Generating self-map monoids of infinite sets*

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Abstract

Let Ω be a countably infinite set, $S = \mathrm{Sym}(\Omega)$ the group of permutations of Ω, and $E = \mathrm{Self}(\Omega)$ the monoid of self-maps of Ω. Given two subgroups $G_1, G_2 \subseteq S$, let us write $G_1 \approx_S G_2$ if there exists a finite subset $U \subseteq S$ such that the groups generated by $G_1 \cup U$ and $G_2 \cup U$ are equal. Bergman and Shelah showed that the subgroups which are closed in the function topology on S fall into exactly four equivalence classes with respect to $\approx_S$. Letting $\approx$ denote the obvious analog of $\approx_S$ for submonoids of E, we prove an analogous result for a certain class of submonoids of E, from which the theorem for groups can be recovered. Along the way, we show that given two subgroups $G_1, G_2 \subseteq S$ which are closed in the function topology on S, we have $G_1 \approx_S G_2$ if and only if $G_1 \approx G_2$ (as submonoids of E), and that $\mathrm{cl}_S(G) \approx \mathrm{cl}_E(G)$ for every subgroup $G \subseteq S$ (where $\mathrm{cl}_S(G)$ denotes the closure of G in the function topology in S and $\mathrm{cl}_E(G)$ its closure in the function topology in E).

1 Introduction

Let Ω be a countably infinite set, let $S = \mathrm{Sym}(\Omega)$ denote the group of all permutations of Ω, and let $E = \mathrm{Self}(\Omega)$ denote the monoid of self-maps of Ω. (Here "E" stands for "endomap.") Given two subgroups $G_1, G_2 \subseteq S$, let us write $G_1 \approx_S G_2$ if there exists a finite subset $U \subseteq S$ such that the group generated by $G_1 \cup U$ is equal to the group generated by $G_2 \cup U$. In [4] Bergman and Shelah show that the subgroups of S that are closed in the function topology on S fall into exactly four equivalence classes with respect to the above equivalence relation. (A subbasis of open sets in the function topology on S is given by the sets $\{f \in S : (\alpha)f = \beta\}$ $(\alpha, \beta \in \Omega)$. This topology is discussed in more detail in Sections 8 and 11.) In this note we investigate properties of an analogous equivalence relation $\approx$ defined for monoids. Two submonoids $M_1, M_2 \subseteq E$ will be considered equivalent if and only if there exists a finite set $U \subseteq E$ such that the monoid generated by $M_1 \cup U$ is equal to the monoid generated by $M_2 \cup U$.

We will show that given two subgroups $G_1, G_2 \subseteq S$ that are closed in the function topology on S, we have $G_1 \approx_S G_2$ if and only if $G_1 \approx G_2$ (as submonoids of E). Writing $\mathrm{cl}_S(G)$ for the closure of the subgroup $G \subseteq S$ in the function topology in S and $\mathrm{cl}_E(G)$ for its closure in the function topology in E, we will show that $\mathrm{cl}_S(G) \approx \mathrm{cl}_E(G)$.

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Our main goal will be to classify into equivalence classes certain closed submonoids of E. In general, unlike the case of groups, the closed submonoids of E fall into infinitely many equivalence classes. To see this, let $\Omega = \omega$, the set of natural numbers, and for each positive integer n let M_n be the submonoid of E generated by all maps whose images are contained in $\{0, 1, \dots, n\}$. It is easy to see that such submonoids are closed in the function topology. Now, let n be a positive integer and consider a monoid word $g_1 g_2 \dots g_k$ in elements of E, where at least one of the $g_i \in M_n$. Then the image of Ω under $g_1 g_2 \dots g_k$ has cardinality at most n. Hence if for some finite subset $U \subseteq E$ we have $M_{n+1} \subseteq \langle M_n \cup U \rangle$ (the monoid generated by $M_n \cup U$), then all the elements of M_{n+1} whose images have cardinality $n+1$ must be in $\langle U \rangle$. This is impossible, since there are uncountably many such elements. Therefore, if $n \neq m$ are two positive integers, then $M_n \not\approx M_m$.

