

INFINITE-DIMENSIONAL DIAGONALIZATION AND SEMISIMPLICITY

MIODRAG C. IOVANOV, ZACHARY MESYAN, AND MANUEL L. REYES

ABSTRACT. We characterize the diagonalizable subalgebras of $\text{End}(V)$, the full ring of linear operators on a vector space V over a field, in a manner that directly generalizes the classical theory of diagonalizable algebras of operators on a finite-dimensional vector space. Our characterizations are formulated in terms of a natural topology (the “finite topology”) on $\text{End}(V)$, which reduces to the discrete topology in the case where V is finite-dimensional. We further investigate when two subalgebras of operators can and cannot be simultaneously diagonalized, as well as the closure of the set of diagonalizable operators within $\text{End}(V)$. Motivated by the classical link between diagonalizability and semisimplicity, we also give an infinite-dimensional generalization of the Wedderburn-Artin theorem, providing a number of equivalent characterizations of left pseudocompact, Jacobson semisimple rings that parallel various characterizations of artinian semisimple rings. This theorem unifies a number of related results in the literature, including the structure of linearly compact, Jacobson semisimple rings and cosemisimple coalgebras over a field.

1. INTRODUCTION

Let V be a vector space over a field K , and let $\text{End}(V)$ be the algebra of K -linear operators on V . Given any basis \mathcal{B} of V , an operator $T \in \text{End}(V)$ is said to be *diagonalizable with respect to \mathcal{B}* (or *\mathcal{B} -diagonalizable*) if every element of \mathcal{B} is an eigenvector for T . It is easy to see that the set $\text{diag}(\mathcal{B})$ of all \mathcal{B} -diagonalizable operators forms a maximal commutative subalgebra of $\text{End}(V)$, isomorphic to the product algebra $K^{\mathcal{B}} = \prod_{\mathcal{B}} K \cong K^{\dim(V)}$ (for details, see Proposition 4.5 below).

A subalgebra $D \subseteq \text{End}(V)$ is said to be *diagonalizable* if $D \subseteq \text{diag}(\mathcal{B})$ for some basis \mathcal{B} of V ; this is evidently equivalent to the condition that, for any basis \mathcal{B} of V , there exists a unit $u \in \text{End}(V)$ such that $uD u^{-1} \subseteq \text{diag}(\mathcal{B})$. Clearly every diagonalizable subalgebra of $\text{End}(V)$ is commutative. Our goal is to characterize which commutative subalgebras of $\text{End}(V)$ are diagonalizable.

For motivation, we recall the classical result from linear algebra characterizing the diagonalizable operators on a finite-dimensional vector space. An individual operator is diagonalizable if and only if its minimal polynomial splits into linear factors over K and has no repeated roots; in the case where K is algebraically closed, this is of course equivalent to the

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property that the minimal polynomial has no repeated roots. For subalgebras, this implies the following.

Classical Diagonalization Theorem. *Let V be a finite-dimensional vector space over a field K , and let C be a commutative subalgebra of $\text{End}(V)$. The following are equivalent:*

- (a) C is diagonalizable;
- (b) $C \cong K^n$ as K -algebras for some integer $1 \leq n \leq \dim(V)$;
- (c) C is spanned by an orthogonal set of idempotents whose sum is 1.

If K is algebraically closed, then these conditions are further equivalent to:

- (d) C is a Jacobson semisimple K -algebra;
- (e) C is a reduced K -algebra (i.e., has no nonzero nilpotent elements).

(The equivalence (a) \Leftrightarrow (b) is mentioned, for instance, in [2, VII.5.7], the equivalence (b) \Leftrightarrow (c) is apparent, and the further equivalence of (b) with (d)–(e) is a straightforward restriction of the Wedderburn-Artin structure theorem to finite-dimensional algebras over algebraically closed fields.)

In particular, the diagonalizability of a commutative subalgebra C of $\text{End}(V)$ is completely determined by the isomorphism class of C as a K -algebra. However, in the case where V is infinite-dimensional, it is impossible to determine whether a commutative subalgebra $C \subseteq \text{End}(V)$ is diagonalizable from purely algebraic properties of C . Indeed, assume for simplicity that K is an infinite field, and let V be a vector space with basis $\{v_n \mid n = 1, 2, 3, \dots\}$. Let $T \in \text{End}(V)$ be diagonalizable with infinitely many eigenvalues, and let $S \in \text{End}(V)$ be the “right shift operator” $S(v_n) = v_{n+1}$. Then $K[T] \cong K[S]$ are both isomorphic to a polynomial algebra $K[x]$ (as neither T nor S satisfies any polynomial relation), while $K[T]$ is diagonalizable but $K[S]$ is not (for instance, because S has no eigenvalues).

Therefore the study of diagonalizability over infinite-dimensional spaces requires us to consider some extra structure on algebras of operators. Our approach will be to consider a natural topology on $\text{End}(V)$, which is then inherited by its subalgebras. (Of course, this is analogous to the use of functional analysis to extend results about operators on Hilbert spaces from finite-dimensional linear algebra to the infinite-dimensional setting.) Specifically, we consider the *finite topology* on the algebra $\text{End}(V)$, also called the *function topology* or the topology of *pointwise convergence*. This is the topology with a basis of open sets of the form $\{S \in \text{End}(V) \mid S(x_i) = T(x_i), i = 1, \dots, n\}$ for fixed $x_1, \dots, x_n \in V$ and $T \in \text{End}(V)$.

It will be shown below (Lemma 2.2) that the closure of a commutative subalgebra of $\text{End}(V)$ is again commutative; in particular, every maximal commutative subalgebra is closed. Thus after replacing a commutative subalgebra with its closure, we may reduce the problem of characterizing diagonalizable subalgebras of $\text{End}(V)$ to that of characterizing which *closed* subalgebras are diagonalizable. The following major result, given in Theorem 4.10, is a generalization of the Classical Diagonalization Theorem, which shows that diagonalizability of a closed subalgebra of $\text{End}(V)$ can be detected from its internal structure as a topological algebra. We say that a commutative topological K -algebra is *K -pseudocompact* if it is an inverse limit (in the category of topological algebras) of finite-dimensional discrete K -algebras; see Section 3 for further details.

Infinite-Dimensional Diagonalization Theorem. *Let V be a vector space over a field K , and let C be a closed commutative subalgebra of $\text{End}(V)$. The following are equivalent:*

- (a) C is diagonalizable;
- (b) $C \cong K^\Omega$ as topological K -algebras, for some cardinal $1 \leq \Omega \leq \dim(V)$;
- (c) C contains an orthogonal set of idempotents $\{E_i\}$ whose linear span is dense in C , such that the net of finite sums of the E_i converges to 1.

If K is algebraically closed, then these conditions are further equivalent to:

- (d) C is K -pseudocompact and Jacobson semisimple;
- (e) C is K -pseudocompact and topologically reduced (i.e., has no nonzero topologically nilpotent elements).

In the classical case of a finite-dimensional vector space V over a field K , because every diagonalizable algebra of operators is isomorphic to K^n , we see that diagonalizable algebras are semisimple. Similarly, in the case where V is infinite-dimensional, every closed diagonalizable algebra of operators on V satisfies a suitable version of semisimplicity. Thus in Theorem 3.10, our other major result, we give a version of the Wedderburn-Artin theorem that is suitable for our context of infinite-dimensional linear algebra, with the dual purpose of extending the classical connection above and preparing for the proof of Theorem 4.10. This result characterizes those topological rings that are isomorphic to $\prod_i \text{End}_{D_i}(V_i)$ where the V_i are arbitrary right vector spaces over division rings D_i . It is presented in the context of *left pseudocompact (topological) rings* (as defined by Gabriel [4]), and the section begins with a basic account of such rings and modules. In Theorem 3.10, among other equivalences, we show the following.

Infinite-Dimensional Wedderburn-Artin Theorem. *Let R be a topological ring. Then the following are equivalent:*

- (a) *As a left topological module, R is a product of simple discrete left modules;*
- (b) *$R \cong \prod \text{End}_{D_i}(V_i)$ as topological rings, where each V_i is a right vector space over a division ring D_i and each $\text{End}_{D_i}(V_i)$ is given the finite topology;*
- (c) *R is left pseudocompact and Jacobson semisimple;*
- (d) *R is left pseudocompact and every pseudocompact left R -module is a product of discrete simple modules.*

This result generalizes the classical Wedderburn-Artin Theorem (which is recovered in the case when R carries the discrete topology), and as we explain after Theorem 3.10, it provides a unifying context for a number of other related results in the literature, including the structure of linearly compact, Jacobson semisimple rings, and the characterization of cosemisimple coalgebras and their dual algebras.

We now give a tour of the major ideas covered in this paper, by way of outlining each section. As the problem of characterizing diagonalizable subalgebras of $\text{End}(V)$ is framed in a purely algebraic context, we anticipate that some readers may not have much familiarity with the topological algebra required to prove our results. Thus we feel that it is appropriate to briefly recall some of the basic theory of topological rings and modules. Most of this review is carried out in Section 2. Selected topics include topological sums of orthogonal collections of idempotents, as well as topological modules that are inverse limits of discrete modules.

Section 3 begins with a basic overview of pseudocompact and linearly compact modules and rings, including a brief review of Gabriel duality for pseudocompact modules. After a discussion of a suitable topological version of the Jacobson radical, we proceed to our infinite Wedderburn-Artin theory which culminates in Theorem 3.10. The remainder of the section consists of various applications of this theorem. This includes a symmetric version of the Infinite-Dimensional Wedderburn-Artin Theorem, which characterizes topological rings of the form $\prod \mathbb{M}_{n_i}(D_i)$, where the D_i are division rings, the $n_i \geq 1$ are integers, and each component matrix ring is equipped with the discrete topology. The section ends with applications to the structure of cosemisimple coalgebras.

The structure and characterization of diagonalizable algebras of operators is addressed in Section 4. If \mathcal{B} is a basis of a vector space V , it is shown that the subalgebra of $\text{End}(V)$ of \mathcal{B} -diagonalizable operators is topologically isomorphic to $K^{\mathcal{B}}$, where K is discretely topologized. Through a detailed study of topological algebras of the form K^X for some set X , we show that such “function algebras” with continuous algebra homomorphisms form a category that is dual to the category of sets. This is applied in Theorem 4.10 to characterize closed diagonalizable subalgebras of $\text{End}(V)$. We also study diagonalizable subalgebras that are not necessarily closed and individual diagonalizable operators, still making use of the finite topology on $\text{End}(V)$. We then turn our attention to the problem of simultaneous diagonalization of operators, showing in that if $C, D \subseteq \text{End}(V)$ are diagonalizable subalgebras that centralize one another, then they are simultaneously diagonalizable. We also present examples of sets of commuting diagonalizable operators that cannot be simultaneously diagonalized. Finally, we describe the closure $\overline{\mathcal{D}}$ of the set $\mathcal{D} = \mathcal{D}(V)$ of diagonalizable operators in $\text{End}(V)$. Here the order of the field of scalars plays an important role. If K is finite, we have $\mathcal{D} = \overline{\mathcal{D}}$; in the case where K is infinite, $\overline{\mathcal{D}}$ is shown to consist of those operators T that are diagonalizable on the torsion part of V when regarded as a $K[x]$ -module via the action of T .

1.A. A reader’s guide to diagonalization. While our major motivation for the present investigation was to characterize the diagonalizable subalgebras of $\text{End}(V)$, we have attempted to place this result in a broader context within infinite-dimensional linear algebra. In the classical finite-dimensional case, diagonalization is connected to the structure theory of finite-dimensional algebras and to algebras of functions (though one typically does not think beyond polynomial functions in classical spectral theory). Our attempt to understand infinite-dimensional diagonalization from both of these perspectives led us to incorporate aspects of the theories of pseudocompact algebras and “function algebras.”

We anticipate that some readers may appreciate a shorter path to understanding only the proof of the characterization of diagonalizability in Theorem 4.10; for such readers, we provide the following outline. The definition of the finite topology on $\text{End}(V)$ is the most important portion of Section 2, and the discussion around Lemmas 2.15 and 2.17 is intended to help the reader better understand the structure of pseudocompact modules. Other results from Section 2 can mostly be referred to as needed, although readers unfamiliar with topological algebra may benefit from a thorough reading of this section. The basic definition of pseudocompactness should be understood from Section 3. The only results from this section that are crucial to later diagonalizability results are Theorem 3.8 and the implication

(e) \Rightarrow (d) from Theorem 3.10, the latter of which is established in [4, Proposition IV.12], or through the implications (e) \Rightarrow (f) \Rightarrow (d) using [15, Theorem 29.7]. With this background, readers can proceed to Section 4. Finally, some readers may wish to note that Theorem 4.7 is only used in Corollary 4.8 to bound the cardinality of an index set; this bound may be achieved through a more elementary argument via topological algebra, an exercise which is encouraged for those who are interested.

1.B. Some conventions. While diagonalizability of an operator on a vector space depends keenly upon the field K of scalars under consideration, we will often suppress explicit reference to K in notation such as $\text{End}(V)$ or $\mathcal{D}(V)$ (and similarly for the case of $\text{End}(V)$ when V is a vector space over a division ring). This does not pose any serious risk of confusion because we work with a single field of scalars K , with no field extensions in sight.

Unless otherwise noted, we follow the convention that all rings are associative with a multiplicative identity, and all ring homomorphisms preserve the identity. Given a ring R , we let $J(R)$ denote its Jacobson radical. Further, we emphasize that a module M is a left (respectively, right) R -module via the notation ${}_R M$ (respectively, M_R).

2. TOPOLOGICAL ALGEBRA IN $\text{End}(V)$

In this section we collect some basic definitions and results from topological algebra that will be used in later sections, beginning with a quick review of the most fundamental notions. We expect that readers are familiar with basic point-set topology, particularly including product topologies and convergence of nets (as in [8, Chapters 2–3], for instance).

Recall that a *topological abelian group* is an abelian group G equipped with a topology for which addition and negation form continuous functions $+: G \times G \rightarrow G$ and $-: G \rightarrow G$, where $G \times G$ is considered with the product topology.

A *topological ring* is a ring R equipped with a topology such that R forms a topological abelian group with respect to its addition, and such that multiplication forms a continuous function $R \times R \rightarrow R$. Given a topological ring R , a *left topological module* over R is a topological abelian group M endowed with a left R -module structure, such that R -scalar multiplication forms a continuous function $R \times M \rightarrow M$. Right topological modules are similarly defined. Any subring or submodule of a topological ring or module may be considered again as a topological ring or module in its subspace topology. The reader is referred to [15] for further background on topological rings and modules.

All topologies that we consider in this paper will be Hausdorff. A topological abelian group is Hausdorff if and only if the singleton $\{0\}$ is closed for its topology [15, Theorem 3.4]. Note that the quotient of a topological ring by an ideal or the quotient of a topological module by a submodule is again a topological ring or module under the quotient topology (the finest topology with respect to which the quotient map is continuous); if the ideal or submodule is closed, then the induced topology on the quotient will consequently be Hausdorff.

We will also require the notion of completeness for topological rings and modules. Let G be a Hausdorff topological abelian group, so that any net in G has at most one limit (see [8], Theorem 3 of Chapter 2). We say that a net $(g_i)_{i \in I}$ in G is *Cauchy* if, for every open neighborhood $U \subseteq G$ of 0, there exists $N \in I$ such that, for all $m, n \geq N$ in the directed set I , one has $g_m - g_n \in U$. We say that G is *complete* if every Cauchy net in

G converges. As closed subsets contain limits of nets, a closed subgroup of a complete topological abelian group is complete. (Note that some references such as [1, III.3] and [15, I.7] define completeness in terms of Cauchy *filters* rather than Cauchy nets. However, one can reconcile these two apparently different notions as equivalent by noting the connection between convergence of nets and convergence of filters, as in [8, Problem 2.L(f)].)

A left topological module M over a topological ring R is said to be *linearly topologized* if it has a neighborhood basis of 0 consisting of submodules of M . We say that a ring R has a *left linear topology* (or is *left linearly topologized*) if ${}_R R$ is a linearly topologized left R -module (i.e., R has a neighborhood basis of 0 consisting of left ideals). Any subring or submodule of a (left) linearly topologized ring is clearly (left) linearly topologized in the induced topology.

2.A. The finite topology. Let X be a set and Y a topological space, and let Y^X denote the set of all functions $X \rightarrow Y$. The *topology of pointwise convergence* on Y^X is the product topology under the identification $Y^X = \prod_X Y$.

If Y has the discrete topology, Y^X has a base of open sets given by sets of the following form, for any fixed finite lists of elements $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in Y$:

$$\{f: X \rightarrow Y \mid f(x_1) = y_1, \dots, f(x_n) = y_n\}.$$

This is often called the *finite topology* on Y^X ; as a product of discrete spaces, this space is locally compact and Hausdorff.

Now let V be a right vector space over a division ring D . Then $\text{End}(V) = \text{End}_D(V) \subseteq V^V$ is a closed subset of V^V (as in the proof of [3, Proposition 1.2.1]). It inherits a topology from the finite topology on V^V ; we will also refer to this as the *finite topology* on $\text{End}(V)$. Under this topology, $\text{End}(V)$ is a topological ring [15, Theorem 29.1]. Alternatively, we may view the finite topology on $\text{End}(V)$ as the left linear topology generated by the neighborhood base of 0 given by the sets of the following form, where $X \subseteq V$ is a finite-dimensional subspace:

$$X^\perp = \{T \in \text{End}(V) \mid T|_X = 0\}.$$

These neighborhoods of 0 coincide with the left annihilators of the form $\text{ann}_\ell(F)$ where $F \in \text{End}(V)$ has finite rank. Furthermore, for every such left ideal $I = X^\perp$, the left $\text{End}(V)$ -module $\text{End}(V)/I$ has finite length. Indeed, one can readily verify that the map $\text{End}(V) \rightarrow \text{Hom}_D(X, V)$ given by $T \mapsto T|_X$ fits into the short exact sequence of left $\text{End}(V)$ -modules

$$0 \rightarrow X^\perp \rightarrow \text{End}(V) \rightarrow \text{Hom}_D(X, V) \rightarrow 0,$$

and that $\text{Hom}_D(X, V) \cong V^{\dim(X)}$ is a semisimple left $\text{End}(V)$ -module of finite length.

