beta S\pi_\gamma \subseteq \pi_\alpha S\pi_\gamma$. Hence, $\rho$ is transitive and therefore a preorder.

Let $f \in E$ be an element with the property that $\pi_\alpha f\pi_\beta \neq 0$ implies $(\alpha, \beta) \in \rho$. Then $\pi_\alpha f\pi_\beta \in \pi_\alpha E\pi_\beta \subseteq S$ for all $\alpha, \beta \in \Omega$, by the previous paragraph. Now, $f$ is in the closure of the set of sums of elements of the form $\pi_\alpha f\pi_\beta$. Hence $f \in S$, by the hypothesis that $S$ is closed in the function topology. This shows that $S = E(\rho)$.

The converse is clear. □

In the previous section we showed that if $M$ is finitely generated and $S \subseteq E$ is a weakly $\aleph_0$-fearing subring, then $S \not\approx E$. It turns out that for a subring $S \subseteq E$ of the form $E(\rho)$ the property of being weakly $\aleph_0$-fearing is not only sufficient but also necessary for $S \not\approx E$. We require a lemma before proceeding to the proof of this statement.

**Lemma 20.** *Let $\Phi$ and $\Gamma$ be sets such that $|\Phi| = |\Gamma| = \aleph_0$, and let $\{\Lambda_\varphi : \varphi \in \Phi\}$ be a collection of infinite subsets of $\Gamma$. Then there is a subset $\{\varphi_j : j \in \mathbb{Z}^+\} \subseteq \Phi$ of distinct elements and a collection of infinite sets $\{\bar{\Lambda}_j : j \in \mathbb{Z}^+\}$, where for each $j \in \mathbb{Z}^+$, $\bar{\Lambda}_j \subseteq \Lambda_{\varphi_j}$, such that one of the following holds:*

(1) *$\bar{\Lambda}_j \subseteq \bar{\Lambda}_{j'}$ for $j \geq j'$,*

(2) *$\bar{\Lambda}_j \cap \bar{\Lambda}_{j'} = \emptyset$ for $j \neq j'$.*

*Proof.* Suppose that $\{\varphi_j : j \in \mathbb{Z}^+\}$ is a sequence of distinct elements of $\Phi$. Let us construct inductively an infinite collection of sets $\{\bar{\Lambda}_j : j \in \mathbb{Z}^+\}$, where for each $j \in \mathbb{Z}^+$, $\bar{\Lambda}_j \subseteq \Lambda_{\varphi_j}$, and $\bar{\Lambda}_{j+1} \subseteq \bar{\Lambda}_j$. Let $\bar{\Lambda}_1 = \Lambda_{\varphi_1}$, and assuming that $\bar{\Lambda}_n$ has been constructed, let $\bar{\Lambda}_{n+1} = \bar{\Lambda}_n \cap \Lambda_{\varphi_{n+1}}$.

If there is a sequence of distinct elements of $\Phi$, $\{\varphi_j : j \in \mathbb{Z}^+\}$, such that the collection constructed above consists of infinite sets, then (1) is satisfied. So suppose that no such sequence exists. Let $\varphi_1, \varphi_2, \ldots, \varphi_n \in \Phi$ be any sequence of elements such that $n \geq 1$, $\Delta_1 := \Lambda_{\varphi_1} \cap \Lambda_{\varphi_2} \cap \cdots \cap \Lambda_{\varphi_n}$ is infinite, and for all $\varphi \in \Phi \setminus \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$, $\Delta_1 \cap \Lambda_\varphi$ is finite. Set $\Phi_1 = \Phi \setminus \{\varphi_1, \varphi_2, \ldots, \varphi_n\}$. Repeating the above process, let $\varphi'_1, \varphi'_2, \ldots, \varphi'_n \in \Phi_1$ be any sequence of elements such that $n \geq 1$, $\Delta_2 := \Lambda_{\varphi'_1} \cap \Lambda_{\varphi'_2} \cap \cdots \cap \Lambda_{\varphi'_n}$ is infinite, and for all $\varphi \in \Phi_1 \setminus \{\varphi'_1, \varphi'_2, \ldots, \varphi'_n\}$, $\Delta_2 \cap \Lambda_\varphi$ is finite. Set $\Phi_2 = \Phi_1 \setminus \{\varphi'_1, \varphi'_2, \ldots, \varphi'_n\}$, etc. Continuing in this fashion, we obtain a subset $\{\varphi_j : j \in \mathbb{Z}^+\} \subseteq \Phi$ of distinct elements and a collection of infinite sets $\{\Delta_j : j \in \mathbb{Z}^+\}$, where for each $j \in \mathbb{Z}^+$, $\Delta_j \subseteq \Lambda_{\varphi_j}$, and $\Delta_j \cap \Delta_{j'}$ is finite for $j \neq j'$.