Classifying all closed submonoids of E into equivalence classes appears to be a very difficult task, so we will for the most part focus on submonoids that have large stabilizers (i.e., submonoids M such that for any finite set $\Sigma \subseteq \Omega$ the pointwise stabilizer of Σ in M is $\approx M$) and the property that the subset of a.e. injective maps is dense (with respect to the function topology). Examples of submonoids that have large stabilizers and dense sets of a.e. injective maps include subgroups of $S \subseteq E$, as well as their closures in the function topology in E. Another class of examples of monoids with these properties arises from preorders on Ω. Given such a preorder ρ, let $E(\rho)$ denote the submonoid of E consisting of all maps f such that for all $\alpha \in \Omega$, one has $(\alpha, (\alpha)f) \in \rho$. We will show that the submonoids $E(\rho)$ have large stabilizers and dense sets of a.e. injective maps. (They are also closed in the function topology on E.)

The Bergman-Shelah theorem can be recovered from our classification of the submonoids described above. A surprising aspect of this classification is that the submonoids in question fall into five equivalence classes, rather than the four predicted by the Bergman-Shelah theorem. Throughout, we will also give a number of examples demonstrating various unusual features of our equivalence relation.

Conditions under which a submonoid $M \subseteq E$ satisfies $M \approx E$ have been considered before. For instance, Howie, Ruškuc, and Higgins show in [8] that $S \approx E$. (We give a shorter proof of this fact below.) These authors and Mitchell also exhibit various submonoids that are $\not\approx E$ in [7]. Related questions, but with other kinds of objects in place of E, are discussed in [5], [7], and [11], as well as in papers referenced therein.

2 Chains

Let Ω be an arbitrary infinite set, and set $E = \mathrm{Self}(\Omega)$, the monoid of self-maps of Ω. Elements of E will be written to the right of their arguments. If $U \subseteq E$ is a subset, then we will write $\langle U \rangle$ to denote the submonoid generated by U. The cardinality of a set Γ will be denoted by $|\Gamma|$. If $\Sigma \subseteq \Omega$ and $U \subseteq E$ are subsets, let $U_{\{\Sigma\}} = \{f \in U : (\Sigma)f \subseteq \Sigma\}$.

Definition 1. *Let M be a monoid that is not finitely generated. Then the* cofinality $c(M)$ *of M is the least cardinal κ such that M can be expressed as the union of an increasing chain of κ proper submonoids.*

The main goal of this section is to show that $c(E) > |\Omega|$, which will be needed later on.

This section is modeled on Sections 1 and 2 of [2], where analogous statements are proved for the group of all permutations of Ω. Ring-theoretic analogs of these statements are proved in [10].

Lemma 2. *Let $U \subseteq E$ and $\Sigma \subseteq \Omega$ be such that $|\Sigma| = |\Omega|$ and the set of self-maps of Σ induced by $U_{\{\Sigma\}}$ is all of* $\mathrm{Self}(\Sigma)$. *Then $E = gUh$, for some $g, h \in E$.*

Proof. Let $g \in E$ be a map that takes Ω bijectively to Σ, and let $h \in E$ be a map whose restriction to Σ is the right inverse of g. Then $E = gU_{\{\Sigma\}}h$. □

We will say that $\Sigma \subseteq \Omega$ is a *moiety* if $|\Sigma| = |\Omega| = |\Omega \backslash \Sigma|$. A moiety $\Sigma \subseteq \Omega$ is called *full* with respect to $U \subseteq E$ if the set of self-maps of Σ induced by members of $U_{\{\Sigma\}}$ is all of $\mathrm{Self}(\Sigma)$. The following two results are modeled on group-theoretic results in [9].

Lemma 3 (cf. [2, Lemma 3])**.** *Let $(U_i)_{i\in I}$ be any family of subsets of E such that $\bigcup_{i\in I} U_i = E$ and $|I| \leq |\Omega|$. Then Ω contains a full moiety with respect to some U_i.*

Proof. Since $|\Omega|$ is infinite and $|I| \leq |\Omega|$, we can write Ω as a union of disjoint moieties Σ_i, $i \in I$. Suppose that there are no full moieties with respect to U_i for any $i \in I$. Then in particular, Σ_i is not full with respect to U_i for any $i \in I$. Hence, for every $i \in I$ there exists a map $f_i \in \mathrm{Self}(\Sigma_i)$ which is not the restriction to Σ_i of any member of $(U_i)_{\{\Sigma_i\}}$. Now, if we take $f \in E$ to be the map whose restriction to each Σ_i is f_i, then f is not in U_i for any $i \in I$, contradicting $\bigcup_{i\in I} U_i = E$. □

Proposition 4. $c(E) > |\Omega|$.