Because the discrete abelian group V is Hausdorff and complete, so are V^V and the closed subspace $\text{End}(V)$. Thus we see that *the topological ring $\text{End}(V)$ is left linearly topologized, Hausdorff and complete.*

Throughout this paper, whenever we refer to $\text{End}(V)$ as a topological ring or algebra, we will always consider it to be equipped with the finite topology. Also, if the underlying division ring $D = K$ is a field, then we will consider V as a left K -vector space as in classical linear algebra.

Remark 2.1. If R is any topological ring and S is a subring of R , then its closure \overline{S} in R is readily verified to be a subring of R as well. For instance, to check that \overline{S} is closed under

multiplication, let $m: R \times R \rightarrow R$ be the multiplication map. Since S is a subring we have $S \times S \subseteq m^{-1}(S)$, and by continuity of m we find that $\overline{S} \times \overline{S} = \overline{S \times S} \subseteq m^{-1}(\overline{S})$, proving that \overline{S} is closed under multiplication.

2.B. Commutative closed subalgebras. The following lemma shows that in the topological algebra $\text{End}(V)$, commutativity behaves well under closure. It is surely known in other settings (for part (1), see [15, Theorem 4.4]), but we include a brief argument for completeness. For a subset X of a ring R , its *commutant* (often called the *centralizer*) is the subring

$$X' = \{r \in R \mid rx = xr \text{ for all } x \in X\}.$$

Lemma 2.2. *Let R be a Hausdorff topological ring.*

- (1) *The closure of any commutative subring of R is commutative.*
- (2) *The commutant of any subset of R is closed in R .*
- (3) *Any maximal commutative subring of R is closed.*

Proof. Define a function $f: R \times R \rightarrow R$ by $f(x, y) = xy - yx$. Because R is a topological ring, f is continuous (where $R \times R$ is given the product topology). By the Hausdorff property, the singleton $\{0\}$ is closed in R . It follows that $f^{-1}(0)$ is closed in $R \times R$.

(1) Suppose that $C \subseteq R$ is a commutative subring. Then $C \times C \subseteq R \times R$ is contained in the closed subset $f^{-1}(0)$. It follows that $\overline{C} \times \overline{C} = \overline{C \times C} \subseteq f^{-1}(0)$, whence \overline{C} is commutative.

(2) Let $X \subseteq R$. For each $x \in X$, define $g_x: R \rightarrow R$ by $g_x(y) = f(x, y)$. Because f is continuous, so is each g_x . So $X' = \bigcap_{x \in X} g_x^{-1}(0)$ is an intersection of closed sets and is therefore closed in R .

(3) If $C \subseteq R$ is a maximal commutative subring, then $C = \overline{C}$ follows from either (1) (by maximality) or from (2) (as maximality implies $C = C'$). \square

If H is a Hilbert space and $B(H)$ is the algebra of bounded operators on H , then there is a well-known characterization of maximal abelian $*$ -subalgebras of $B(H)$ in measure-theoretic terms [12, Theorem 1], which can be used to describe their structure quite explicitly (for instance, see the proof of [5, Lemma 6.7]). Lest one hope that maximal commutative subalgebras of $\text{End}(V)$ should admit such a simple description, Proposition 2.3 below shows that a full-blown classification of such subalgebras is a wild problem.

For the next few results, we restrict to vector spaces V over a field K . Suppose that A is a subalgebra of $\text{End}(V)$. Then the commutant $A' \subseteq \text{End}(V)$ is the set of those operators whose action commutes with that of every element of A . That is to say, A' is the endomorphism ring (acting from the left) of V considered as a left A -module; in symbols, one might write $A' = \text{End}_A(V) \subseteq \text{End}_K(V)$.

Proposition 2.3. *Let V be a K -vector space and A be a K -algebra with $\dim(A) = \dim(V)$. Then $\text{End}(V)$ contains a closed subalgebra (even a commutant) that is topologically isomorphic to A with the discrete topology. If A is commutative, then this subalgebra may furthermore be chosen to be a maximal commutative subalgebra.*

Proof. If $\phi: A \rightarrow V$ is a vector space isomorphism, we can identify $\text{End}(A) = \text{End}_K(A)$ with $\text{End}_K(V)$, and we may prove everything in $\text{End}(A)$. The left regular representation $\lambda: A \rightarrow \text{End}(A)$, where $\lambda(a): x \mapsto ax$, and the right regular representation $\rho: A \rightarrow \text{End}(A)$,

where $\rho(a): x \mapsto xa$, respectively define a homomorphism and an anti-homomorphism of algebras. Certainly ρ and λ are injective because $\rho(a)$ and $\lambda(a)$ both send $1 \mapsto a$; furthermore, the intersection of the open set $\{T \in \text{End}(A) \mid T(1) = a\}$ with either $\lambda(A)$ or $\rho(A)$ is the singleton $\{a\}$. Therefore, the induced topologies from $\text{End}(A)$ on the algebras $\lambda(A)$ and $\rho(A)$ are both discrete, and the maps λ and ρ are continuous.

By the comments preceding this proposition, the commutants $\lambda(A)'$ and $\rho(A)'$ are respectively the endomorphism rings of A considered as a left and right module over itself. It is well known that these endomorphism rings are $\lambda(A)' = \rho(A)$ (that is, left A -module endomorphisms of A are right-multiplication operators) and $\rho(A)' = \lambda(A)$; for instance, see [9, Example 1.12]. Thus $\lambda(A)'' = \rho(A)' = \lambda(A)$, and $\lambda(A)$ is closed in $\text{End}(V)$ thanks to Lemma 2.2(2). Consequently, $\lambda(A)$ a discrete closed subalgebra of $\text{End}(A)$.

Finally, note that in the case where A is commutative we have $\lambda(a) = \rho(a)$ for all $a \in A$. So $\lambda(A)' = \rho(A) = \lambda(A)$, proving that the image of A in $\text{End}(V)$ is a maximal commutative subalgebra. \square

The property of being a discrete subalgebra of $\text{End}(V)$ translates nicely into representation-theoretic terms. First, having a subalgebra $A \subseteq \text{End}(V)$ is equivalent to saying that V is a faithful A -module. Faithfulness of V is equivalent to the existence of a set $\{v_i \mid i \in I\} \subseteq V$ such that, for any $a \in A$, if $av_i = 0$ for all i then $a = 0$. But the existence of such a set is further equivalent to the existence of a left A -module embedding $A \hookrightarrow V^I$, given by $a \mapsto (av_i)_{i \in I}$. The next proposition shows that being a discrete subalgebra amounts the requirement that this embedding is into a finite power of V .

Proposition 2.4. *Let $A \subseteq \text{End}(V)$ be a subalgebra. Then A is discrete if and only if there is a left A -module embedding of A into V^n for some positive integer n .*

Proof. The subalgebra A is discrete if and only if $\{0\}$ is open in A , which is further equivalent to $\{0\} = A \cap \{T \in \text{End}(V) \mid T(W) = 0\}$, for some finite-dimensional subspace W of V . Thus if A is discrete then there are $w_1, \dots, w_n \in V$ such that $aw_i = 0$ ($i = 1, \dots, n$) implies $a = 0$ for all $a \in A$, so that the map $\varphi: A \rightarrow V^n$ given by $\varphi(a) = (aw_i)_i \in V^n$ is an injective morphism of A -modules. Conversely, assume that there is an injective A -module homomorphism $\phi: A \rightarrow V^n$, and denote $\phi(1) = (w_1, \dots, w_n)$. Then $aw_i = 0$ implies $a = 0$ for all $a \in A$ thanks to injectivity of ϕ , which translates as above to A being a discrete subalgebra of $\text{End}(V)$. \square

Corollary 2.5. *Let V be an infinite-dimensional K -vector space and A a K -algebra. Then $\text{End}(V)$ contains a closed subalgebra that is topologically isomorphic to A with the discrete topology if and only if $\dim(A) \leq \dim(V)$.*

Proof. Suppose that $\dim(A) \leq \dim(V)$, and let B be any K -algebra with $\dim(B) = \dim(V)$. Then $\dim(A \otimes B) = \dim(A) \dim(B) = \dim(B)$, since $\dim(B) = \dim(V)$ is infinite. So, by Proposition 2.3, $\text{End}(V)$ has closed subalgebra that is topologically isomorphic to $A \otimes B$ with the discrete topology, which in turn contains the closed discrete subalgebra $A \otimes 1 \cong A$.

Converely, if $A \subseteq \text{End}(V)$ is a discrete subalgebra, then $A \hookrightarrow V^n$ with n finite by Proposition 2.4. This shows that $\dim(A) \leq n \cdot \dim(V) = \dim(V)$ since $\dim(V)$ is infinite. \square

We use the above to illustrate a few examples of discrete maximal commutative subalgebras of $\text{End}(V)$ generated by “shift operators” when V has countably infinite dimension.

Example 2.6. Let $A = K[x]$ be the polynomial algebra in a single indeterminate. Let V be a K -vector space with basis $\{v_0, v_1, v_2, \dots\}$. Then Proposition 2.3 implies that $\text{End}(V)$ has a discrete maximal commutative subalgebra isomorphic to A . Under the K -linear isomorphism $A \cong V$ that sends $x^i \mapsto v_i$, the embedding $A \rightarrow \text{End}(V)$ of Proposition 2.3 sends x to the right-shift operator $S \in \text{End}(V)$, with

$$S(v_i) = v_{i+1} \quad \text{for all } i \geq 0.$$

Example 2.7. Let $A = K[x, x^{-1}]$ be the Laurent polynomial ring over K . If V is a K -vector space with basis $\{v_i \mid i \in \mathbb{Z}\}$, then the vector space isomorphism $A \cong V$ sending $x^i \mapsto v_i$ gives an embedding $A \rightarrow \text{End}(V)$ of A onto a discrete maximal commutative subalgebra of $\text{End}(V)$. One can check that the embedding provided in the proof of Proposition 2.3 sends $x \in A$ to the invertible “infinite shift” operator $T \in \text{End}(V)$, with

$$T(v_i) = v_{i+1} \quad \text{for all } i \in \mathbb{Z}.$$

2.C. Summability of idempotents. Another topic that will play a role in diagonalizability of subalgebras is the ability to form the “sum” of an infinite set of orthogonal idempotents.

Definition 2.8. [15, Section 10] Let G be a Hausdorff topological abelian group, and let $\{g_i \mid i \in I\} \subseteq G$. If the net of finite sums of the g_i , indexed by finite subsets of I , converges to a limit in G , then we write this limit as $\sum g_i$, and we say for brevity that $\sum g_i$ *exists* (in G) or that the set $\{g_i\}$ is *summable*.

Perhaps the most basic nontrivial example of a summable set of elements is as follows: let I be a set, and consider the product space K^I as a topological K -algebra. Let $e_i \in K^I$ denote the idempotent whose i th coordinate is 1 and whose other coordinates are 0. Then for any $\lambda_i \in K$, the set $\{\lambda_i e_i\}$ is summable; in fact, it is clear that each element $x \in K^I$ has a unique expression of the form $x = \sum \lambda_i e_i$.

Recall that there are three preorderings for idempotents in a ring R : given idempotents $e, f \in R$, one defines

$$\begin{aligned} e \leq_\ell f &\iff ef = e &&\iff Re \subseteq Rf \\ e \leq_r f &\iff fe = e &&\iff eR \subseteq fR \\ e \leq f &\iff ef = e = fe &&\iff eRe \subseteq fRf. \end{aligned}$$

While the last of these is a partial ordering, the first two generally are not. (For instance, the idempotents $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ in $\mathbb{M}_2(\mathbb{Q})$ satisfy $e \leq_\ell f \leq_\ell e$ with $e \neq f$.)

Lemma 2.9. *Let R be a Hausdorff topological ring with an orthogonal set of idempotents $\{e_i \mid i \in I\} \subseteq R$. Suppose that $\sum e_i$ exists. Then $\sum e_i$ is an idempotent that is the least upper bound for the set $\{e_i\}$ in the preorderings \leq_ℓ and \leq_r and the partial ordering \leq .*

Proof. Write $e = \sum e_i$. Given a finite subset $J \subseteq I$, write $e_J = \sum_{i \in J} e_i$, so e is the limit of the net (e_J) . Because each e_J lies in the zero set of the continuous function $R \rightarrow R$ given by $x \mapsto x - x^2$, it follows that the limit e also lies in this set. So $e = e^2$ is idempotent. Further, $e_k \leq e$ for all $k \in I$ (and $e_k \leq_\ell e$, $e_k \leq_r e$) because for any finite $J \subseteq I$ with $J \supseteq \{k\}$, $e_k e_J = e_J e_k = e_k$, and the continuity of the functions $x \mapsto e_k x$, $x \mapsto x e_k$ shows

that the relation holds in the limit as well: $e_k e = e e_k = e$. Thus e is an upper bound for the idempotents e_i with respect to \leq_ℓ and \leq_r (and consequently, \leq).

To see that e is the least upper bound of the e_i , let $f \in R$ be idempotent, and suppose that $e_i \leq_\ell f$ for all i . Then for all finite $J \subseteq I$, we have $e_J f = \sum_{j \in J} e_j f = \sum_{j \in J} e_j = e_J$. Thus all e_J lie in the closed subset $\text{ann}_\ell(1-f) \subseteq R$, so we also have $e = \lim e_J \in \text{ann}_\ell(1-f)$, which translates to $e \leq_\ell f$. Similarly, if $f = f^2 \in R$ satisfies $e_i \leq_r f$ (respectively, $e_i \leq f$) for all i , then $e \leq_r f$ (respectively, $e \leq f$). So e is a least upper bound for the e_i in all of these (pre)orderings. \square

Suppose that $\{E_i \mid i \in I\}$ is an orthogonal set of idempotents in $\text{End}(V)$. Notice that the sum of subspaces $\sum \text{range}(E_i) \subseteq V$ is direct. Indeed, if $\sum r_i = 0$ in V where $r_i \in \text{range}(E_i)$ are almost all zero, then for every $j \in I$ we have $0 = E_j(\sum r_i) = E_j(\sum E_i(r_i)) = E_j(r_j) = r_j$ as desired. Similarly, it is straightforward to show that the following sum of subspaces in V is direct:

$$(2.10) \quad \left(\bigoplus \text{range}(E_i) \right) \oplus \left(\bigcap \ker(E_i) \right) \subseteq V.$$

Whether or not this direct sum is equal to the whole space V strictly controls the existence of $\sum E_i$ in $\text{End}(V)$.

Lemma 2.11. *Let $R = \text{End}(V)$, and let $\{E_i \mid i \in I\} \subseteq R$ be an orthogonal set of idempotents. The following are equivalent:*

- (a) $\sum E_i$ exists in R ;
- (b) $V = \left(\bigoplus \text{range}(E_i) \right) \oplus \left(\bigcap \ker(E_i) \right)$ (i.e., the containment (2.10) is an equality);
- (c) For every $v \in V$, the set $\{i \in I \mid v \notin \ker(E_i)\}$ is finite;
- (d) For every finite-dimensional subspace $W \subseteq V$, there exists a finite subset $J \subseteq I$ such that $(1 - \sum_{j \in J} E_j)W \subseteq \bigcap \ker(E_i)$.

When the above conditions hold, then $\{E_i \mid i \in I_0\}$ is also summable for any subset $I_0 \subseteq I$, and $\sum_{i \in I} T_i$ exists for any elements $T_i \in E_i R E_i$.

Proof. First assume that (a) holds; we will verify (d). Let $W \subseteq V$ be any finite-dimensional subspace. Denote $E = \sum E_i$, the limit of the finite sums of the E_i . Because $\{T \in \text{End}(V) \mid T|_W = E|_W\}$ is an open neighborhood of E in $\text{End}(V)$, there exists a finite subset $J \subseteq I$ such that $\sum_{j \in J} E_j$ lies in this set. Then for every $i \in I$, since $E_i \leq E$ gives $E_i(1 - E) = 0$, we conclude that $E_i(1 - \sum_{j \in J} E_j)W = E_i(1 - E)W = 0$.

Now assume (d) holds. To prove (c), let $v \in V$ and consider the subspace $W = \text{Span}(v)$ of V . Then there exists a finite subset $J \subseteq I$ such that, for $E_J = \sum_{j \in J} E_j$ we have $(1 - E_J)W \subseteq \bigcap \ker(E_i)$. Thus for all $i \in I$, $E_i(1 - E_J)(v) = 0$, from which it follows that

$$E_i(v) = E_i \sum_{j \in J} E_j(v) = \sum_{j \in J} E_i E_j(v).$$

The last sum is 0 whenever $i \notin J$, and so $\{i \in I \mid v \notin \ker(E_i)\} \subseteq J$ is finite, as desired.

Next suppose that (c) holds; we will verify (b). Let $v \in V$, so that the set $S = \{i \in I \mid v \notin \ker(E_i)\}$ is finite by hypothesis. Then $E_i v = 0$ for all $i \notin S$. With this, one can readily verify that $v = (\sum_{i \in S} E_i)v + (1 - \sum_{i \in S} E_i)v$ is a sum of elements in $\bigoplus \text{range}(E_i)$ and $\bigcap \ker(E_i)$, respectively. So equality indeed holds in (2.10).