Let $\bar{\Lambda}_1 = \Delta_1$, and for each $j > 1$, let $\bar{\Lambda}_j = \Delta_j \setminus \bigcup_{i=1}^{j-1}(\Delta_i \cap \Delta_j)$. Then the elements of $\{\bar{\Lambda}_j : j \in \mathbb{Z}^+\}$ satisfy (2). □

**Proposition 21.** *Suppose that $M$ is finitely generated, $\rho$ is a preordering of $\Omega$, and $S = E(\rho)$. If $S$ is not weakly $\aleph_0$-fearing, then $S \approx E$.*

*Proof.* For each $\alpha \in \Omega$ $(= \mathbb{Z}^+)$, denote the support of $(M_\alpha)S$ by $\mathrm{supp}(M_\alpha S)$, and consider the set $\{\mathrm{supp}(M_\alpha S) : |\mathrm{supp}(M_\alpha S)| = \aleph_0\}$. This is an infinite collection of infinite subsets of $\Omega$, since $S$ is not weakly $\aleph_0$-fearing. By the previous lemma, there is a set $\Sigma = \{\alpha_j : j \in \mathbb{Z}^+\}$ of distinct elements of $\Omega$ and a collection of infinite sets $\{\bar{\Lambda}_j : j \in \mathbb{Z}^+\}$, where for each $j \in \mathbb{Z}^+$, $\bar{\Lambda}_j \subseteq \mathrm{supp}(M_{\alpha_j} S)$, and either

(1) $\bar{\Lambda}_j \subseteq \bar{\Lambda}_{j'}$ for $j \geq j'$, or

(2) $\bar{\Lambda}_j \cap \bar{\Lambda}_{j'} = \emptyset$ for $j \neq j'$.

We will now treat the two cases individually.

Suppose that (1) holds. We begin by constructing a sequence $\{\beta_j : j \in \mathbb{Z}^+\}$ of elements of $\bar{\Lambda}_1$. Pick $\beta_1 \in \bar{\Lambda}_1$ arbitrarily. Let $\beta_2 \in \bar{\Lambda}_2$ be such that $\beta_2 \neq \beta_1$. Then let $\beta_3 \in \bar{\Lambda}_3$ be such that $\beta_3 \neq \beta_2$ and $\beta_3 \neq \beta_1$, and so on. Now, let $S' \subseteq S$ be the subset consisting of all endomorphisms $f$ such that for each $j \in \mathbb{Z}^+$, $(M_{\alpha_j})f \subseteq M^{\Gamma_j}$, where $\Gamma_j = \{\beta_i : i \geq j\} \subseteq \bar{\Lambda}_j$. Let $g \in E$ be the endomorphism that maps $N$ to $M^{\Sigma}$ by sending $M_j$ identically to $M_{\alpha_j}$ for each $j \in \mathbb{Z}^+ = \Omega$. Also, let $h \in E$ be an endomorphism that takes $M^{\Gamma_1}$ to $N$ by sending $M_{\beta_j}$ identically to $M_j$ for each $j \in \mathbb{Z}^+ = \Omega$. Then $\bar{T} = gS'h$, since $S = E(\rho)$. Hence, by Corollary 13, $S \approx E$.

Suppose that (2) holds. Then $\sum_{j\in\mathbb{Z}^+} M^{\bar{\Lambda}_j}$ is direct, and so there is an endomorphism $h \in E$ that simultaneously maps each $M^{\bar{\Lambda}_j}$ isomorphically to $N$. Let $S' \subseteq S$ be the subset consisting of all endomorphisms $f$ such that for each $j \in \mathbb{Z}^+$, $(M_{\alpha_j})f \subseteq M^{\bar{\Lambda}_j}$. Then $\pi_\Sigma \mathrm{Hom}_R(M^{\Sigma}, N) \subseteq S'h$, since $S = E(\rho)$. Now, let $g \in E$ be an endomorphism that maps $N$ isomorphically to $M^{\Sigma} \subseteq N$. Then $E \subseteq gS'h$, and hence $S \approx E$. □

**Corollary 22.** *Suppose that $M$ is finitely generated, and let $\rho$ be a preordering of $\Omega$. Then $E(\rho) \approx E$ if and only if $E(\rho)$ is not weakly $\aleph_0$-fearing.*

*Proof.* The forward implication was proved in Lemma 15, while the backward implication was proved in Proposition 21. □

Putting together Corollary 22, Theorem 17, and the remarks after Definition 18 we obtain the following result.

**Theorem 23.** *Suppose that $M$ is finitely generated, and let $\rho$ be a preordering of $\Omega$. Then exactly one of the following holds:*

(1) $E(\rho) \approx D$,

(2) $E(\rho) \approx E$.

If we assume that $\mathrm{End}_R(M)$ is a simple ring, then this theorem can be stated without reference to preorders.

**Corollary 24.** *Suppose that $M$ is finitely generated and $\mathrm{End}_R(M)$ is a simple ring. If $S \subseteq E$ is a subring that is closed in the function topology and $D \subseteq S$, then exactly one of the following holds:*

(1) $S \approx D$,

(2) $S \approx E$.