Proof. Suppose that $(M_i)_{i\in I}$ is a chain of submonoids of E such that $\bigcup_{i\in I} M_i = E$ and $|I| \leq |\Omega|$. We will show that $E = M_i$ for some $i \in I$.

By the preceding lemma, Ω contains a full moiety with respect to some M_i. Thus, Lemma 2 implies that $E = \langle M_i \cup \{g, h\}\rangle$ for some $g, h \in E$. But, by the hypotheses on $(M_i)_{i\in I}$, $M_i \cup \{g, h\} \subseteq M_j$ for some $j \in I$, and hence $E = \langle M_i \cup \{g, h\}\rangle \subseteq M_j$, since M_j is a submonoid. □

This result is proved by a very different method in [3].

3 Equivalence classes

Throughout this note we will be primarily interested in submonoids of $E = \mathrm{Self}(\Omega)$. However, we begin this section with a definition applicable to submonoids of an arbitrary monoid.

Definition 5. *Let M be a monoid, κ an infinite cardinal, and M_1, M_2 submonoids of M. We will write $M_1 \preccurlyeq_{\kappa,M} M_2$ if there exists a subset $U \subseteq M$ of cardinality $< \kappa$ such that $M_1 \subseteq \langle M_2 \cup U\rangle$. If $M_1 \preccurlyeq_{\kappa,M} M_2$ and $M_2 \preccurlyeq_{\kappa,M} M_1$, we will write $M_1 \approx_{\kappa,M} M_2$, while if $M_1 \preccurlyeq_{\kappa,M} M_2$ and $M_2 \not\preccurlyeq_{\kappa,M} M_1$, we will write $M_1 \prec_{\kappa,M} M_2$. The subscripts M and κ will be omitted when their values are clear from the context.*

It is clear that $\preccurlyeq_{\kappa,M}$ is a preorder on submonoids of M, and hence $\approx_{\kappa,M}$ is an equivalence relation. This equivalence relation and many of the results below are modeled on those in [4]. Ring-theoretic analogs of these results can be found in [11].

We record the following result for future use.

Theorem 6 (Sierpiński, cf. [12], [1], [7])**.** *Every countable subset of E is contained in a subsemigroup generated by two elements of E.*

Proposition 7. *Let $M_1, M_2 \subseteq E$ be submonoids.*
(i) *$M_1 \preccurlyeq_{\aleph_0} M_2$ if and only if $M_1 \preccurlyeq_{\aleph_1} M_2$ (and hence $M_1 \approx_{\aleph_0} M_2$ if and only if $M_1 \approx_{\aleph_1} M_2$).*
(ii) *$M \approx_{\aleph_0} E$ if and only if $M \approx_{|\Omega|^+} E$ (where $|\Omega|^+$ is the successor cardinal of $|\Omega|$).*

Proof. (i) follows from Theorem 6. (ii) follows from Proposition 4. For, if $M \approx_{|\Omega|^+} E$, then among subsets $U \subseteq E$ of cardinality $\leq |\Omega|$ such that $\langle M \cup U \rangle = E$, we can choose one of least cardinality. Let us write $U = \{f_i : i \in |U|\}$. Then the submonoids $M_i = \langle M \cup \{f_j : j < i\} \rangle$ ($i \in |U|$) form a chain of $\leq |\Omega|$ proper submonoids of E. If $|U|$ were infinite, this chain would have union E, contradicting Proposition 4. Hence U is finite, and $M \approx_{\aleph_0} E$. □

Definition 8. *Let $E_{\leq} \subseteq \mathrm{Self}(\omega)$ denote the submonoid of all maps decreasing with respect to the usual ordering of ω. Specifically, $f \in E_{\leq}$ if and only if for all $\alpha \in \omega$, $(\alpha)f \leq \alpha$*

The following result will be our main tool for separating various equivalence classes of submonoids of $\mathrm{Self}(\Omega)$ throughout the paper.