Now assume that (b) holds, and let $T_i \in E_i R E_i$ for all i . We will prove that $\sum T_i$ exists in R ; this will imply that (a) holds (the particular case where all $T_i = E_i$). Define $T \in R$ to be the linear operator on V with kernel $L = \bigcap \ker(E_i)$ whose restriction to each $U_i = E_i(V)$ agrees with T_i . A basic open neighborhood of T in R has the form $\{S \in R \mid S|_X = T|_X\}$ for some finite set $X \subseteq V$. There is a finite set $J \subseteq I$ such that $X \subseteq \left(\bigoplus_{j \in J} U_j\right) \oplus L$. It follows that for any finite subset $J' \subseteq I$ with $J' \supseteq J$, the operators T and $\sum_{i \in J'} T_i$ have the same restrictions to $\left(\bigoplus_{j \in J} U_j\right) \oplus L \supseteq X$. Thus the net of finite sums of the T_i converges to T .

Finally, to see that $\sum_{i \in I_0} E_i$ exists for any $I_0 \subseteq I$, notice that condition (c) still holds when I is replaced with I_0 . \square

In the proof above, the idempotent sum $\sum E_i$ is explicitly described as the projection of V onto the subspace $\bigoplus \text{range}(E_i)$ with kernel equal to $\bigcap \ker(E_i)$.

Corollary 2.12. *For an orthogonal set of idempotents $\{E_i \mid i \in I\} \subseteq \text{End}(V)$, the following are equivalent:*

- (a) $\{E_i\}$ is summable and $\sum E_i = 1 \in \text{End}(V)$;
- (b) $\bigoplus \text{range}(E_i) = V$;
- (c) For every finite-dimensional subspace $W \subseteq V$, there exists a finite subset $J \subseteq I$ such that $\sum_{j \in J} E_j$ restricts to the identity on W .

Proof. This follows from a straightforward translation of conditions (a), (b), and (d) in Lemma 2.11, realizing that if $\{E_i\}$ is summable, then $\sum E_i = 1$ if and only if $\bigcap \ker(E_i) = \{0\}$. \square

The following example shows that an arbitrary set of orthogonal idempotents in $\text{End}(V)$ need not be summable.

Example 2.13. Let V be a K -vector space with countably infinite basis $\{v_0, v_1, \dots\}$. Consider the K -algebra A generated by countably many idempotents $x_1, x_2, \dots \in A$ subject to the conditions $x_i x_j = 0$ when $i \neq j$. Then A has K -basis given by $\{1, x_1, x_2, \dots\}$. Under the isomorphism (identification) $\phi: A \rightarrow V$ given by $\phi(1) = v_0$ and $\phi(x_i) = v_i$ for $i \geq 1$, the proof of Proposition 2.3 provides an injective homomorphism $\lambda: A \rightarrow \text{End}(V)$ such that $\lambda(A)$ is a closed discrete maximal commutative subalgebra of $\text{End}(V)$. Denote each $\lambda(x_i) = E_i \in \text{End}(V)$. (If we let $\{E_{ij} \mid i, j \geq 0\}$ denote the infinite set of matrix units of $\text{End}(V)$ with respect to the basis $\{v_i \mid i \geq 0\}$, then we may explicitly describe $E_i = E_{i0} + E_{ii}$.) One may quickly deduce from Lemma 2.11(c) (with $v = v_0$) that $\sum E_i$ does not exist in $\text{End}(V)$.

Lemma 2.14. *Suppose that $\{E_i \mid i \in I\}$ and $\{F_j \mid j \in J\}$ are orthogonal sets of idempotents in $\text{End}(V)$ such that $E = \sum E_i$ and $F = \sum F_j$ exist. Furthermore, suppose that all of the E_i and F_j commute pairwise. Then the orthogonal set of idempotents $\{E_i F_j \mid (i, j) \in I \times J\}$ is also summable, and $\sum_{i,j} E_i F_j = EF$.*

Proof. We begin by remarking that $E_i F_j = F_j E_i$ for all i and j implies that E and F commute with each other and with each E_i and F_j .

Fix a finite-dimensional subspace $W \subseteq V$. Then there exist finite sets $I_0 \subseteq I$ and $J_0 \subseteq J$ such that $E - \sum_{i \in I_0} E_i$ and $F - \sum_{j \in J_0} F_j$ both restrict to zero on W . Then

$$\begin{aligned} EF - \sum_{I_0 \times J_0} E_i F_j &= EF - \left(\sum_{I_0} E_i \right) F + \left(\sum_{I_0} E_i \right) F - \left(\sum_{I_0} E_i \right) \left(\sum_{J_0} F_j \right) \\ &= F \left(E - \sum_{I_0} E_i \right) + \sum_{I_0} E_i \left(F - \sum_{J_0} F_j \right). \end{aligned}$$

The latter expression also restricts to zero on W , and since W was arbitrary, we conclude that $\sum E_i F_j$ exists and is equal to EF . \square

2.D. Topological modules. In the remainder of this section, we record some results relating to topological modules that will be of use when considering pseudocompact modules. The first is a link between open submodules and discrete modules.

Lemma 2.15. *Let M be a topological module over a topological ring R , and let N be a submodule of M . Then the following are equivalent:*

- (a) N is open;
- (b) M/N is discrete in the quotient topology;
- (c) N is the kernel of a continuous homomorphism from M to a discrete topological R -module.

Furthermore, any open submodule of M is closed, and any submodule of M containing an open submodule is itself open.

Proof. That (b) implies (c) follows from the fact that the canonical surjection $M \rightarrow M/N$ is continuous when M/N is equipped with the quotient topology. To see that (c) implies (a), simply consider a continuous module homomorphism $f: M \rightarrow D$ as in (c) and note that $N = f^{-1}(0)$ where $\{0\}$ is open in D . Finally, suppose (a) holds and let $\pi: M \rightarrow M/N$ be the canonical surjection. For any coset $x + N \in M/N$, one has $\pi^{-1}(x + N) = x + N \subseteq M$. As N is open, the same is true of the translate $x + N$. It follows that each singleton of M/N is open, whence it is discrete.

That any open submodule of M is closed follows, for instance, from characterization (c) above, as the kernel of a continuous homomorphism to a Hausdorff module is closed. For the final claim, assume that $N_0 \subseteq N \subseteq M$ are submodules where N_0 is open. One may readily verify that N is open as it is the kernel of the composite $M \rightarrow M/N_0 \rightarrow (M/N_0)/(N/N_0) \cong M/N$ of surjective continuous homomorphisms, where the latter two modules are discrete. \square

The next result characterizes modules that are inverse limits of discrete modules. We first make some general remarks on inverse limits of topological modules. (These remarks also apply to limits of general diagrams, but we restrict to inversely directed systems for notational simplicity as these are the only systems we require.) Given a topological ring R , let ${}_R\mathbf{Mod}$ denote the category of left topological R -modules with continuous module homomorphisms. Given an inversely directed system $\{M_j \mid j \in J\}$ with connecting morphisms $\{f_{ij}: M_j \rightarrow M_i \mid i \leq j \text{ in } J\}$ in ${}_R\mathbf{Mod}$, its limit can be constructed via the usual ‘‘product-equalizer’’ method as in [10, V.2]:

$$\varprojlim M_j = \{(m_j) \in \prod M_j \mid f_{ij}(m_j) = m_i \text{ for all } i \leq j \text{ in } J\} \subseteq \prod M_j.$$

(The same product-equalizer construction yields both the inverse limit of the M_j as R -modules and the inverse limit as topological groups.) The topology on the inverse limit is the *initial topology* with respect to the canonical morphisms $g_i: \varprojlim M_j \rightarrow M_i$ for all $i \in J$; this is the topology generated by the subbasis of sets of the form $g_i^{-1}(U)$ for any $i \in J$ and open $U \subseteq M_i$. (In fact, thanks to the inverse directedness of the system M_j , the sets of the form $g_i^{-1}(U)$ are closed under intersection and actually form a basis for the topology: given $i, j \leq k$ in J and open $U \subseteq M_i$ and $V \subseteq M_j$, we have $g_i^{-1}(U) \cap g_j^{-1}(V) = g_k^{-1}(f_{ik}^{-1}(U) \cap f_{jk}^{-1}(V))$.)

Suppose that all of the M_j in the inverse system above are Hausdorff. Then the product of any subset of the M_j will also be Hausdorff. Since the above presentation of the limit can be viewed as the equalizer (i.e., kernel of the difference) of two continuous maps $\prod M_j \rightarrow \prod_{f_{ij}} M_i$ (see [10, Theorem V.2.2]), and the equalizer of morphisms between Hausdorff spaces is closed in the domain, we see that $\varprojlim M_j$ forms a closed submodule of $\prod M_j$. (See also [1, III.7.2].)

Because the universal property of an (inverse) limit $\varprojlim M_j$ identifies it within ${}_R\mathbf{TMod}$ uniquely up to a unique isomorphism, when a topological module L is isomorphic to such a limit (i.e., satisfies the universal property) then we will write $L = \varprojlim M_j$ without danger of confusion.

Remark 2.16. Suppose that \mathcal{P} is any property of Hausdorff topological modules that is preserved when passing to products and closed submodules of modules satisfying \mathcal{P} . Then the (inverse) limit of any system of Hausdorff modules satisfying \mathcal{P} will again have property \mathcal{P} (as it is a closed submodule of a product of modules satisfying \mathcal{P} thanks to the discussion above). In particular, this holds when \mathcal{P} is taken to be either of the properties of being *complete* or *linearly topologized*.

We are now ready to characterize inverse limits of discrete modules. (Note that special cases of the following were given in [16, Theorem 3] and [15, Corollary 5.22].)

Lemma 2.17. *For a left topological module M over a topological ring R , the following are equivalent:*

- (a) M is Hausdorff, complete, and linearly topologized;
- (b) $M = \varprojlim M/N$ in the category ${}_R\mathbf{TMod}$, where N ranges over any neighborhood basis \mathcal{U} of 0 consisting of open submodules of M (and the connecting homomorphisms for $N \supseteq N'$ are the canonical surjections $M/N' \twoheadrightarrow (M/N')/(N/N') \cong M/N$);
- (c) M is an inverse limit of discrete topological R -modules.

When M satisfies the above conditions and \mathcal{U} is any neighborhood basis of 0 consisting of open submodules, the topology on M is induced from the product topology of $\prod_{N \in \mathcal{U}} M/N$ via the usual inclusion of the inverse limit.

Proof. Suppose that (a) holds and let \mathcal{U} be a neighborhood basis of 0 consisting of open submodules of M . To deduce (b), first note that the Hausdorff property of M ensures that natural map $\phi: M \rightarrow \prod_{N \in \mathcal{U}} M/N$ is an embedding. The image of this map lies in the closed submodule $\varprojlim_{N \in \mathcal{U}} M/N$ of $\prod M/N$. To see that ϕ is surjective, let $x \in \varprojlim_{N \in \mathcal{U}} M/N$ be represented by the compatible family $(x_N)_{N \in \mathcal{U}}$ with $x_N \in M/N$ for all $N \in \mathcal{U}$. For each $N \in \mathcal{U}$, fix $y_N \in M$ such that $y_N + N = x_N$. Then it is straightforward to show from the

compatibility condition on the x_N that $(y_N)_{N \in \mathcal{U}}$ forms a Cauchy net, which converges to some $y \in M$ by completeness. We claim that this element satisfies $y + N = y_N + N = x_N$ for all $N \in \mathcal{U}$. Indeed, as $y = \lim y_N$, for fixed $N \in \mathcal{U}$ there exists $N' \subseteq N$ such that $y - y_{N'} \in N$. But compatibility of the family $(x_N) = (y_N + N)$ implies that $y_{N'} + N = y_N + N$, so that $y + N = y_{N'} + N = y_N + N$. It follows that y has image $\phi(y) = (y + N)_{N \in \mathcal{U}} = (x_N)_{N \in \mathcal{U}} = x$, as desired. Finally, to see that ϕ is a homeomorphism onto the inverse limit, it suffices to show that upon identifying M with the inverse limit, they share a common neighborhood basis of 0. As each of the components of the product $\prod_{N \in \mathcal{U}} M/N$ is discrete by Lemma 2.15, the comments preceding this lemma imply that a neighborhood basis of $\varprojlim M/N$ is given by $\pi_N^{-1}(0)$, where each π_N is the projection of the product onto the corresponding factor for $N \in \mathcal{U}$. But under the isomorphism $M \cong \varprojlim M/N$, each $\pi_N^{-1}(0)$ corresponds to $N \in \mathcal{U}$, so the claim is proved. Moreover, this establishes the last claim about the topology on M being induced by that of $\prod_{N \in \mathcal{U}} M/N$.

Clearly (b) implies (c). Finally, assume that (c) holds, so that $M = \varprojlim M_i$ is the limit of a directed system of discrete topological modules M_i . Because the M_i are Hausdorff, complete, and linearly topologized, we see from Remark 2.16 that the limit $M = \varprojlim M_i$ satisfies condition (a). \square

Definition 2.18. A topological module that satisfies the equivalent conditions of the previous lemma will be called *pro-discrete*. A topological ring R is *left pro-discrete* if ${}_R R$ is pro-discrete as a topological R -module, and right pro-discrete rings are similarly defined. Further, R is said to be *pro-discrete* if it is Hausdorff, complete, and has a neighborhood basis of zero consisting of two-sided ideals.

Remark 2.19. As the terminology suggests, a topological ring R is left and right pro-discrete if and only if it is pro-discrete. Clearly every pro-discrete ring is left and right pro-discrete. Conversely, suppose that R is left and right pro-discrete, and let N be an open left ideal. Since R is right pro-discrete, there is an open right ideal M such that $M \subseteq N$, and again, by left pro-discreteness there is an open left ideal N' with $N' \subseteq M$. Then $I = RM \subseteq N$ since N is a left ideal. Now I contains $RN' = N'$, so that I is open by Lemma 2.15. Hence, every open neighborhood of 0 contains an open two-sided ideal I , making R pro-discrete.

In the next lemma we note that the property of pro-discreteness is inherited by closed subrings. (A similar statement holds for topological modules, but we will not make use of this fact.)

Lemma 2.20. *Every closed subring of a (left) pro-discrete ring is (left) pro-discrete.*

Proof. Suppose B is a (left) pro-discrete ring with a closed subring $A \subseteq B$. Then A is certainly Hausdorff in its inherited topology, and because A is closed in the complete ring B we find that A is also complete. Finally, any open (left) ideal I of B intersects to an open (left) ideal $I \cap A$ of A . Because such I form a neighborhood base of zero in B , the intersections $I \cap A$ form a neighborhood base of zero in A . Thus A is (left) linearly topologized, showing that it is (left) pro-discrete. \square

In particular, our discussion of the finite topology above makes it clear that the topological ring $\text{End}(V)$ for a right D -vector space V is left pro-discrete. Thus every closed subring of $\text{End}(V)$ is left pro-discrete.

In the case when $D = K$ is a field, this raises an interesting question about representations of topological algebras. Fixing an infinite-dimensional K -vector space V , to what extent can one characterize those left pro-discrete K -algebras that can be realized as closed subalgebras of $\text{End}(V)$? Corollary 2.5 characterizes exactly which *discrete* algebras have such a representation, purely in terms of their dimension. A topological characterization would necessarily extend this result.

3. INFINITE WEDDERBURN-ARTIN THEOREM

As mentioned in the introduction, the structure theory of artinian rings plays an important role in the theory of diagonalizability for operators on a finite-dimensional vector space V . Every subalgebra of $\text{End}(V)$ is finite-dimensional and therefore artinian; diagonalizable subalgebras are furthermore semisimple.

If V is infinite-dimensional, we have seen in Proposition 2.3 that $\text{End}(V)$ contains a wild array of discrete subalgebras. But it turns out that $\text{End}(V)$ itself, as well as its diagonalizable subalgebras, satisfy a well-known condition of being “almost left artinian” (in a topological sense) called *pseudocompactness*. In fact, they satisfy an even stronger “almost finite-dimensional” property that we shall call *K -pseudocompactness*. Further, both of these algebras of interest are Jacobson semisimple, so that they satisfy a suitable infinite-dimensional version of semisimplicity.

In this section we aim to present a Wedderburn-Artin theorem for left pseudocompact, Jacobson semisimple rings in Theorem 3.10, which gathers some known and somewhat independent results on topological semisimplicity along with some new ones. Using our methods, we recover some results from [16, 17], along with the classical Wedderburn-Artin theorem and similar types of theorems for algebras and coalgebras.

3.A. Pseudocompact and linearly compact modules and rings. Pseudocompactness is an important property in topological algebra that expresses a particular way for a module or ring to be “close to having finite length.” We recall the definition after giving a few equivalent formulations of this property.

We say that a submodule N of a module M has *finite colength* if M/N is a module of finite length.

Lemma 3.1. *Let M be a left topological module over a topological ring R . The following are equivalent:*

- (a) M is Hausdorff, complete, and has a neighborhood basis of 0 consisting of open submodules of finite colength;
- (b) M is pro-discrete and every open submodule of M has finite colength;
- (c) M is an inverse limit in ${}_R\mathbf{TMod}$ of discrete topological R -modules of finite length.

Proof. The equivalence of (a) and (c) and the implication (b) \Rightarrow (c) follow directly from Lemma 2.17 using the basis \mathcal{U} of open submodules of M that have finite colength. Now assume that (a) holds; we will deduce (b). It follows from Lemma 2.17 that M is pro-discrete. Given any open submodule N of M , we are given that there exists an open submodule $L \subseteq N$ of finite colength. It follows immediately that N has finite colength, as desired. \square

A module satisfying the equivalent conditions above is called a *pseudocompact* module. A topological ring R is said to be *left pseudocompact* if ${}_R R$ is a pseudocompact topological module. We also recall that a complete Hausdorff topological ring R which has a basis of neighborhoods of 0 consisting of two-sided ideals I such that R/I has finite length both on the left and on the right (i.e., R/I is a two-sided artinian ring) is called a *pseudocompact ring*.

Remark 3.2. Notice that a topological ring R is left and right pseudocompact if and only if it is pseudocompact. The argument is identical to the one given in Remark 2.19, taking into account that if the one-sided ideals M, N, N' there have finite colength, then the ideal I has finite colength as both a left and a right ideal, making R/I an artinian ring.