*Proof.* This follows from Theorem 23 and Proposition 19. □

It would be desirable to relax the condition $D \subseteq S$ in the above statement, say, by instead considering closed subrings $S$ satisfying $I \subseteq S$, where $I$ is the diagonally embedded copy of $\mathrm{End}_R(M)$ in $E$. However, doing so makes the situation much more messy, though we can show that in general such subrings fall into at least four equivalence classes. In the next result, let $C \subseteq E$ denote the subring $\langle I \cup \mathrm{Hom}_R(N, M_1)\iota_1 \rangle = I + \mathrm{Hom}_R(N, M_1)\iota_1$, where $\iota_1$ is the inclusion of $M_1$ in $N$.

**Proposition 25.** *Suppose that $M$ is simple and $\mathrm{End}_R(M)$ has countable dimension as a vector space over its center. Then $I \prec C \prec D \prec E$.*

*Proof.* By Theorem 10, $D \prec E$, and, by Theorem 17, $C \preccurlyeq D$. Also, $I \subseteq C$. Thus, it suffices to show that $I \not\approx C \not\approx D$.

Let $Z$ denote the center of $\mathrm{End}_R(M)$. Also, let $U \subseteq E$ be a finite subset. Then $\langle I \cup U \rangle$ has countable dimension as a vector space over $Z$, and hence $C \not\subseteq \langle I \cup U \rangle$, since $C$ has uncountable dimension. Therefore $I \not\approx C$.

It remains to show that $C \not\approx D$. By Proposition 12, it suffices to prove that given a finite set $U \subseteq E$, we have $T \not\subseteq \langle C \cup U \rangle$. Now, for any subset $\Sigma \subseteq \Omega$, let $\iota_\Sigma$ denote the inclusion of $M^\Sigma$ in $N$, and set $F = \{h \in E : \exists \Sigma \subseteq \Omega \text{ finite, such that } h \in \mathrm{Hom}_R(N, M^\Sigma)\iota_\Sigma\}$ (i.e., the set of matrices with zeros in all but finitely many columns). We will first show that any $f \in \langle C \cup U \rangle$ can be written as $f = g + h$, where $g \in \langle I \cup U \rangle$ and $h \in F$.

Setting $H = \mathrm{Hom}_R(N, M_1)\iota_1$, we have $C = I + H$. Hence, every element $f \in \langle C \cup U \rangle$ can be expressed as $f = g + \bar{h}$, where $g \in \langle I \cup U \rangle$ and $\bar{h}$ is a sum of products of elements of $I$, $H$, and $U$, such that each product contains an element of $H$. Now, $EH \subseteq H$, so $\bar{h}$ is a sum of products of the form $h'g'$, where $h' \in H$ and $g' \in \langle I \cup U \rangle$. But, such elements $h'g'$ belong to $F$, and hence $f$ can be written as $f = g + h$, where $g \in \langle I \cup U \rangle$ and $h \in F$ (since $F$ is closed under addition).

Now, for each $f \in T$ pick some $h_f \in F$. Then $\{f - h_f : f \in T\}$ generates a $Z$-vector space of uncountable dimension and is therefore not contained in any subring of $E$ that has countable dimension, regardless of how the elements $h_f$ are picked. In particular, $\{f - h_f : f \in T\} \not\subseteq \langle I \cup U \rangle$. Hence $T \not\subseteq \langle C \cup U \rangle$, by our description of the elements of $\langle C \cup U \rangle$ above, completing the proof. □

# References


[1] J. Araújo, J. D. Mitchell, and N. Silva, *On generating countable sets of endomorphisms,* Algebra Universalis **50** (2003), no. 1, 61–67.

[2] George M. Bergman and Saharon Shelah, *Closed subgroups of the infinite symmetric group,* to appear, Algebra Universalis.

[3] Fred Galvin, *Generating countable sets of permutations,* J. London Math. Soc. (2) **51** (1995), 230–242.

[4] Graham Higman, B. H. Neumann, and Hanna Neumann, *Embedding theorems for groups,* J. London Math. Soc. **24** (1949), 247–254.

[5] H. D. Macpherson and Peter M. Neumann, *Subgroups of infinite symmetric groups,* J. London Math. Soc. (2) **42** (1990), 64–84.

[6] K. D. Magill, Jr., *The countability index of the endomorphism semigroup of a vector space,* Linear and Multilinear Algebra **22** (1988), no. 4, 349–360.

[7] Zachary Mesyan, *Generating endomorphism rings of infinite direct sums and products of modules,* J. Algebra **283** (2005), no. 1, 364–366.

[8] Zachary Mesyan, *Generating self-map monoids of infinite sets,* preprint.

[9] K. C. O'Meara, *Embedding countable rings in* 2*-generator rings,* Proc. Amer. Math. Soc. **100** (1987), no. 1, 21–24.

[10] K. C. O'Meara, C. I. Vinsonhaler, and W. J. Wickless, *Identity-preserving embeddings of countable rings into* 2*-generator rings,* Rocky Mountain J. Math. **19** (1989), no. 4, 1095–1105.

[11] B. H. Neumann and Hanna Neumann, *Embedding theorems for groups,* J. London Math. Soc. **34** (1959), 465–479.

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