Theorem 9. (i) *Let κ be a regular infinite cardinal $\leq |\Omega|$ and $T \subseteq \mathrm{Self}(\Omega)$ a subset. If $|(\alpha)T| < \kappa$ for all $\alpha \in \Omega$, then $\langle T \rangle \prec_{|\Omega|^+} \mathrm{Self}(\Omega)$.*

(ii) *Let $T \subseteq \mathrm{Self}(\omega)$ be a subset. If there exists a finite λ such that $|(\alpha)T| \leq \lambda$ for all $\alpha \in \omega$, then $\langle T \rangle \prec_{\aleph_0} E_{\leq}$.*

Proof. (i) Let $U \subseteq \mathrm{Self}(\Omega)$ be a subset of cardinality $\leq |\Omega|$. We will show that $\mathrm{Self}(\Omega) \not\subseteq \langle T \cup U \rangle$. Without loss of generality we may assume that $1 \in T \cap U$. For all $j \in \omega$ and $u_0, \ldots, u_j \in U$, we define

$$(1) \qquad B(u_0, \ldots, u_j) = \{u_0 t_0 \ldots u_j t_j : t_0, \ldots, t_j \in T\}.$$

Then the monoid $\langle T \cup U \rangle$ can be written as the union of the sets $B(u_0, \ldots, u_j)$ (using the assumption that $1 \in T \cap U$).

Next, we show by induction on j that for all $\alpha \in \Omega$, $j \in \omega$, and $u_0, \ldots, u_j \in U$, we have $|(\alpha)B(u_0, \ldots, u_j)| < \kappa$. If $j = 0$, $|(\alpha)B(u_0)| = |((\alpha)u_0)T| < \kappa$, by our hypothesis on T. Now, $(\alpha)B(u_0, \ldots, u_{j+1}) = ((\Sigma)u_{j+1})T$, where $\Sigma = (\alpha)B(u_0, \ldots, u_j)$. Assuming inductively that $|\Sigma| < \kappa$, and hence $|(\Sigma)u_{j+1}| < \kappa$, the set $(\alpha)B(u_0, \ldots, u_{j+1})$ can be written as the union of $< \kappa$ sets of cardinality $< \kappa$. By the regularity of κ, $|(\alpha)B(u_0, \ldots, u_{j+1})| < \kappa$.

Let us write $\Omega = \bigcup_{j \in \omega} \Omega_j$, where the union is disjoint and each Ω_j has cardinality $|\Omega|$. Also, for each $n \in \omega$ let $h_n : \Omega_n \to \prod_{j=0}^{n} U$ be a surjection. By the previous paragraph, there is a map $f \in \mathrm{Self}(\Omega)$ such that for all $\alpha \in \Omega_j$, $(\alpha)f \in \Omega \setminus (\alpha)B(u_0, \ldots, u_j)$, where $(u_0, \ldots, u_j) = (\alpha)h_j$. We conclude the proof by showing that $f \notin \langle T \cup U \rangle$.

Suppose that $f \in \langle T \cup U \rangle$. Then $f \in B(u_0, \ldots, u_j)$ for some $j \in \omega$ and $u_0, \ldots, u_j \in U$. Let $\alpha \in \Omega_j$ be such that $(u_0, \ldots, u_j) = (\alpha)h_j$. Then $(\alpha)f \in (\alpha)B(u_0, \ldots, u_j)$, contradicting our definition of f. Hence $f \notin \langle T \cup U \rangle$.

(ii) Let $T \subseteq \mathrm{Self}(\omega)$ and $\lambda \in \omega$ be such that $|(\alpha)T| \leq \lambda$ for all $\alpha \in \omega$, and let $U \subseteq \mathrm{Self}(\omega)$ be a finite subset. We note that for all $\alpha \in \omega$, $|(\alpha)(T \cup U)| \leq \lambda + |U| < \aleph_0$. Hence $T \cup U$ satisfies our hypotheses on T. Therefore, to show that $E_{\leq} \nsubseteq \langle T \cup U \rangle$ for all T and U, it suffices to show that $E_{\leq} \nsubseteq \langle T \rangle$ for all T as in the statement.

Let $f \in E_{\leq}$ be any element such that

$$(\lambda^j + 1)f \notin (\lambda^j + 1)T^j \text{ for all } j \geq 1, \tag{2}$$

where $T^j = \{t_0 \ldots t_{j-1} : t_0, \ldots, t_{j-1} \in T\}$. Such a map exists, since $|(\lambda^j + 1)T^j| \leq \lambda^j$, while there are $\lambda^j + 1$ possible values for $(\lambda^j + 1)f$. Then $f \notin T^j$ for all $j \geq 1$, and hence $f \notin \langle T \rangle$. It remains to be shown that $\langle T \rangle \preccurlyeq_{\aleph_0} E_{\leq}$.