The following key example is of particular interest to us.

Example 3.3. Let D be a division ring and V a right D -vector space. As shown in the discussion of the finite topology in Section 2, the topological ring $\text{End}(V)$ is Hausdorff, complete, and has a basis of open left ideals having finite colength. Thus $\text{End}(V)$ is left pseudocompact.

Remark 3.4. We also observe that in the example above, if V has infinite dimension over D , then $\text{End}(V)$ is not right pro-discrete and thus is not right pseudocompact. Indeed, if $\text{End}(V)$ is pro-discrete, then it has a neighborhood basis of 0 consisting of two-sided ideals. It is well known that the left socle Σ of $\text{End}(V)$ (the sum of all the simple left ideals of $\text{End}(V)$) is equal to the ideal of finite rank operators on V (see [9, Exercise 11.8]), and that Σ is the smallest nonzero two-sided ideal of $\text{End}(V)$ (see [9, Exercise 3.16]). Hence $\Sigma \subseteq I$ for every non-zero open ideal I , and since $\text{End}(V)$ is Hausdorff, it follows that 0 must be open for $\text{End}(V)$ to be pro-discrete. Thus there is a finite-dimensional subspace $W \subseteq V$ with $W^\perp = 0$, which shows that $V = W$ must be finite-dimensional.

Given a topological ring R , let ${}_R \text{PC}$ denote the full subcategory of ${}_R \text{TMod}$ whose objects are the left pseudocompact topological R -modules. For the portions of our Wedderburn-Artin theorem that mimic the structure of module categories over semisimple rings (e.g., every short exact sequence splits), we will make use of Gabriel duality for categories of pseudocompact modules. To this end, we recall that a *Grothendieck category* is an abelian category with a *generator* (i.e., an object G such that, for any pair of morphisms $f, g: X \rightarrow Y$ with the same domain and codomain, if $f \neq g$ then there exists $h: G \rightarrow X$ such that $f \circ h \neq g \circ h$) in which all small coproducts exist, and with exact direct limits. Also, an abelian category is said to be *locally finite* if every object is the (directed) colimit of its finite-length subobjects. We recommend [14, Chapters IV–V] as a basic reference on abelian and Grothendieck categories.

Remark 3.5. By a well-known result of Gabriel, the opposite category ${}_R \text{PC}^{\text{op}}$ is Grothendieck and locally finite for any left pseudocompact ring R . What is not so well-known is that essentially the same proof of [4, Théorème IV.3] applies more generally to show that, *for any (not necessarily left pseudocompact) topological ring R , the category ${}_R \text{PC}$ is abelian and $\mathcal{C} = {}_R \text{PC}^{\text{op}}$ is Grothendieck and locally finite.* Gabriel's proof makes use of [4, Propositions IV.10–11], both of which are purely module-theoretic results whose proofs do not make reference to the base ring itself. The only small adjustment that one needs to make is in identifying a

cogenerator for the category ${}_R\mathbf{PC}$, as follows. We claim that a family of cogenerators for ${}_R\mathbf{PC}$ is given by the discrete finite-length left R -modules. Indeed, if $f, g: M \rightarrow N$ are morphisms in ${}_R\mathbf{PC}$ with $f \neq g$, then there exists an element $x \in M$ and an open submodule $U \subseteq N$ such that the composite

$$M \xrightarrow{f-g} N \twoheadrightarrow N/U$$

sends x to a nonzero element. Because N/U is a discrete finite-length left R -module, we conclude that such modules indeed form a cogenerating set for ${}_R\mathbf{PC}$. The isomorphism classes of discrete finite-length R -modules form a set (as all of them are isomorphic to R^n/U for some integer $n \geq 1$ and some open submodule $U \subseteq R^n$), so the product of a set of such representatives forms a cogenerator of ${}_R\mathbf{PC}$ (and a generator in the dual category ${}_R\mathbf{PC}^{\text{op}}$).

An elementary observation that shall be used frequently below is that a simple left pseudocompact (or even linearly topologized and Hausdorff) module is discrete, and conversely, every simple discrete left module is pseudocompact.

We will also make use of the notion of linear compactness. A left topological module M over a topological ring R is said to be *linearly compact* if it is linearly topologized, Hausdorff, and every family of closed cosets in M with the finite intersection property (i.e., the intersection of any finite subfamily is nonempty) has nonempty intersection. It is known that every linearly compact module is also complete [15, Theorem 28.5], so linearly compact modules are pro-discrete. Furthermore, it is known that linear compactness is preserved when passing to products and closed submodules of linearly compact modules; see [15, Theorems 28.5–6]. Thus by Remark 2.16 we see that linearly compact modules are closed under (inverse) limits in the category ${}_R\mathbf{TMod}$. Furthermore, since the quotient of a linearly compact module by a closed submodule is again linearly compact [15, Theorem 28.3], we have the following immediate consequence of Lemma 2.17, part of which is found in [15, Theorem 28.15].

Lemma 3.6. *Let M be a left topological module over a topological ring R . The following are equivalent:*

- (a) M is linearly compact;
- (b) M is Hausdorff, complete, and has a neighborhood basis of 0 consisting of open submodules N such that M/N is linearly compact for the discrete topology;
- (c) M is pro-discrete and every open submodule N of M is such that M/N is linearly compact for the discrete topology;
- (d) M is an inverse limit in ${}_R\mathbf{TMod}$ of discrete linearly compact left R -modules.

A discrete artinian module is linearly compact [15, Theorem 28.12], with the following consequence.

Corollary 3.7. *Every pseudocompact left module M over a topological ring R is linearly compact.*

Proof. By Lemma 3.1, M is an inverse limit of discrete modules of finite length. These modules are linearly compact, whence Lemma 3.6 gives that M is linearly compact. \square

Applying this terminology directly to a topological ring R , we say that R is *left linearly compact* if ${}_R R$ is linearly compact, *right linearly compact* if R_R is linearly compact, and *linearly compact* if it is both left and right linearly compact.

3.B. The Jacobson radical of pro-discrete rings. We note that certain results of [4] on the Jacobson radical of pseudocompact rings generalize to the pro-discrete case. These will also be used to derive alternate proofs of results from [16, 17] on Jacobson radicals of prolimits of rings and Jacobson semisimple, linearly compact rings.

We will make use of a natural topological version of the Jacobson radical for a left pro-discrete ring R : we let $J_0(R)$ denote the intersection of all of the open maximal left ideals of R . In the case where R is (two-sided) pro-discrete, this definition is left-right symmetric. For if I is an open ideal of R , all of the left or right ideals containing I are open and R/I is discrete by Lemma 2.15. Thus the two radicals J and J_0 coincide for R/I and there is an open ideal J_I of R such that $J(R/I) = J_0(R/I) = J_I/I$. Since every open maximal left or right ideal contains an open ideal I , we find that the intersection of all open maximal left ideals equals the intersection of all these J_I , which also equals the intersection of all open maximal right ideals.

The following is likely well known in some other form, but we include it for convenience:

Theorem 3.8. *Let R be a pro-discrete ring.*

- (1) *As topological rings, $R = \varprojlim R/I$ where I ranges over the open ideals of R . The topology on R is induced from the product topology of $\prod_{\text{open } I} R/I$ via the usual inclusion of the inverse limit.*
- (2) *An element of R is invertible in R if and only if it is invertible modulo I for every open ideal I .*
- (3) *$J(R) = J_0(R)$, and $J(R)$ is a closed ideal. Moreover, $x \in J(R)$ if and only if $x + I \in J(R/I)$ for all open ideals I , so that under the isomorphism of (1), we have $J(R) = \varprojlim_{\text{open } I} J(R/I)$.*
- (4) *$R/J(R) = \varprojlim_{\text{open } I} R/J_I$, where $I \subseteq J_I \subseteq R$ is such that $J_I/I = J(R/I)$, and $R/J(R)$ is pro-discrete with a neighborhoods basis of 0 consisting of the ideals $J_I/J(R)$.*

Proof. (1) This is a standard argument used, for example, on complete valuation rings, profinite groups, algebras, etc. (see also [4]). It follows as in the proof of Lemma 2.17, noting that the connecting homomorphisms in the inversely directed system of the R/I for open ideals I are (continuous) ring homomorphisms.

(2) We need only prove the if clause. Assume $y + I \in R/I$ is invertible for all open I . Let $u_I \in R$ be such that $u_I + I$ is the inverse of $y + I$ (in R/I). If $I \subseteq L$ are open ideals and $\pi_{I,L} : R/I \rightarrow R/L$ is the canonical surjection, then obviously $\pi_{I,L}(u_I + I) = u_L + L$, since the inverse of $y + L \in R/L$ is unique. Thus we have an element $(u_I) \in \varprojlim_{\text{open } I} R/I$, and under the identification $R = \varprojlim R/I$ this element forms an inverse to y .

(3) Obviously, $J(R) \subseteq J_0(R)$. Conversely, let $x \in J_0(R)$ and let $a \in R$. For each open I , we have $x + I \in J(R/I)$ since every maximal left (or right) ideal of the discrete ring R/I is open; hence, $(1 - ax) + I$ is invertible in R/I . By (2), $1 - ax$ is invertible in R . Thus the ideal $J_0(R)$ must be contained in $J(R)$, yielding the desired equality. Because $J_0(R) = J(R)$ is an intersection of open, hence closed, left ideals, it is closed. An argument similar to the one

above shows that $x \in J(R)$ if and only if $x + I \in J(R/I)$ for all open ideals I . In particular, $J(R) = \varprojlim_{I \text{ open}} J(R/I)$ follows.

(4) There is a canonical map $R = \varprojlim_{I \text{ open}} R/I \longrightarrow \varprojlim_{I \text{ open}} R/J_I$. The kernel of this morphism is $\bigcap_{I \text{ open}} J_I$ and this is equal to $J(R)$ by the discussion preceding the theorem. Consequently, the isomorphism of (4) follows. From this isomorphism we can deduce as in the proof of Lemma 2.17 that $R/J(R)$ is pro-discrete with the described neighborhood basis of 0. \square

We note that the above generalizes [16, Lemma 5], which describes the Jacobson radical of a left linearly compact ring.

3.C. An infinite Wedderburn-Artin theorem. Given a closed submodule N of a topological module M , we will say that N has a *closed complement* if there exists a closed submodule N' of M such that $M = N \oplus N'$. It is straightforward to verify that this is equivalent to the condition that there exists a continuous idempotent endomorphism e of M such that $N = e(M)$. Note that if N is an open submodule of M that is a (“purely algebraic”) direct summand of M , then any complementary submodule of N is closed; for if $M = N \oplus N'$ then $M \setminus N' = \bigcup_{x \in N \setminus \{0\}} (x + N')$ is a union of open cosets and thus is open in M .

Given an object X of a Grothendieck category \mathcal{C} , we let $\text{soc}(X)$ denote the *socle* of X in \mathcal{C} , which is the colimit (“sum”) of all simple subobjects of X .

Lemma 3.9. *Let M be a left topological module over a topological ring R . The following are equivalent:*

- (a) M is a product in ${}_R\mathbf{TMod}$ of simple discrete modules;
- (b) M is pseudocompact and every closed submodule of M has a closed complement;
- (c) M is pseudocompact and every open submodule of M has a complement.

Proof. To begin, note that any module M satisfying (a) is a product of pseudocompact modules and therefore is pseudocompact. Thus to prove the equivalence of the three conditions, we may assume that M is pseudocompact throughout. As mentioned in Remark 3.5, the category $\mathcal{C} = {}_R\mathbf{PC}^{\text{op}}$ is locally finite and Grothendieck. Let M' denote the object in \mathcal{C} “opposite” to M . Noting that products in ${}_R\mathbf{PC}$ are the same as in ${}_R\mathbf{TMod}$, the three conditions on M above translate to:

- (a') M' is a direct sum of simple objects in \mathcal{C} ;
- (b') Every subobject of M' is a direct summand of M' in \mathcal{C} ;
- (c') Every finite-length subobject of M' is a direct summand of M' in \mathcal{C} .

The equivalence of (a') and (b') follows from well known generalizations of the usual module-theoretic argument to Grothendieck categories, as in [14, Section V.6]. Clearly (b') \Rightarrow (c'). Finally, assume (c') holds; we will deduce (a'). Let L be a finite-length subobject of M' . Because all subobjects of L also have finite length, the hypothesis implies that every subobject of L is a direct summand of M' , including L itself, so that every subobject of L is in fact a summand of L . Thus $L = \text{soc}(L) \oplus L'$ for a subobject L' with $\text{soc}(L') = 0$. But L' also has finite length, so that its socle being zero means that $L' = 0$. Thus $L = \text{soc}(L)$ is a (direct) sum of simple objects. Since \mathcal{C} is locally finite, M' is the sum of all of its

finite-length subobjects $L = \text{soc}(L)$, and we deduce that $M' = \text{soc}(M')$ is a (direct) sum of simple objects. \square

The lemma above raises an interesting question: to what extent is it possible to characterize the structure of pro-discrete left modules over a topological ring R in which every closed submodule has a closed complement?

We now present the following infinite-dimensional version of the Wedderburn-Artin theorem for left linearly topologized rings.

Theorem 3.10. *Let R be a topological ring. Then the following are equivalent:*

- (a) R is a product in ${}_R\mathbf{TMod}$ of simple discrete left modules;
- (b) R is left pro-discrete and every closed left ideal of R has a closed complement;
- (c) R is left pro-discrete and every open left ideal of R is a direct summand;
- (d) $R \cong \prod \text{End}_{D_i}(V_i)$ as topological rings, where each V_i is a right vector space over a division ring D_i and each $\text{End}_{D_i}(V_i)$ is given the finite topology;
- (e) R is left pseudocompact and $J_0(R)$ (equivalently, $J(R)$) is zero;
- (f) R is left linearly compact and $J_0(R)$ (equivalently, $J(R)$) is zero;
- (g) R is left pseudocompact and the abelian category ${}_R\mathbf{PC}$ (equivalently, ${}_R\mathbf{PC}^{\text{op}}$) has all short exact sequences split (i.e., the category is spectral);
- (h) R is left pseudocompact and every object in ${}_R\mathbf{PC}$ is a product of simple objects (equivalently, every object in ${}_R\mathbf{PC}^{\text{op}}$ is a direct sum of simple objects, i.e., ${}_R\mathbf{PC}^{\text{op}}$ is a semisimple category).

Proof. That (b) \Rightarrow (c) follows from the fact that every open left ideal is closed. Assuming that either condition (a) or (c) holds, we will show that R is left pseudocompact; it will then follow from Lemma 3.9 that (c) \Rightarrow (a) \Rightarrow (b). If R satisfies condition (a), so that $R = \prod_{i \in I} S_i$ for simple discrete left modules S_i , then R is a product (hence inverse limit) of discrete modules and therefore is left pro-discrete by Lemma 2.17. Furthermore, R has a neighborhood basis of 0 consisting of open left ideals L for which R/L is a finite direct product of the simple S_i ; these are the kernels of the projections $\prod_{i \in I} S_i \rightarrow \prod_{j \in J} S_j$ for any finite subset $J \subseteq I$. Thus R is pseudocompact. Similarly assume (c) holds, and let L be an open left ideal of R . Since every left ideal containing L is also open, and consequently a direct summand, we see that R/L is a semisimple R -module. This semisimple module R/L is finitely generated, and therefore has finite length. By Lemma 3.1, R is left pseudocompact as claimed.

Again assume that (a) holds. We have established that $R = \prod S_i$ is left pseudocompact in this case, and the intersection of the open maximal left ideals corresponding to the kernel of each canonical projection $R \twoheadrightarrow S_i$ is zero. Thus $J_0(R) = 0$ (which is equivalent to $J(R) = 0$ by Lemma 3.8), establishing (a) \Rightarrow (e). Also (e) \Rightarrow (f) by Corollary 3.7, and (f) \Rightarrow (d) follows from the characterization of linearly compact, Jacobson semisimple rings given in [15, Theorem 29.7]. For (d) \Rightarrow (a), it suffices to show that any ring of the form $\text{End}(V)$, for V_D a vector space over a division ring, is a product of simple discrete modules. Fix a basis $\{v_i \mid i \in I\}$ of V and let E_i denote the projection onto $\text{Span}(v_i)$ with kernel spanned by $\{v_j\}_{j \neq i}$. Then $\text{End}(V) \cong \prod \text{End}(V)E_i \cong \prod_{i \in I} V$ is a product of simple discrete left modules in ${}_R\mathbf{TMod}$.

Assume that (a) holds, so that once more R is left pseudocompact; we will verify (h). The simple objects of ${}_R\mathbf{PC}$ are precisely the simple discrete R -modules in ${}_R\mathbf{TMod}$, which are

all of the form R/U for an open maximal left ideal U of R , thanks to Lemma 2.15. Since we have already shown (a) \Rightarrow (c), we see that every such U is a direct summand. Noting from [4, Corollaire IV.1] that R is a projective object in ${}_R\text{PC}$, because $R \cong R/U \oplus U$ we find that the direct summand R/U is projective in ${}_R\text{PC}$. So every simple object in ${}_R\text{PC}$ is projective, and dually we have that every simple object in the Grothendieck category $\mathcal{C} = {}_R\text{PC}^{\text{op}}$ is injective. Because \mathcal{C} is (locally finite and therefore) locally noetherian, injectives are closed under direct sums [14, Proposition V.4.3]. Then every semisimple object in \mathcal{C} , being a direct sum of simple (hence injective) objects, must be injective. Thus for every object X in \mathcal{C} , the socle $\text{soc}(X)$ is injective and therefore is a direct summand of X , so that $X = \text{soc}(X) \oplus X'$ for some subobject X' of X . This X' cannot have any simple subobjects as it intersects $\text{soc}(X)$ trivially. Because \mathcal{C} is locally finite, every nonzero object has a nonzero socle, from which we deduce that $X' = 0$. Thus $X = \text{soc}(X)$ is a direct sum of simple objects (dually, every object of ${}_R\text{PC}$ is a product of simple objects), establishing (h). Conversely, if (h) holds then R is an object of ${}_R\text{PC}$ and thus is a product of simple objects in ${}_R\text{PC}$. That is to say, R is a product of simple discrete left modules, and (a) holds.

For (h) \Rightarrow (g), note that in the Grothendieck category $\mathcal{C} = {}_R\text{PC}^{\text{op}}$, all objects are semisimple, and therefore all short exact sequences split by essentially the same argument as in the case of a module category [14, Section V.6]. Conversely, if (g) holds and X is an object of \mathcal{C} , then the monomorphism $\text{soc}(X) \hookrightarrow X$ splits, so that $X = \text{soc}(X) \oplus X'$ for some subobject X' of X . Just as in the previous paragraph, local finiteness of \mathcal{C} implies that $X' = 0$. Thus $X = \text{soc}(X)$, so that (h) holds. \square

The algebras above provide a suitable substitute for semisimple rings in the context of topological algebra in which we find ourselves. Thus we introduce the following terminology.

Definition 3.11. A topological ring is called *left pseudocompact semisimple* if it satisfies the equivalent conditions of the previous theorem.

3.D. Some applications of the Wedderburn-Artin theorem. We now present some special cases of Theorem 3.10, some of which recover earlier results on semisimplicity in the literature.

To begin, we note that Theorem 3.10 can be indeed considered as an infinite generalization of the classical Wedderburn-Artin theorem. For if R is a left artinian ring, then it is left pseudocompact when equipped with its discrete topology. So a semisimple (left) artinian ring is left pseudocompact and Jacobson semisimple. In the decomposition $R \cong \prod_{i \in I} \text{End}_{D_i}(V_i)$, the topology on R is discrete if and only if I is finite and each V_i is finite-dimensional. Also, viewing R as a product of simple discrete left R -modules, we again have that R carries the discrete topology if and only if this set of simple modules is finite.

In the situation of a left and right pseudocompact ring R , there is a more refined version of the structure theorem above. This will be presented after the following lemma that extends [6, Lemma 2.5], which is the analogous statement for the case where the dual category ${}_R\text{PC}$ is locally finite-dimensional over a fixed basefield. We will only need to use it in the pseudocompact semisimple case, but we state it in full generality and include details.

Lemma 3.12. *Let R be a topological ring, $(M_i)_{i \in I}$ be pseudocompact left modules and let $P = \prod_i M_i$. Assume that for every simple pseudocompact left R -module S , only finitely*

many M_i have S as a quotient in ${}_R\mathbf{PC}$ (in particular, this is true if the intersection of all open $M \subseteq P$ for which $P/M \cong S$ is of finite colength in P). Then the coproduct of the family $(M_i)_i$ in ${}_R\mathbf{PC}$ is P , with the obvious canonical maps.

Proof. We will show that P with the canonical morphisms $\sigma_i: M_i \hookrightarrow P$ satisfies the universal property of the coproduct in ${}_R\mathbf{PC}$. Let $\Sigma = \bigoplus_i M_i$ denote the usual direct sum of the family $(M_i)_i$, forming the coproduct in the category of left R -modules. Regard $\Sigma \subseteq P$ as a submodule of the product.

Let N be a pseudocompact module with continuous maps $f_i: M_i \rightarrow N$, and let $f: \Sigma \rightarrow N$ be the unique canonical R -module map with $f = f_i$ on M_i (i.e., $f\sigma_i = f_i$). We will first show that f is continuous (with respect to the topology on Σ inherited from P), then that f extends to a continuous homomorphism $P \rightarrow N$ satisfying the desired universal property.

Because f is linear, to show that f is continuous it suffices to show that, for any open submodule H of N , $f^{-1}(H)$ is open in M . We claim that $M_i \subset f^{-1}(H)$ for all but finitely many i . Indeed, let \mathcal{S} be the set of simple modules that occur in the Jordan-Hölder series of N/H . These simple modules are discrete and therefore belong to ${}_R\mathbf{PC}$. If M_i is such that $M_i \not\subseteq f^{-1}(H)$ then $f^{-1}(H) \cap M_i = f_i^{-1}(H)$ is open (by continuity of f_i) and proper in M_i . Now the composite map

$$M_i \xrightarrow{f_i} N \twoheadrightarrow N/H$$

has image isomorphic to $M_i/f_i^{-1}(H) = M_i/(f^{-1}(H) \cap M_i)$, making the latter a nonzero module that embeds in N/H . Being nonzero, this finite-length module has some simple quotients that lie in the family \mathcal{S} . We conclude that M_i has some simple module $S \in \mathcal{S}$ as a quotient. Because \mathcal{S} is finite, the hypothesis on the M_i ensures that we can have at most finitely many $M_i \not\subseteq f^{-1}(M)$, as claimed. Thus $f^{-1}(H)$ contains $\bigoplus_{i \notin F} M_i$ for some finite subset $F \subseteq I$, from which it readily follows that $f^{-1}(H) \supseteq \Sigma \cap (\prod_{i \in F} f_i^{-1}(H) \times \prod_{i \notin F} M_i)$. The latter submodule is open in Σ (being the restriction of an open submodule of P), so that $f^{-1}(H)$ is open by Lemma 2.15. Hence f is continuous.

Note that every element $x \in P$ of the product is the limit of the Cauchy net (x_J) in Σ indexed by the finite subsets $J \subseteq I$, where x_J has the same entries as x at each index in J and all other entries zero. (In particular, Σ is dense in P .) Because $f: \Sigma \rightarrow N$ is linear and continuous, each such Cauchy net (x_J) is mapped to a Cauchy net $(f(x_J))$ in N . Setting $\bar{f}(x) = \lim_J f(x_J)$ to be the limit of this net, one may verify that $\bar{f}: P \rightarrow N$ is R -linear (using the fact that the assignment $x \mapsto (x_J)$ is R -linear) and continuous (see also [15, Theorem 7.19]). This map satisfies $\bar{f} \circ \sigma_i = f_i$ for all i . Further, it is the unique continuous homomorphism satisfying this condition: any such map restricts to f on the dense subset $\Sigma \subseteq P$, and a continuous function $P \rightarrow N$ is uniquely determined by its restriction to a dense subset thanks to the Hausdorff property of N . \square

Theorem 3.13. *Let R be a topological ring. The following assertions are equivalent:*

- (a) $R = \prod_{i \in I} S_i^{n_i}$ in ${}_R\mathbf{TMod}$ for some pairwise non-isomorphic discrete simple left R -modules S_i and integers $n_i \geq 1$;
- (b) R is pseudocompact and ${}_R R$ is a coproduct of simple objects in the category ${}_R\mathbf{PC}$;
- (c) $R \cong \prod_i \mathbb{M}_{n_i}(D_i)$ as topological rings for some division rings D_i and integers $n_i \geq 1$, where each matrix ring is given the discrete topology;

- (d) R is pro-discrete and every open (respectively, closed) left ideal has a (closed) complement;
- (e) R is left pseudocompact semisimple and right linearly topologized;
- (f) R is pseudocompact and $J(R)$ (equivalently, $J_0(R)$) is zero;
- (g) R is linearly compact and $J(R)$ (equivalently, $J_0(R)$) is zero;
- (h) The left-right symmetric statements of (a), (b), (d), and (e).

Proof. First note that if (c) holds, then R is a product of pseudocompact semisimple (hence pro-discrete and linearly compact) rings. Thus (c) implies all of (d), (e), (f), and (g). Conversely, if any one of the conditions (d)–(g) holds, then R is right linearly topologized and by Theorem 3.10 we have $R \cong \prod_{i \in I} \text{End}_{D_i}(V_i)$ as topological rings for some right vector spaces V_i over division rings D_i . In particular, each of the factors $\text{End}_{D_i}(V_i)$ in the product is right linearly topologized. It follows from Remark 3.4 that each V_i has finite dimension (say $n_i = \dim_{D_i}(V_i)$), in which case the finite topology on $\text{End}_{D_i}(V_i) \cong \mathbb{M}_{n_i}(D_i)$ is discrete. This establishes the equivalence of (c)–(g).

Next, assume that (c) holds. Let $S_i = D_i^{n_i}$, which is a simple discrete right R -module via the projection $\pi_i: R \rightarrow \mathbb{M}_{n_i}(D_i)$. It is straightforward to show that the annihilator of ${}_R S_i$ is the kernel of the projection π_i . As $\ker(\pi_i) \neq \ker(\pi_j)$ for distinct $i, j \in J$, we deduce that the S_i are pairwise non-isomorphic. Then because each $\mathbb{M}_{n_i}(D_i) \cong S_i^{n_i}$ as left R -modules, and because the isomorphism in (c) is also an isomorphism in ${}_R \text{TMod}$, we find that (c) \Rightarrow (a).

Next let R satisfy (a); we will deduce (b). Certainly such R is left pseudocompact semisimple, and in ${}_R \text{PC}$ we have $R = \coprod S_i^{n_i}$ is a coproduct of simple objects according to Lemma 3.12. To see that R is right pseudocompact, let $J_i \subseteq R$ denote the kernel of the canonical projection $R \twoheadrightarrow S_i^{n_i}$. Each J_i is an open left ideal of finite colength, and the J_i form a basis of open neighborhoods of zero thanks to the structure of R . It suffices to show that each J_i is also a right ideal, for then R/J_i is a semisimple artinian ring. To this end, fix $r \in R$ and let $\rho: R \rightarrow R$ be the continuous endomorphism defined by $\rho(x) = xr$. Note that each $S_j^{n_j}$ is the isotypic component of R corresponding to S_j (i.e., the sum of all ${}_R \text{PC}$ -subobjects of R that are isomorphic to S_j). It is clear from the construction of J_i that $J_i = \coprod_{j \neq i} S_j^{n_j}$ is the coproduct in ${}_R \text{PC}$ of the isotypic components of R corresponding to the simple objects S_j with $j \neq i$. Since each isotypic component $S_j^{n_j}$ is invariant under ρ , the same is true of J_i . It follows that $J_i r = \rho(J_i) \subseteq J_i$, making J_i a right ideal as desired.

For (b) \Rightarrow (e), suppose R is pseudocompact and $R = \coprod L_t$ is a coproduct of simple objects in ${}_R \text{PC}$. Certainly R is right linearly topologized. To see that it is Jacobson semisimple, let M_t denote the open maximal left ideal that is the kernel of the projection $R \rightarrow L_t$. It follows from $R = \coprod L_t$ that $\bigcap M_t = 0$, so that $J(R) = 0$.

This establishes the equivalence of conditions (a)–(g). Finally, condition (h) is equivalent to the rest because properties (f) and (g) are left-right symmetric. \square

We note that the equivalence (b) \Leftrightarrow (g) above recovers [17, Theorem 1], which in turn generalized [7, Theorem 16] using the methods of [16].

We will also briefly demonstrate that the above results can be particularized to yield the characterization of cosemisimple coalgebras. For the definitions of coalgebras and their comodules we refer the reader to [3, 11].

The following is a more refined notion of pseudocompactness for K -algebras, which is satisfied both by the diagonalizable algebras that we consider in Section 4, as well as dual algebras of coalgebras.

Definition 3.14. Given a topological algebra A over a field K , we say that a left topological A -module M is K -pseudocompact if it is pseudocompact and its open submodules have finite K -codimension. We say that A is *left K -pseudocompact* (or *left pseudocompact as a K -algebra*) if ${}_A A$ is K -pseudocompact. (Right K -pseudocompact algebras are defined similarly.) We say that A is K -pseudocompact if it pseudocompact and every open ideal of A has finite K -codimension.

Because a module of finite length over an algebra is finite-dimensional if and only if all of its simple composition factors are finite-dimensional, one can see that a topological algebra A is left K -pseudocompact if and only if it is a left pseudocompact topological ring and every simple discrete left A -module has finite K -dimension.

Note that, as in Remark 3.2, a topological K -algebra is K -pseudocompact if and only if it is both left and right K -pseudocompact. If A is a left K -pseudocompact algebra, then every left pseudocompact A -module is K -pseudocompact, so that ${}_A \text{PC}$ coincides with the category of K -pseudocompact left A -modules (with continuous module homomorphisms).

The Wedderburn-Artin theorems proved above restrict to K -pseudocompact algebras in the following way.

Proposition 3.15. *Let A be a topological K -algebra. Then the following are equivalent:*

- (a) A is left K -pseudocompact and Jacobson semisimple;
- (b) A is K -pseudocompact and Jacobson semisimple;
- (c) $A \cong \prod_{i \in I} \mathbb{M}_n(D_i)$ as topological algebras, where each D_i is a finite-dimensional division algebra over K and each $\mathbb{M}_n(D_i)$ is given the discrete topology.

Proof. Clearly we have (c) \Rightarrow (b) \Rightarrow (a). Finally, assuming (a) holds, it follows from Theorem 3.10 that $A \cong \prod_{i \in I} \text{End}_{D_i}(V_i)$ as topological algebras for some division K -algebras D_i and right D_i -vector spaces V_i . Note that each V_i is a discrete simple left A -module via the projection $A \twoheadrightarrow \text{End}_{D_i}(V_i)$. If there exists i such that either D_i is infinite-dimensional over K or V_i is infinite-dimensional over D_i , then the corresponding discrete simple left A -module V_i is infinite-dimensional over K , contradicting left K -pseudocompactness of A . Thus all of the D_i are finite-dimensional K -algebras and all V_i are finite-dimensional D_i -vector spaces. Now (c) readily follows. \square

As a corollary, using the duality between pseudocompact algebras and coalgebras, one can also deduce known results in the basic theory of coalgebras, namely, the characterization of cosemisimple coalgebras; see [3, Theorem 3.1.5].

The category of K -pseudocompact algebras (with continuous algebra homomorphisms) is in duality with the category of coalgebras over K ; we refer readers to [13] for details, but we sketch the ideas here. Given a coalgebra C , its K -dual algebra C^* is pseudocompact with open ideals being those of the following form, where H is a finite-dimensional subcoalgebra of C :

$$H^\perp = \{\phi \in C^* \mid \phi(H) = 0\}.$$

Conversely, if A is a K -pseudocompact algebra, then one can form its “finite dual” coalgebra

$$A^\circ = \varinjlim (A/I)^*,$$

where I ranges over the open ideals of A . The assignments $C \mapsto C^*$ and $A \mapsto A^\circ$ are functors that provide a duality between the categories of K -coalgebras (with coalgebra morphisms) and K -pseudocompact algebras (with continuous homomorphisms) [13, Theorem 3.6].

Similarly, if M is a left comodule over a coalgebra C , then its dual M^* naturally carries the structure of a left C^* -module. When M^* is equipped with the topology whose open submodules are of the form

$$N^\perp = \{\phi \in M^* \mid \phi(N) = 0\}$$

for finite-dimensional subcomodules $N \subseteq M$, it becomes a K -pseudocompact left C^* -module [3, Corollary 2.2.13]. In fact, this provides an duality between the category of left C -comodules and the category ${}_A\mathbf{PC}$ for the K -pseudocompact algebra $A = C^*$; see [13, Theorem 4.3].

Corollary 3.16. *Let C be a coalgebra over a field K . The following assertions are equivalent:*

- (a) C is a direct sum of simple left (equivalently, right) comodules, i.e., it is cosemisimple;
- (b) Every left (equivalently, right) C -comodule is semisimple;
- (c) Every short exact sequence of left (equivalently, right) C -comodules splits;
- (d) $C^* \cong \prod_i \mathbb{M}_{n_i}(D_i)$ as topological algebras, for some finite-dimensional division algebras D_i ;
- (e) C is a direct sum of coalgebras of the form $\mathbb{M}_n(D)^*$ (dual to the algebras $\mathbb{M}_n(D)$), with D a finite-dimensional division algebra over K .

Proof. The dual algebra $A = C^*$ is K -pseudocompact, and under the duality between C -comodules and pseudocompact C^* -modules, conditions (a)–(c) above translate to the following:

- (a') A is a product of simple pseudocompact left (equivalently, right) modules;
- (b') Every object in ${}_A\mathbf{PC}$ (equivalently, \mathbf{PC}_A) is a product of simple objects;
- (c') Every short exact sequence in ${}_A\mathbf{PC}$ (equivalently, \mathbf{PC}_A) splits.

Thus the equivalence of (a)–(d) follows from Theorem 3.10 and Proposition 3.15. Also, (e) is the dual of (d), with $\mathbb{M}_n(D)^*$ being the finite-dimensional coalgebra dual to the matrix algebra $\mathbb{M}_n(D)$. \square

Remark 3.17. We note that if A is a K -pseudocompact algebra, asking that every pseudocompact left A -module be semisimple as an object in ${}_A\mathbf{PC}$ is not equivalent to A being semisimple in the above sense. This fails for set-theoretic reasons that cannot be avoided. Translated into the dual category of comodules over $C = A^*$, the statement that the category ${}_A\mathbf{PC}$ is semisimple would mean that every C -comodule is a direct product of simple comodules. However, taking S to be a simple comodule, the comodule $L = S^{(\aleph_0)}$ is not a product of simple comodules. If it were so, it is easy to see that we would have $L \cong S^\Omega$ for some infinite cardinal Ω . Hence, its dimension over K would be at least 2^{\aleph_0} since $\Omega \geq \aleph_0$. But $\dim_K(S^{(\aleph_0)}) = \aleph_0$ since $\dim_K(S) < \infty$, and this is a contradiction.

4. DIAGONALIZABLE ALGEBRAS OF OPERATORS

This final section begins with a detailed study of the structure of diagonalizable algebras of operators. We then prove our major theorem characterizing diagonalizable subalgebras in terms of their structure as topological algebras. At the end of the section, we investigate the closure of the set of diagonalizable operators.