We can find a set $\omega' \subseteq \omega$ and a collection of disjoint sets $\Delta_\alpha \subseteq \omega$ $(\alpha \in \omega')$ such that $|\omega'| = |\omega|$, $|\Delta_\alpha| = \lambda$, and $\Delta_\alpha \subseteq (\alpha)E_{\leq}$. (Specifically, we can take $\omega' = \{\lambda, 2\lambda, 3\lambda, \ldots\}$ and $\Delta_{i\lambda} = \{(i-1)\lambda, (i-1)\lambda + 1, (i-1)\lambda + 2, \ldots, i\lambda - 1\}$ for $i \geq 1$.) Now, let $g \in \mathrm{Self}(\omega)$ be an injective map from ω to ω', and let $h \in \mathrm{Self}(\omega)$ be a map that takes each $\Delta_{(\alpha)g}$ $(\alpha \in \omega')$ onto $(\alpha)T$. Then $T \subseteq gE_{\leq}h$, and hence $\langle T \rangle \subseteq \langle E_{\leq} \cup \{g, h\} \rangle$. □

While we will not use the following two results in the future, they are of interest in their own right.

Corollary 10. *Let κ be a regular infinite cardinal $\leq |\Omega|$ and $\{T_i\}_{i \in \kappa}$ subsets of $E = \mathrm{Self}(\Omega)$ such that $|(\alpha)T_i| < \kappa$ for all $\alpha \in \Omega$ and $i \in \kappa$. Then $\langle \bigcup_{i \in \kappa} T_i \rangle \prec_\kappa E$.*

Proof. Suppose that $\langle \bigcup_{i \in \kappa} T_i \rangle \approx_\kappa E$. Then there is a set $U \subseteq E$ of cardinality $< \kappa$ such that $E = \langle \bigcup_{i \in \kappa} T_i \cup U \rangle$. For each $j \in \kappa$ let $N_j = \langle \bigcup_{i \leq j} T_i \cup U \rangle$. Then $\langle \bigcup_{i \in \kappa} T_i \cup U \rangle = \bigcup_{i \in \kappa} N_i$, and so $E = \bigcup_{i \in \kappa} N_i$. Hence, by Proposition 4, $E = N_n$ for some $n \in \kappa$. For each $\alpha \in \Omega$, set $|(\alpha) \bigcup_{i \leq n} T_i| = \lambda_\alpha$. Then each $\lambda_\alpha < \kappa$, and for all $\alpha \in \Omega$, we have $|(\alpha) \bigcup_{i \leq n} T_i \cup U| \leq \lambda_\alpha + |U| < \kappa$. Thus, $E = N_n$ contradicts Theorem 9. □

In view of Proposition 7(ii), $\prec_\kappa$ can be replaced with $\prec_{|\Omega|^+}$ in the above corollary. This result can be viewed as a generalization to arbitrary Ω of [7, Corollary 2.2]. In [7] a subset $T \subseteq \mathrm{Self}(\omega)$ is said to be *dominated* (by U) if there exists a countable subset $U \subseteq \mathrm{Self}(\omega)$ having the property that for each $f \in T$ there exists $h \in U$ such that $(\alpha)f \leq (\alpha)h$ for all $\alpha \in \omega$. Rewritten using our notation, Corollary 2.2 states that if $T \subseteq \mathrm{Self}(\omega)$ is a dominated subset, then $\langle T \rangle \prec_{\aleph_1} \mathrm{Self}(\omega)$. This can be deduced from Corollary 10 as follows. Let $U \subseteq \mathrm{Self}(\omega)$ be a countable subset that dominates $T \subseteq \mathrm{Self}(\omega)$, and write $U = \{h_i : i \in \omega\}$. Then $T = \bigcup_{i \in \omega} T_i$, where $T_i \subseteq T$ is a subset dominated by $\{h_i\}$. For all $\alpha \in \omega$, we have $|(\alpha)T_i| \leq (\alpha)h_i + 1 < \aleph_0$. Corollary 10 then implies that $\langle T \rangle = \langle \bigcup_{i \in \omega} T_i \rangle \prec_{\aleph_0} \mathrm{Self}(\omega)$, which is equivalent to $\langle T \rangle \prec_{\aleph_1} \mathrm{Self}(\omega)$.