4.A. Structure of diagonalizable subalgebras. Let V be a finite-dimensional vector space over a field K , and let $T \in \text{End}(V)$. Suppose that T is diagonalizable, so that its minimal polynomial $\mu(x) = (x - \lambda_1) \cdots (x - \lambda_d)$ is a product of distinct linear factors. The subalgebra $K[T] \subseteq \text{End}(V)$ is then isomorphic to $K[x]/(\mu(x)) \cong K^d$ (the latter isomorphism following readily from the Chinese Remainder Theorem). In terms of algebraic geometry, the induced surjective homomorphism $K[x] \rightarrow K[x]/(\mu(x)) \cong K[T]$ is dual to the inclusion of algebraic varieties $\{\lambda_1, \dots, \lambda_d\} \hookrightarrow \mathbb{A}_K^1$ of the d distinct eigenvalues of T into the affine line over K . Thus the subalgebra $K[T]$ can be viewed in algebro-geometric terms as being isomorphic to the algebra of K -valued functions on a set of d points. In particular, the prime spectrum of $K[T]$ is a discrete space consisting of d distinct points, so that $\text{Spec}(K[T]) \cong \{\lambda_1, \dots, \lambda_d\}$.

This is certainly not the typical picture of spectral theory presented in linear algebra textbooks, but we will show in this section that such a view of diagonalizable algebras of operators as algebras of functions generalizes to the infinite-dimensional case. In particular, every diagonalizable subalgebra of $\text{End}(V)$ is isomorphic to an algebra K^X of K -valued functions on a (possibly infinite) set X , even as topological algebras. We work at the level of categories and functors, to present a kind of duality between functions and underlying sets as in the brief illustration above in terms of algebraic geometry. We refer readers to [10, Chapter I] for the basic category theory required here. In this subsection, we shall follow standard category-theoretic practice and view a *contravariant* (“arrow-reversing”) functor $F: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ as a covariant functor $F: \mathcal{C}_1^{\text{op}} \rightarrow \mathcal{C}_2$ out of the opposite category.

The “underlying set” of one of these function algebras can be viewed as a kind of prime spectrum, but suitably modified to fit with the context of topological algebra in which we work. Given a commutative topological ring A , we let $\text{Spec}_0(A)$ denote the set of open prime ideals of A . Let \mathbf{TRing} denote the category of topological rings with continuous ring homomorphisms, and let \mathbf{cTRing} denote the full subcategory of commutative topological rings. If $f: A \rightarrow B$ is a morphism in \mathbf{cTRing} and $\mathfrak{p} \in \text{Spec}_0(B)$, then it is clear that $f^{-1}(\mathfrak{p}) \in \text{Spec}_0(A)$. In this way, the assignment $A \mapsto \text{Spec}_0(A)$ forms a functor $\text{Spec}_0: \mathbf{cTRing}^{\text{op}} \rightarrow \mathbf{Set}$. We will show in Theorem 4.7 below that this becomes an equivalence when restricted to a suitable full subcategory. We begin with a more detailed account of the Jacobson radical and semisimplicity for commutative pseudocompact rings.

In the case where A above is pseudocompact, if $\mathfrak{m} \in \text{Spec}_0(A)$ then A/\mathfrak{m} is an artinian integral domain and therefore is a field, so that \mathfrak{m} is maximal. Thus when A is pseudocompact, $\text{Spec}_0(A)$ is the set of open *maximal* ideals of A , and the assignment sending a commutative pseudocompact topological ring to its set of open maximal ideals is functorial (coinciding with Spec_0).

Recall that every element of the Jacobson radical of a commutative artinian ring is nilpotent; we shall show that the radical of a commutative pseudocompact ring satisfies a similar

condition. An element x of a topological ring is *topologically nilpotent* if the sequence $(x^n)_{n=1}^\infty$ converges to zero. Given a commutative topological ring A , let $N_0(A)$ denote the set of all topologically nilpotent elements of A .

Lemma 4.1. *Let A be a commutative linearly topologized ring.*

- (1) $N_0(A) = \bigcap \text{Spec}_0(A)$; consequently, $N_0(A)$ is a closed ideal of A contained in $J(A)$.
- (2) If A is pseudocompact, then $N_0(A) = J(A)$.

Proof. (1) Set $N = N_0(A)$. Let $x \in N$ and $\mathfrak{p} \in \text{Spec}_0(A)$. Since \mathfrak{p} is an open neighborhood of 0 and $\lim_{n \rightarrow \infty} x^n = 0$, there is an integer $n \geq 1$ such that $x^n \in \mathfrak{p}$. Since \mathfrak{p} is prime, this means that $x \in \mathfrak{p}$. We conclude that $N \subseteq \bigcap \text{Spec}_0(A)$. Now suppose that $x \in A \setminus N$; then there is an open neighborhood $I \subseteq A$ of 0, which we may assume is an ideal, such that set $S = \{x^n \mid n \geq 1\}$ is disjoint from I . By a familiar application of Zorn's Lemma, there is an ideal \mathfrak{p} of A with $I \subseteq \mathfrak{p}$ (making \mathfrak{p} open) that is maximal with respect to $\mathfrak{p} \cap S = \emptyset$. Since S is a multiplicatively closed set, this \mathfrak{p} is prime by a well known argument from commutative algebra. Thus $\mathfrak{p} \in \text{Spec}_0(A)$ with $x \notin \mathfrak{p}$, so that $x \notin \bigcap \text{Spec}_0(A)$. Now we see that $N_0(A)$ is closed as it is an intersection of open, hence closed, ideals.

(2) If A is pseudocompact, then $\text{Spec}_0(A)$ is the set of open maximal ideals of A , as noted above. Thus $N_0(A) = \bigcap \text{Spec}_0(A) = J_0(A) = J(A)$ thanks to part (1) above and Theorem 3.8(3). \square

We say that a topological ring is *topologically reduced* if its only topologically nilpotent element is zero.

Lemma 4.2. *Let A be a commutative topological ring. The following are equivalent:*

- (a) A is pseudocompact semisimple;
- (b) A is pseudocompact and topologically reduced;
- (c) The open maximal ideals of A intersect to zero and form a neighborhood subbasis for 0, and A is complete;
- (d) As a topological ring, A is a product of discrete fields;
- (e) The canonical map $A \rightarrow \prod_{\mathfrak{m} \in \text{Spec}_0(A)} A/\mathfrak{m}$ is an isomorphism of topological rings.

Proof. It is clear that (e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (a), and (a) \Leftrightarrow (b) follows from the previous lemma. Theorem 3.10 gives (a) \Rightarrow (d), as the endomorphism ring of a vector space over a division ring is commutative if and only if the division ring is a field and the vector space has dimension 1. Finally, assume (d) holds; we will show (e). Suppose $A \cong \prod_{i \in I} K_i$ for some (discrete) fields K_i , and for any $j \in I$ let \mathfrak{m}_j denote the kernel of the projection $A \cong \prod K_i \twoheadrightarrow K_j$. Then every open ideal of A must contain a finite intersection of the \mathfrak{m}_i . In particular, any open prime ideal of A contains a finite intersection of these \mathfrak{m}_i and therefore (contains and) is equal to one of the \mathfrak{m}_i . So $\text{Spec}_0(A) = \{\mathfrak{m}_i \mid i \in I\}$, from which (e) readily follows. \square

Given a (discrete) field K , topological K -algebras with continuous algebra homomorphisms form a category TAlg_K . We denote the hom-sets of this category simply by $\text{Hom}(A, B)$.

We next concern ourselves with a study of topological K -algebras that are isomorphic to K^X . For any set X , the natural identification

$$K^X = \text{Set}(X, K)$$

can be viewed as an identification of topological K -algebras, where the topology on K^X is the product topology and the topology of $\mathbf{Set}(X, K)$ is the topology of pointwise convergence as described in Section 2 (in both cases, K is endowed with the discrete topology). This makes it clear that the assignment $X \mapsto K^X$ is the same as (the object part of) the representable functor $\mathbf{Set}(-, K): \mathbf{Set}^{\text{op}} \rightarrow \mathbf{TAlg}_K$ from the category of sets to the category of topological K -algebras. Thus we shall interchange notation at the level of functors: $K^- = \mathbf{Set}(-, K)$.

Given a set X and $x \in X$, let $\mathfrak{m}_x \subseteq K^X$ denote the open maximal ideal consisting of those functions that vanish at x , and let $\text{ev}_x: K^X \rightarrow K$ denote the continuous homomorphism given by evaluating at x , so that $\text{ev}_x(f) = f(x)$ for $f \in K^X$. (These are the same as the canonical projections when K^X is viewed as a product of sets, but reinterpreted in the language of functions in order to evoke appropriate imagery from algebraic geometry.) Clearly each $\ker(\text{ev}_x) = \mathfrak{m}_x$. Further, the topology on K^X has a neighborhood basis of open ideals given by the finite intersections of the \mathfrak{m}_x . In particular, each open ideal contains a finite product of the \mathfrak{m}_x , so every open prime ideal must be equal to some \mathfrak{m}_x . Thus $\text{Spec}_0(K^X) = \{\mathfrak{m}_x \mid x \in X\}$, and we obtain a canonical bijection $\text{Spec}_0(K^X) \xrightarrow{\sim} \text{Hom}(K^X, K)$ given by $\mathfrak{m}_x \mapsto \text{ev}_x$.

Lemma 4.3. *Given a field K and a topological K -algebra A , the following are equivalent:*

- (a) *A is pseudocompact semisimple, and all of its open maximal ideals have K -codimension equal to 1 in A ;*
- (b) *$A \cong K^X$ as topological algebras for some set X ;*
- (c) *The natural map $A \rightarrow K^{\text{Hom}(A, K)}$ given by $f \mapsto (\psi(f))_\psi$ is an isomorphism of topological algebras;*

and for such an algebra A the canonical map $\text{Hom}(A, K) \rightarrow \text{Spec}_0(A)$ given by $\psi \mapsto \ker(\psi)$ is a bijection.

If K is algebraically closed, then the above are further equivalent to:

- (d) *A is K -pseudocompact and Jacobson semisimple;*
- (e) *A is K -pseudocompact and topologically reduced.*

Proof. We have (c) \Rightarrow (b), and (b) \Rightarrow (a) follows by combining Lemma 4.2 with the facts that $\text{Spec}_0(K^X) = \{\mathfrak{m}_x \mid x \in X\}$ and $K^X = (K \cdot 1) \oplus \mathfrak{m}_x$. Now assuming (a), we shall derive (c). Under this hypothesis, for each $\mathfrak{m} \in \text{Spec}_0(A)$ we have $A = (K \cdot 1) \oplus \mathfrak{m}$, yielding an isomorphism $A/\mathfrak{m} \cong K$ via $\lambda + \mathfrak{m} \mapsto \lambda$. So every such \mathfrak{m} is the kernel of some $\psi_{\mathfrak{m}} \in \text{Hom}(A, K)$ (the composite $A \twoheadrightarrow A/\mathfrak{m} \cong K$), and this $\psi_{\mathfrak{m}}$ is unique thanks to the decomposition $A = (K \cdot 1) \oplus \mathfrak{m}$. Conversely, every $\psi \in \text{Hom}(A, K)$ is of the form $\psi = \psi_{\mathfrak{m}}$ for $\mathfrak{m} = \ker(\psi) \in \text{Spec}_0(A)$. Now let $\alpha: A \rightarrow \prod_{\mathfrak{m} \in \text{Spec}_0(A)} A/\mathfrak{m}$ be the canonical isomorphism provided by Lemma 4.2, and consider the isomorphism $\beta: \prod_{\mathfrak{m} \in \text{Spec}_0(A)} A/\mathfrak{m} \rightarrow \prod_{\psi \in \text{Hom}(A, K)} K$ defined by $(\lambda_{\mathfrak{m}} + \mathfrak{m})_{\mathfrak{m}} \mapsto (\lambda_{\mathfrak{m}})_{\psi_{\mathfrak{m}}}$. Then the composite isomorphism in \mathbf{TAlg}_K

$$A \xrightarrow{\alpha} \prod_{\mathfrak{m} \in \text{Spec}_0(A)} A/\mathfrak{m} \xrightarrow{\beta} \prod_{\psi \in \text{Hom}(A, K)} K$$

coincides with the natural map described in (c). Thus (a) \Rightarrow (c).

Finally, assume that K is algebraically closed. Note that (d) \Leftrightarrow (e) thanks to Lemma 4.2. Certainly (a) \Rightarrow (d). Conversely, suppose that (d) holds. To verify (a), let $\mathfrak{m} \in \text{Spec}_0(A)$.

Then \mathfrak{m} has finite codimension, making A/\mathfrak{m} a finite field extension of K . Because K is algebraically closed, we must have $A/\mathfrak{m} \cong K$ as K -algebras, meaning that \mathfrak{m} has codimension 1 as desired. \square

Definition 4.4. We will refer to a topological algebra over a field K satisfying the equivalent conditions above as a *function algebra over K* , or a *K -function algebra*.

The importance of function algebras when studying diagonalizable algebras of operators is due to the following two facts.

Proposition 4.5. *Let V be a K -vector space and let \mathcal{B} be a basis for V . The commutative subalgebra $A \subseteq \text{End}(V)$ consisting of \mathcal{B} -diagonalizable operators is a function algebra, isomorphic as a topological K -algebra to $K^{\mathcal{B}}$. Furthermore, A is a maximal commutative subalgebra.*

Proof. For each element $f \in K^{\mathcal{B}}$ considered as a function $f: \mathcal{B} \rightarrow K$, there is a corresponding operator T_f on V defined by $T_f(b) = f(b) \cdot b$. It is quite clear that this defines an algebra isomorphism $\phi: K^{\mathcal{B}} \rightarrow A$ by $\phi(f) = T_f$. To see that this is a homeomorphism, we simply note that the basic open neighborhoods of zero in $K^{\mathcal{B}}$ of the form $\mathfrak{m}_{b_1} \cap \cdots \cap \mathfrak{m}_{b_n}$ for $b_1, \dots, b_n \in \mathcal{B}$ correspond under ϕ to the basic open neighborhoods of zero in A of the form $\{T \in A \mid T(b_1) = \cdots = T(b_n) = 0\}$.

To see that A is a maximal commutative subalgebra of $\text{End}(V)$, fix an element $T \in A'$ of the commutant of A in $\text{End}(V)$; we will show that $T \in A$. For each $b \in \mathcal{B}$, let $E_b \in A$ denote the projection of V onto Kb with kernel spanned by $\mathcal{B} \setminus \{b\}$. Then T centralizes the E_b , from which we deduce that the 1-dimensional subspaces $Kb \subseteq V$ are invariant under T for all $b \in \mathcal{B}$. It follows that T is \mathcal{B} -diagonalizable, which is to say that $T \in A$, as desired. \square

Lemma 4.6. *Every closed K -subalgebra of a K -function algebra is again a K -function algebra.*

Proof. Let B be a K -function algebra and let $A \subseteq B$ be a closed subalgebra. We will show that condition (a) of Lemma 4.3 passes from B to A . Because A is closed in B and B is complete, A is also complete. Also, A is Hausdorff because B is. Now let $\{\mathfrak{m}_i\}$ denote the open maximal ideals of B and set $\mathfrak{m}'_i = \mathfrak{m}_i \cap A$. These are open ideals of A , which still intersect to zero (since $\bigcap \mathfrak{m}_i = 0$) and form a neighborhood subbasis of 0 in A , making A Jacobson semisimple. Furthermore, as each \mathfrak{m}_i has codimension 1 in B , the same remains true of the \mathfrak{m}'_i in A . In particular, each \mathfrak{m}'_i is an open maximal ideal of A . To see that A is pseudocompact, let \mathcal{U} denote the neighborhood basis of 0 in A consisting of the intersections of finite subfamilies of $\{\mathfrak{m}'_i\}$. By the Chinese Remainder Theorem, for each $U \in \mathcal{U}$ the ring A/U is a finite direct product of fields (hence artinian). Thus A is pseudocompact by Lemma 3.1(a). As we have shown that A is Jacobson semisimple with every open maximal ideal of codimension 1, we find that A is a function algebra. \square

Now let $\text{Func}(K)$ denote the category of K -function algebras with continuous K -algebra homomorphisms. The discrete algebra K is an object of $\text{Func}(K)$, and the kernel of each homomorphism $A \rightarrow K$ in $\text{Func}(K)$ is an open maximal ideal of A , which is to say an element of $\text{Spec}_0(A)$. Conversely, if $\mathfrak{m} \in \text{Spec}_0(A)$ for a K -function algebra A , then $A/\mathfrak{m} \cong K$ as

topological K -algebras, giving a continuous K -algebra morphism $A \rightarrow K$ with kernel \mathfrak{m} . Thus we have a natural isomorphism between the functors

$$\begin{aligned} \text{Spec}_0 &: \text{Func}(K)^{\text{op}} \rightarrow \text{Set} \quad \text{and} \\ \text{Hom}(-, K) &: \text{Func}(K)^{\text{op}} \rightarrow \text{Set} \end{aligned}$$

given (in one direction) by sending $f \in \text{Hom}(A, K)$ to $\ker(f) \in \text{Spec}_0(A)$. (Note that this generalizes the case where $A = K^X$ discussed before Lemma 4.3.)

The formalities developed above allow us to describe K -function algebras and the morphisms relating them in the following precise way.

Theorem 4.7. *Let $\text{Func}(K)$ denote the category of K -function algebras with continuous K -algebra homomorphisms. Then the representable functors*

$$\begin{aligned} \text{Spec}_0 &\cong \text{Hom}(-, K): \text{Func}(K)^{\text{op}} \rightarrow \text{Set} \\ K^- &= \text{Set}(-, K): \text{Set}^{\text{op}} \rightarrow \text{Func}(K) \end{aligned}$$

form a contravariant equivalence between $\text{Func}(K)$ and Set .

Proof. Given a set X , as previously described we have a bijection $\eta_X: X \xrightarrow{\sim} \text{Hom}(K^X, K)$ given by $x \mapsto \text{ev}_x$ for $x \in X$, which we argue is natural in X . Fixing a morphism $\phi: X \rightarrow Y$ in Set , we have an induced morphism $\phi^* = K^\phi: K^Y \rightarrow K^X$ in \mathbf{TAAlg}_K , which precomposes each function in K^X with ϕ . To describe the function $\text{Hom}(\phi^*, K): \text{Hom}(K^X, K) \rightarrow \text{Hom}(K^Y, K)$, fix $x \in X$ and the corresponding evaluation map $\text{ev}_x \in \text{Hom}(K^X, K)$. Then $\text{Hom}(\phi^*, K)(\text{ev}_x) = \text{ev}_x \circ \phi^*: K^X \rightarrow K$. Given $f \in K^X$, we have

$$\text{ev}_x \circ \phi^*(f) = \text{ev}_x(\phi^*(f)) = \text{ev}_x(f \circ \phi) = f(\phi(x)) = \text{ev}_{\phi(x)}(f).$$

So $\text{Hom}(\phi^*, K)(\text{ev}_x) = \text{ev}_{\phi(x)}$, verifying that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \text{Hom}(K^X, K) \\ \phi \downarrow & & \downarrow \text{Hom}(K^\phi, K) \\ Y & \xrightarrow{\eta_Y} & \text{Hom}(K^Y, K) \end{array}$$

commutes. Thus the η_X form components of a natural isomorphism $\eta: \mathbf{1}_{\text{Set}} \rightarrow \text{Hom}(K^-, K)$ of endofunctors of Set .

Now suppose that A is a K -function algebra, and let $\varepsilon_A: A \rightarrow K^{\text{Hom}(A, K)}$ denote the natural map to the product in \mathbf{TAAlg}_K , given by $\varepsilon_A(f) = (\psi(f))_{\psi \in \text{Hom}(A, K)}$. This is an isomorphism according to Lemma 4.3(c). Let $g: A \rightarrow B$ be a morphism in $\text{Func}(K)$. Denote $g^* = \text{Hom}(g, K): \text{Hom}(B, K) \rightarrow \text{Hom}(A, K)$, given by precomposition with g . Given $f \in A$, we now alternately view $\varepsilon_A(f) \in K^{\text{Hom}(A, K)}$ as the function $\varepsilon_A(f): \text{Hom}(A, K) \rightarrow K$ given by $\psi \mapsto \psi(f)$. So fixing $\psi \in \text{Hom}(B, K)$, we have

$$(K^{g^*} \circ \varepsilon_A(f))(\psi) = (\varepsilon_A(f) \circ g^*)(\psi) = \varepsilon_A(f)(\psi \circ g) = \psi(g(f)) = \varepsilon_B(g(f))(\psi).$$

It follows that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varepsilon_A} & K^{\mathrm{Hom}(A,K)} \\ g \downarrow & & \downarrow K^{\mathrm{Hom}(g,K)} \\ B & \xrightarrow{\varepsilon_B} & K^{\mathrm{Hom}(B,K)} \end{array}$$

commutes, making the ε_A into the components of a natural isomorphism $\varepsilon: \mathbf{1}_{\mathrm{Func}(K)} \rightarrow K^{\mathrm{Hom}(-,K)}$ of endofunctors of $\mathrm{Func}(K)$. This establishes the desired contravariant equivalence between Set and $\mathrm{Func}(K)$. \square

For us, the key application of the previous theorem is to determine the structure of an arbitrary closed subalgebra of a K -function algebra.

Corollary 4.8. *Let K be a field, I a set, and $A \subseteq K^I$ a closed K -subalgebra. Then $A \cong K^J$, as topological K -algebras, for some set J with $|J| \leq |I|$.*

Proof. Lemmas 4.3 and 4.6 yield that $A \cong K^{\mathrm{Spec}_0(A)}$ is a function algebra, so that the inclusion map $A \hookrightarrow K^I$ is a monomorphism in $\mathrm{Func}(K)$. Thus applying Spec_0 yields an epimorphism in Set (i.e., a surjection) $I \cong \mathrm{Spec}_0(K^I) \rightarrow \mathrm{Spec}_0(A)$. This implies that $|\mathrm{Spec}_0(A)| \leq |I|$. The claim follows by setting $J = \mathrm{Spec}_0(A)$. \square

Even more precisely than the above, Theorem 4.7 allows one to characterize the closed subalgebras of K^I (and their containments) in terms of equivalence relations \sim on I (and their refinements), as closed subalgebras of K^I are dual to the surjections $I \rightarrow I/\sim$. We do not include further details as we will not make use of this observation.

We also note that the above statement can be easily translated into a coalgebra statement and proved this way: a quotient of the coalgebra $C = K^{(I)}$ has a basis of grouplike elements $(g_i)_{i \in I}$ and thus is isomorphic to $K^{(J)}$ for some set of cardinality $|J| \leq |I|$. Indeed, if $p: C \rightarrow D$ is such a quotient, then each grouplike $g_i \in C$ produces a grouplike $p(g_i) \in D$, and D is spanned by these grouplike elements. Extracting a basis, we see that $D \cong K^{(J)}$ as coalgebras.

In general, the conclusion of the previous result fails for subalgebras of K^I that are not closed.

Example 4.9. Let K be an infinite field, and let $(\lambda_i)_{i \in I}$ be a tuple of elements in K for which the set $\{\lambda_i \mid i \in I\}$ is infinite. Then the element $\theta = (\lambda_i)_{i \in I} \in K^I$ satisfies no polynomial in $K[x]$, since $\{\theta^n : n \geq 0\}$ is linearly independent over K , and therefore the subalgebra $K[\theta] \subseteq K^I$ is isomorphic to $K[x]$. In particular, $K[\theta]$ is not isomorphic to K^Ω for any cardinal Ω , since $K[\theta]$ is an integral domain and K^Ω is not (for $\Omega > 1$).

4.B. Characterizations of diagonalizable subalgebras. We are now ready for our main result about diagonalization of algebras of operators.

Theorem 4.10. *Let K be a field, V a K -vector space, and $A \subseteq \mathrm{End}(V)$ a closed subalgebra. Then the following are equivalent:*

- (a) A is diagonalizable;
- (b) $A \cong K^\Omega$ as topological K -algebras;

(c) A is the closed subalgebra of $\text{End}(V)$ generated by a summable set of orthogonal idempotents $\{E_i \mid i \in I\}$ with $\sum E_i = 1$;

and the cardinal Ω in (b) above must satisfy $\Omega \leq \dim(V)$.

Furthermore, if K is algebraically closed, then the above are also equivalent to:

(d) A is K -pseudocompact and Jacobson semisimple.

(e) A is K -pseudocompact and topologically reduced.

Proof. Suppose that (a) holds. Let \mathcal{B} be a basis of V such that A is \mathcal{B} -diagonalizable, so that A is a closed subalgebra of the algebra C of all \mathcal{B} -diagonalizable operators. Now $C \cong K^{\mathcal{B}}$ as topological algebras according to Proposition 4.5. It follows from Corollary 4.8 that $A \cong K^{\Omega}$ as topological K -algebras for some cardinal $\Omega \leq |\mathcal{B}| = \dim(V)$, and (b) is established.

Suppose that (b) holds. Since K^{Ω} is the closure of the subalgebra generated by a summable set of orthogonal idempotents, summing to 1 (namely, the characteristic functions on the singletons of Ω), the same is true of A , and hence (c) holds.

Suppose that (c) holds. Let $A_0 = \text{Span}\{E_i \mid i \in I\} \subseteq \text{End}(V)$, which is a commutative subalgebra. Then $A = \overline{A_0}$ is commutative by Lemma 2.2(1). Furthermore, $V = \bigoplus E_i(V)$, by Corollary 2.12. Note that each $V_i = E_i(V)$ is a simultaneous eigenspace for A_0 ; we claim that these are also eigenspaces for A . Indeed, given any $i \in I$, let $T \in A$ and $v \in V_i$. By the density of A_0 in A , there exists $T_0 \in A_0$ that belongs to the open neighborhood $\{S \in A \mid T(v) = S(v)\}$ of T in A . But since V_i is an eigenspace of T_0 , this means that $T(v) = T_0(v) = \lambda v$ for some $\lambda \in K$. This confirms that each V_i is a simultaneous eigenspace for A , from which (a) follows.

Assuming that K is algebraically closed, the equivalence of (b), (d), and (e) follows from Lemma 4.3. \square

In Corollary 4.12 below, we apply the above theorem to characterize diagonalizable subalgebras of $\text{End}(V)$ that are not necessarily closed, still in terms of the restriction of the finite topology. We make use of the following preparatory fact.

Lemma 4.11. *Let B be a topological ring with a dense subring $A \subseteq B$. For any open ideal I of B , the canonical map $A/(A \cap I) \rightarrow B/I$ is a (topological) isomorphism.*

Proof. The canonical map $A/(A \cap I) \rightarrow B/I$ given by $a + (A \cap I) \mapsto a + I$ for $a \in A$ is certainly injective. As Lemma 2.15 implies that both factor rings have the discrete topology (because the ideals are respectively open in A and B), it suffices to prove that this map is surjective. To this end, fix a coset $b + I \in B/I$. Note that this coset is open as it is a translate of the open set I , so there exists an element $a \in A \cap (b + I)$ by the density of A . But then $a + (A \cap I) \mapsto a + I = b + I$ as desired. \square

Corollary 4.12. *Let V be a vector space over a field K , and let A be a subalgebra of $\text{End}(V)$, considered as a topological algebra with the topology inherited from the finite topology. Then the following are equivalent:*

(a) A is diagonalizable;

(b) $\overline{A} \cong K^{\Omega}$ as topological algebras for some cardinal Ω , which necessarily satisfies $\Omega \leq \dim(V)$;

(c) For every open ideal I of A , there is an integer $n \geq 1$ such that $A/I \cong K^n$ as K -algebras.

Proof. First we show (a) \Leftrightarrow (b). Assuming (a), fix a basis \mathcal{B} such that A is \mathcal{B} -diagonalizable, and let C denote the set of \mathcal{B} -diagonalizable operators on V . This is a maximal commutative subalgebra of $\text{End}(V)$ by Proposition 4.5, hence closed thanks to Lemma 2.2(3). Thus $\overline{A} \subseteq C$ is also diagonalizable, and (b) (along with the bound $\Omega \leq \dim(V)$) follows from Theorem 4.10. Conversely, if (b) holds then \overline{A} is diagonalizable by Theorem 4.10. As $A \subseteq \overline{A}$, we conclude that (a) holds.

Next we show (b) \Rightarrow (c). Assuming (b), let I be an open ideal of A . Because the topology of A is induced from that of \overline{A} , there is an open ideal J of \overline{A} such that $J \cap A \subseteq I$ and $\overline{A}/J \cong K^m$ for some integer $m \geq 1$ (thanks to the structure of K^Ω as a topological algebra). Using Lemma 4.11, this means that we have a surjection $K^m \cong \overline{A}/J \cong A/(J \cap A) \twoheadrightarrow A/I$, from which it follows that $A/I \cong K^n$ for some integer $n \leq m$. This verifies (c).

Conversely, assume (c) holds. Note that \overline{A} is pro-discrete by Lemma 2.20. Let $J \subseteq \overline{A}$ be an open ideal. Then $\overline{A}/J \cong A/(J \cap A) \cong K^n$ as K -algebras thanks to Lemma 4.11 and the hypothesis. It follows that \overline{A} is K -pseudocompact, that every open maximal ideal of \overline{A} has K -codimension equal to 1 (as K^n is a field if and only if $n = 1$), and that every open ideal of \overline{A} is an intersection of open maximal ideals (as the intersection of the maximal ideals in the discrete algebra $\overline{A}/J \cong K^n$ is zero). Because \overline{A} is pro-discrete, the intersection of all of its open ideals is zero; as each of these open ideals is an intersection of open maximal ideals, we deduce that $J_0(\overline{A}) = 0$. Now Lemma 4.3 implies that (b) holds. \square

The conditions on diagonalizability of a commutative subalgebra may also be translated into a condition for an operator $T \in \text{End}(V)$ to be diagonalizable. Condition (b) below is a topological substitute for the characterization in the finite-dimensional case that the minimal polynomial of an operator splits.

Proposition 4.13. *Let V be a vector space over a field K , and let $T \in \text{End}(V)$. Let $\sigma = \sigma(T)$ denote the spectrum of eigenvalues of T . Then the following are equivalent:*

- (a) T is diagonalizable;
- (b) The net of finite products of the form $(T - \lambda_1) \cdots (T - \lambda_n)$ for distinct $\lambda_i \in \sigma$ (indexed by the finite subsets $\{\lambda_1, \dots, \lambda_n\} \subseteq \sigma$) converges to zero;
- (c) For every finite-dimensional subspace $W \subseteq V$, there are distinct $\lambda_1, \dots, \lambda_n \in \sigma$ such that the restriction of $(T - \lambda_1) \cdots (T - \lambda_n)$ to W is zero;
- (d) The closed K -subalgebra $\overline{K[T]} \subseteq \text{End}(V)$ generated by T is isomorphic to K^Ω as a topological K -algebra for some cardinal $\Omega \leq \dim(V)$.

Proof. Note that T is diagonalizable if and only if the subalgebra $K[T] \subseteq \text{End}(V)$ is diagonalizable. Thus (a) \Leftrightarrow (d) follows from Corollary 4.12. Note also that (b) \Leftrightarrow (c) because (c) is simply a reformulation of (b) in terms of the finite topology.

For (a) \Rightarrow (b), suppose that T is \mathcal{B} -diagonalizable for some basis \mathcal{B} of V . To prove (b), note that the set of open neighborhoods of $\text{End}(V)$ of the form

$$U = \{S \in \text{End}(V) \mid S(b_1) = \cdots = S(b_n) = 0\}$$

for some finite subset $\{b_1, \dots, b_n\} \subseteq \mathcal{B}$ forms a neighborhood basis of zero (since any finite-dimensional subspace of V is contained in the span of a sufficiently large but finite subset of \mathcal{B}). Given such $b_1, \dots, b_n \in \mathcal{B}$, let $\alpha_1, \dots, \alpha_n \in \sigma$ denote the (possibly repeated) eigenvalues of T associated to the b_i . Then for any finite subset $X \subseteq \sigma$ containing $\{\alpha_1, \dots, \alpha_n\}$, set $S = \prod_{\lambda \in X} (T - \lambda)$. Because the factors $T - \lambda I$ commute with one another and each $\alpha_i \in X$, we have $S(b_i) = 0$ for $i = 1, \dots, n$ so that $S \in U$. Thus (b) is satisfied.

Now suppose that (b) holds; we verify (d). Let $I \subseteq K[T]$ be an open ideal. By hypothesis there exist distinct $\lambda_1, \dots, \lambda_n \in \sigma$ such that, for the polynomial $p(x) = (x - \lambda_1) \cdots (x - \lambda_n) \in K[x]$, we have $p(T) \in I$. An easy application of the Chinese Remainder Theorem implies that $K[x]/(p(x)) \cong K^n$ as K -algebras. Thus there is a surjection $K^n \cong K[x]/(p(x)) \twoheadrightarrow K[T]/(p(T)) \twoheadrightarrow K[T]/I$ of K -algebras, from which it follows that $K[T]/I \cong K^m$ for some $m \leq n$. Then (d) holds by Corollary 4.12. \square

4.C. Simultaneously diagonalizable operators. It is well known that if two diagonalizable operators on a finite-dimensional vector space V commute, then they are simultaneously diagonalizable. An immediate corollary (taking into account the finite-dimensionality of $\text{End}(V)$) is that an arbitrary commuting set of diagonalizable operators on V is simultaneously diagonalizable.

A carefully formulated analogue of this statement passes to the infinite-dimensional case, but a counterexample shows that the statement does not fully generalize in the strongest sense. We begin with the positive results. The following may be well-known in other contexts, as it can also be proved with an adaptation of the classical argument that two commuting diagonalizable operators on a finite-dimensional vector space can be simultaneously diagonalized. We provide an alternative argument via summability of idempotents.

Theorem 4.14. *Let $C, D \subseteq \text{End}(V)$ be diagonalizable subalgebras that centralize one another. Then C and D are simultaneously diagonalizable, in the sense that the subalgebra $K[C \cup D] \subseteq \text{End}(V)$ generated by both sets is diagonalizable.*

Proof. Because a subalgebra of $\text{End}(V)$ is diagonalizable if and only if its closure is diagonalizable, and because $K[\overline{C \cup D}] \subseteq \overline{K[C \cup D]}$, we may assume without loss of generality that C and D are closed. In this case, by Theorem 4.10 both C and D are respectively generated by orthogonal sets of idempotents $\{E_i \mid i \in I\}$ and $\{F_j \mid j \in J\}$ such that $\sum E_i = 1 = \sum F_j$.

Consider the set of pairwise products $\{E_i F_j \mid (i, j) \in I \times J\}$. By hypothesis, the E_i and F_j pairwise commute, so that each $E_i F_j$ is again idempotent. For $(i, j) \neq (m, n)$ we have that $E_i F_j$ is orthogonal to $E_m F_n$. Furthermore, from Lemma 2.14 we find that

$$\begin{aligned} \sum_j E_i F_j &= E_i \quad \text{for each } i \in I, \\ \sum_i E_i F_j &= F_j \quad \text{for each } j \in J, \text{ and} \\ \sum_{i,j} E_i F_j &= 1. \end{aligned}$$

So $\{E_i F_j\}$ is an orthogonal set of idempotents whose sum is 1, and it follows from Theorem 4.10 that the closed subalgebra $A \subseteq \text{End}(V)$ generated thereby is diagonalizable. But also each $E_i = \sum_j E_i F_j \in A$ and each $F_j = \sum_i E_i F_j \in A$. So $C, D \subseteq A$ and it follows that $K[C \cup D] \subseteq A$ is diagonalizable. \square

Specializing to the case when one or both of the subalgebras is generated by a single operator, we immediately have the following.

Corollary 4.15. *Any finite set of commuting diagonalizable operators in $\text{End}(V)$ is simultaneously diagonalizable. If $C \subseteq \text{End}(V)$ and $T \in C'$ are diagonalizable, then the subalgebra $C[T] \subseteq \text{End}(V)$ generated by C and T is diagonalizable.*

Of course, an infinite set of commuting diagonalizable operators need not be simultaneously diagonalizable. Such a situation is provided by Example 2.13. That maximal commutative subalgebra A contains orthogonal idempotents $\{E_n \mid n = 0, 1, 2, \dots\}$; these are diagonalizable and commute. If they were simultaneously diagonalizable, then the algebra A would be diagonalizable as it is generated by the E_n . Hence, we would have $\overline{A} \cong K^\Omega$ for Ω an infinite cardinal. But $A = \overline{A}$ by Lemma 2.2(3), so \overline{A} is countable-dimensional. This yields a contradiction, because $\dim(K^\Omega)$ is uncountable.

Next we will present in Example 4.17 another construction of an infinite set of commuting diagonalizable operators need not be simultaneously diagonalizable, of a somewhat more subtle nature. For any integer $n \geq 0$, we let $\mathbf{2}^n = \{0, 1\}^n$ denote the set of strings of length n in the alphabet $\{0, 1\}$. For instance, we have $\mathbf{2}^0 = \{\emptyset\}$, $\mathbf{2}^1 = \{0, 1\}$, and $\mathbf{2}^2 = \{00, 01, 10, 11\}$. Given $i \in \mathbf{2}^n$ and $j \in \mathbf{2}^p$, we let $ij \in \mathbf{2}^{n+p}$ denote the concatenated string that consists of i followed by j .

Lemma 4.16. *Let V be a vector space over a field K with a countable basis $\{v_1, v_2, v_3, \dots\}$. For every integer $n \geq 0$, there exist subspaces $V_i \subseteq V$ for all $i \in \mathbf{2}^n$ with $V_\emptyset = V$ satisfying the following conditions:*

- (a) $\text{Span}(v_1, \dots, v_n) \cap V_i = 0$ for every $i \in \mathbf{2}^n$;
- (b) $V_i = V_{i_0} \oplus V_{i_1}$ for every $i \in \mathbf{2}^n$;
- (c) $\dim(V_i) = \aleph_0$ for every $i \in \mathbf{2}^n$;
- (d) There is $w \in V$ such that $w \notin \bigoplus_{j \in \mathbf{2}^n - \{i\}} V_j$, for all $i \in \mathbf{2}^n$.

Consequently, $V = \bigoplus_{i \in \mathbf{2}^n} V_i$ for any $n \geq 0$, and for any sequence of bits $b_1, b_2, \dots \in \{0, 1\}$, we have $\bigcap_{n=1}^\infty V_{b_1 b_2 \dots b_n} = \{0\}$.

Proof. We proceed by induction. The case $n = 0$ is covered by simply setting $V_\emptyset = V$, and choosing $w = v_1$.

For the inductive step, assume that we have constructed V_i for all strings i of length less than n . Let $i \in \mathbf{2}^{n-1}$; we will define V_{i_0} and V_{i_1} . Since $\text{Span}(v_1, \dots, v_{n-1}) \cap V_i = 0$, it follows that $\text{Span}(v_1, \dots, v_n) \cap V_i$ can be at most 1-dimensional. Let also $w = \sum_{i \in \mathbf{2}^{n-1}} w_i$ be the decomposition of w with respect to $V = \bigoplus_{i \in \mathbf{2}^{n-1}} V_i$; by the inductive hypothesis, all w_i are nonzero. In order to construct V_{i_0} and V_{i_1} in such a way as to satisfy condition (d), we distinguish two cases:

- (1) $\text{Span}(v_1, \dots, v_n) \cap V_i \subseteq \text{Span}(w_i)$. We write $w_i = a + b$ inside V_i , with a, b independent, and we choose V_{i_0} and V_{i_1} to be (countably) infinite-dimensional subspaces of V_i which split V_i such that $a \in V_{i_0}$ and $b \in V_{i_1}$ (by completing $\{a, b\}$ to a basis and splitting the basis appropriately).
- (2) $\text{Span}(v_1, \dots, v_n) \cap V_i = \text{Span}(g)$ and g, w_i are linearly independent. In this case, let $a, b \in V_i$ be such that $\{g, w_i, a, b\}$ are linearly independent. Then the set $\{g -$

$a, w_i - b, a, b\}$ is also linearly independent, and we can choose V_{i_0}, V_{i_1} be infinite-dimensional with $V = V_{i_0} \oplus V_{i_1}$ and $g - a, w_i - b \in V_{i_0}$ and $a, b \in V_{i_1}$. This shows that $w_i = (w_i - b) + (b)$ (and also $g = (g - a) + (a)$) has non-zero components in V_{i_0} and V_{i_1} , and $\text{Span}(v_1, \dots, v_n) \cap V_{i_0} = \text{Span}(v_1, \dots, v_n) \cap V_i \cap V_{i_0} = \text{Span}(g) \cap V_{i_0} = 0$ and similarly $\text{Span}(v_1, \dots, v_n) \cap V_{i_1} = \{0\}$.

This completes the proof by induction.

It remains to verify the final two claims of the statement. For any $n \geq 1$, the direct sum decomposition $V = \bigoplus_{i \in \mathbf{2}^n} V_i$ follows from $V_\emptyset = V$ and condition (b). Finally, given $b_1, b_2, \dots \in \{0, 1\}$, using condition (a) we get that $V_{b_1 \dots b_n} \cap \text{Span}(v_1, \dots, v_n) = \{0\}$. Since $V = \bigcup_{n=1}^{\infty} \text{Span}(v_1, \dots, v_n)$, we conclude that $\bigcap_{n=1}^{\infty} V_{b_1 b_2 \dots b_n} = \{0\}$. \square

Let $\mathbf{2}^* = \bigsqcup_{n=0}^{\infty} \mathbf{2}^n$ denote the set of all words in the alphabet $\mathbf{2} = \{0, 1\}$.

A version of the lemma above holds for a vector space V of arbitrarily large infinite dimension, if we delete condition (d). This can be shown by decomposing $V = \bigoplus_{j \in \dim(V)} V_j$ as a direct sum of vector spaces of dimension \aleph_0 , constructing subspaces $W_{j,i} \subseteq V_j$ for each $i \in \mathbf{2}^*$ that satisfy (a)–(c), and then setting $V_i = \bigoplus_j W_{j,i}$.

Example 4.17. Let V be a K -vector space of countably infinite dimension. Fix subspaces $V_i \subseteq V$ for all $i \in \mathbf{2}^*$ as in Lemma 4.16. Fixing $n \geq 0$ and $i \in \mathbf{2}^n$, let $E_i \in \text{End}(V)$ be the idempotent whose range is V_i and whose kernel is $W_i = \bigoplus_{j \in \mathbf{2}^n \setminus \{i\}} V_j$. Then the $\{E_i \mid i \in \mathbf{2}^n\}$ form orthogonal sets of idempotents such that $1 = \sum_{i \in \mathbf{2}^n} E_i$. Set $A_n = \bigoplus_{i \in \mathbf{2}^n} K E_i \subseteq \text{End}(V)$, the commutative subalgebra generated by the idempotents indexed by $\mathbf{2}^n$. Condition (b) of Lemma 4.16 furthermore ensures that each $E_i = E_{i_0} + E_{i_1}$. Thus we have $A_n \subseteq A_{n+1}$ for all n . Then $A = \bigcup A_n$ is a commutative subalgebra of $\text{End}(V)$, generated by the infinite set $\{E_i \mid i \in \mathbf{2}^n, n \geq 0\}$ of commuting idempotents. Being idempotent operators, these generators are diagonalizable.

We claim that *the algebra A is not diagonalizable*, and so the idempotents E_i for $i \in \bigcup_{n \geq 0} \mathbf{2}^n$ cannot be simultaneously diagonalized. In fact, they have no common eigenvector (i.e. V has no 1-dimensional A -submodule). Indeed, assume for contradiction that $v \in V \setminus \{0\}$ is an eigenvector for all of the idempotents E_i . The eigenvalue λ_i of E_i corresponding to the vector v is either 0 or 1. From $1_V = \sum_{i \in \mathbf{2}^n} E_i$ we see that, for $i \in \mathbf{2}^n$, exactly one of the E_i has eigenvalue 1 and the rest have eigenvalue 0; this means that $v \in V_{i_n}$ for exactly one $i_n \in \mathbf{2}^n$. Note that each $i_n = i_{n-1} b_n$ for some $b_n \in \{0, 1\}$. This means that there is a sequence $b_1, b_2, b_3, \dots \in \{0, 1\}$ such that $v \in \bigcap_{n=0}^{\infty} V_{b_1 \dots b_n}$, which contradicts the fact that the latter intersection is zero by condition (b) of Lemma 4.16.

Finally, we claim that *A is a discrete subalgebra of $\text{End}(V)$* . Indeed, let w denote the vector provided by Lemma 4.16(d). Let E be an idempotent such that $w \in \ker(E)$ and $\ker(E)$ is finite-dimensional. Then the left ideal $I = \text{End}(V)E$ is open. We show that $I \cap A = 0$, which will imply that 0 is open, so A is discrete. If $I \cap A \neq 0$ then $I \cap A_n \neq 0$ for some n . Let $T = \sum_{i \in \mathbf{2}^n} \lambda_i E_i \in I \cap A_n$. If $F = \{i \in \mathbf{2}^n \mid \lambda_i \neq 0\}$, then $F \neq \mathbf{2}^n$ since otherwise T would be invertible. Also $\ker(\sum_{i \in F} \lambda_i E_i) = \bigoplus_{i \notin F} V_i$, and since $E(w) = 0$, it follows that $w \in \bigoplus_{i \notin F} V_i$. But by construction, w has non-zero components in all terms $\bigoplus_{i \in \mathbf{2}^n} V_i$. This contradicts the fact that $F \neq \mathbf{2}^n$, completing the argument.

We note a few more properties of the algebra A . We remark that as A is the union of the A_n which are von Neumann regular, the same property holds for A . In fact, algebraically each $A_n \cong K^{2^n}$, and the inclusion $A_n \subseteq A_{n+1}$ corresponds to the diagonal embedding of K^{2^n} into $K^{2^{n+1}} = K^{2^n 0 \sqcup 2^n 1} \cong K^{2^n} \times K^{2^n}$ (i.e., with each $K \rightarrow K \times K$, $1 \mapsto (1, 1)$). Now because A is commutative and von Neumann regular, it is reduced. Being both reduced and discrete, we see that A is also topologically reduced.

We leave open the interesting question of what is the closure of A within $\text{End}(V)$, especially to what extent this depends on the particular choice of the subspaces V_i for $i \in \mathbf{2}^*$. It would also be interesting to understand to what extent the closure would change if one omitted condition (d) from Lemma 4.16, which forced A to be discrete.

4.D. Closure of the set of diagonalizable operators. Finally, we examine the closure of the set $\mathcal{D}(V)$ of diagonalizable operators within $\text{End}(V)$. If $T \in \text{End}(V)$, let us denote by $H(T)$ the set of all $v \in V$ for which $p(T)v = 0$ for some nonzero polynomial $p(x) \in K[x]$. If we consider V as a $K[x]$ -module with x acting as T , this means that $H(T)$ is the torsion part of V , or equivalently, the sum of all of its finite-dimensional $K[x]$ -submodules. We will show that the T -invariant subspace $H(T)$ plays an important role in the characterization of $\overline{\mathcal{D}(V)}$ in the case of vector spaces over infinite fields.

We need the following easy remark: if W is a finite-dimensional vector space over an infinite field K with basis v_0, v_1, \dots, v_n , then there is $w \in W$ such that the linear map $L \in \text{End}(W)$ defined by $L(v_i) = v_{i+1}$ for $i < n$ and $L(v_n) = w$ is diagonalizable; we show that in fact we can choose $w = \sum_{i=0}^n a_i v_i$ with $a_0, a_1, \dots, a_n \in K$. Indeed, for this it is enough to show that the characteristic polynomial f_L of L has all simple roots in K . But f_L has exactly the coefficients $-a_i$, and we can choose the a_i as the coefficients of the polynomial $\prod_{i=0}^n (x - \lambda_i)$, where λ_i are pairwise distinct elements in K . We have the following.

Theorem 4.18. *Let K be an infinite field and $T \in \text{End}(V)$. Then $T \in \overline{\mathcal{D}(V)}$ if and only if T is diagonalizable on $H(T)$.*

Proof. If $T \in \overline{\mathcal{D}(V)}$, then for every finite-dimensional T -invariant subspace W of $H(T)$, we can find $D \in \mathcal{D}(V)$ such that $D = T$ on W . Since W is T -invariant, it is D -invariant, and D is diagonalizable on W since it is on V (W is a submodule of the semisimple D -module V). Hence T is diagonalizable on W , and W is a (possibly trivial) sum of 1-dimensional T -invariant subspaces. As $H(T)$ is the sum of all its finite-dimensional subspaces, it follows that it is a sum of 1-dimensional T -invariant subspaces. This is to say that $H(T)$ is spanned by a set of T -eigenvectors. This spanning set contains a basis of $H(T)$. Thus $H(T)$ has a basis consisting of T -eigenvectors, and T is diagonalizable on $H(T)$.

Conversely, assume T is diagonalizable on $H(T)$. Let $W \subseteq V$ be finite-dimensional, and write $W = (H(T) \cap W) \oplus U$. Consider V as a $K[x]$ -module under the action of T . Then $M = K[x]W$ is finitely generated, and its torsion part is $H(T) \cap M = H(T) \cap W$. Hence, we may find a free $K[x]$ -module N (T -invariant subspace) such that $M = (M \cap H(T)) \oplus N$, with $N \cong K[x]^t$. Let w_1, \dots, w_t be a $K[x]$ -basis of N . Then a K -basis for N is given by $\{T^i(w_k) \mid i \geq 0, 1 \leq k \leq t\}$, and because W is finite-dimensional there exists n such that $W \subseteq (W \cap H(T)) \oplus \text{Span}\{T^i(w_k) \mid i < n, 1 \leq k \leq t\}$. Let $D \in \text{End}(V)$ be the linear map defined as follows:

- D equals T on $H(T) \cap W$;
- D equals T on the finite set $\{T^i(w_k) \mid i < n, 1 \leq k \leq t\}$ and $D(T^n(w_k)) = u_k$ where u_k is chosen as above such that $u_k \in W_k = \text{Span}\{T^i(w_k) \mid i \leq n\}$ and D is diagonalizable on W_k ;
- D equals 0 on a complement L of $(H(T) \cap W) \oplus (\bigoplus W_k)$.

By construction we have T equal to D on $W \subseteq (H(T) \cap W) \oplus (\bigoplus W_k)$. Since D is diagonalizable on each of the invariant subspaces $H(T) \cap W$, W_k , and L , it follows that D is diagonalizable on $V = (H(T) \cap W) \oplus (\bigoplus W_k) \oplus L$. This shows that every open neighborhood of T contains some diagonalizable D (since $D = T$ on W), and the proof is finished. \square

The above shows that in fact the closure of the set of diagonalizable operators coincides with the closure of the set of diagonalizable operators of *finite rank* on V .

We give an example below to show that the above characterization of the closure of $\mathcal{D}(V)$ fails if the field K is finite. Before doing so, we will in fact show that $\mathcal{D}(V)$ is closed if the field of scalars is finite. Given a prime power q , we let \mathbb{F}_q denote the field with q elements.

Proposition 4.19. *Let V be a vector space over a finite field \mathbb{F}_q . An operator $T \in \text{End}(V)$ is diagonalizable if and only if it satisfies $T^q = T$. Consequently, the set $\mathcal{D}(V)$ of diagonalizable operators is closed in $\text{End}(V)$.*

Proof. Consider the polynomial $p(x) = x^q - x \in \mathbb{F}_q[x]$. As every element of \mathbb{F}_q is a root of this polynomial, the same is certainly true for any diagonalizable operator on V . Thus if $T \in \text{End}(V)$ is diagonalizable then $T^q = T$. Conversely, suppose that T satisfies $T^q = T$, so that $p(T) = 0$ in the algebra $\text{End}(V)$. It follows that T has a minimal polynomial $\mu(x)$ (in the usual sense) that divides $p(x) = \prod_{\lambda \in \mathbb{F}_q} (x - \lambda)$. Thus $\mu(x)$ splits into distinct linear factors over \mathbb{F}_q , and it follows from Proposition 4.13 that T is diagonalizable.

Finally, $\mathcal{D}(V)$ is closed because it is the zero set of $p(x)$ interpreted as a continuous function $p: \text{End}(V) \rightarrow \text{End}(V)$. \square

Example 4.20. Let V be a vector space over a finite field with basis $\{v_i \mid i = 1, 2, \dots\}$, and consider the right shift operator $S \in \text{End}(V)$ with $S(v_i) = v_{i+1}$ for all i . It is clear from Example 2.6 that $H(S) = 0$, so that S is diagonalizable on $H(S)$. However, as S is not diagonalizable (having no eigenvectors), we have $S \notin \mathcal{D}(V) = \overline{\mathcal{D}(V)}$ thanks to the proposition above.

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UNIVERSITY OF IOWA, DEPARTMENT OF MATHEMATICS, McLEAN HALL, IOWA CITY, IA, 52242, USA

UNIVERSITY OF BUCHAREST, STR. ACADEMIEI 14, BUCHAREST, ROMANIA

E-mail address: yovanov@gmail.com; miodrag-iovanov@uiowa.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, COLORADO SPRINGS, CO, 80918, USA

E-mail address: zmesyan@uccs.edu

DEPARTMENT OF MATHEMATICS, BOWDOIN COLLEGE, BRUNSWICK, ME 04011–8486, USA

E-mail address: reyes@bowdoin.edu