The next result shows that if the κ in the statement of Theorem 9 is assumed to be uncountable, then a stronger conclusion can be obtained (with less work).

Proposition 11. *Let κ be a regular uncountable cardinal $\leq |\Omega|$ and $T \subseteq E$ a subset.*

(i) *If $|(\alpha)T| < \kappa$ for all $\alpha \in \Omega$, then $|(\alpha)\langle T \rangle| < \kappa$ for all $\alpha \in \Omega$.*

(ii) *If there exists* $\aleph_0 \leq \lambda < \kappa$ *such that* $|(\alpha)T| \leq \lambda$ *for all* $\alpha \in \Omega$, *then* $|(\alpha)\langle T\rangle| \leq \lambda$ *for all* $\alpha \in \Omega$.

Proof. For each $\beta \in \Omega$, let $\lambda_\beta < \kappa$ be such that $|(\beta)T| \leq \lambda_\beta$. Also, let $\alpha \in \Omega$ be any element. Then, by definition,

$$(3) \qquad (\alpha)\langle T\rangle = \bigcup_{j=1}^{\infty} (\alpha)T^j.$$

We claim that $|(\alpha)T^j| < \kappa$ for all $j \geq 1$. This is true, by hypothesis, for $j = 1$. Assuming inductively that $\Sigma = (\alpha)T^{j-1}$ has cardinality $< \kappa$, we have

$$(4) \qquad |(\alpha)T^j| = |(\Sigma)T| = |\bigcup_{\sigma\in\Sigma} \{(\sigma)f : f \in T\}| \leq \sum_{\sigma\in\Sigma} \lambda_\sigma.$$

This sum has $< \kappa$ summands, each $< \kappa$; therefore $|(\alpha)T^j| < \kappa$, by the regularity of κ. Finally, $|(\alpha)T^j| < \kappa$ $(j \geq 1)$ implies that $|(\alpha)\langle T\rangle| < \kappa$, since κ is uncountable.

If there exists $\aleph_0 \leq \lambda < \kappa$ such that $|(\alpha)T| \leq \lambda$ for all $\alpha \in \Omega$, then each λ_β can be taken to be λ. Let us assume inductively that for some $j > 1$, $\Sigma = (\alpha)T^{j-1}$ has cardinality $\leq \lambda^{j-1}$. Then, the above argument shows that

$$(5) \qquad |(\alpha)T^j| \leq \sum_{\sigma\in\Sigma} \lambda \leq \sum_{\lambda^{j-1}} \lambda = \lambda^j.$$

Since $\aleph_0 \leq \lambda$, we have $|\lambda^j| = \lambda$ for each $j \geq 1$. Therefore $|(\alpha)\langle T\rangle| \leq \sum_\omega \lambda = \lambda$. □

We conclude the section by noting that $S = \mathrm{Sym}(\Omega)$, the group of all permutations of Ω, is equivalent to $E = \mathrm{Self}(\Omega)$, with respect to our equivalence relation. This result is known (cf. [8, Theorem 3.3]), but we provide a quick proof, for the convenience of the reader.

Theorem 12 (Howie, Ruškuc, and Higgins)**.** *There exist* $g_1, g_2 \in E$ *such that* $E = g_1 S g_2$. *In particular,* $E \approx_{\aleph_0} S$.

Proof. Since Ω is infinite, we can write $\Omega = \bigcup_{\alpha\in\Omega} \Sigma_\alpha$, where the union is disjoint, and for each $\alpha \in \Omega$, $|\Sigma_\alpha| = |\Omega|$. Let $g_1 \in E$ be an injective map such that $|\Omega \setminus (\Omega)g_1| = |\Omega|$, and let $g_2 \in E$ be the map that takes each Σ_α to α.

Now, let $f \in E$ be any element. For each $\alpha \in \Omega$, let Δ_α denote the preimage of α under f. Let $h \in E$ be an injective self-map that embeds Δ_α in Σ_α, for each $\alpha \in \Omega$, and such that for some $\alpha \in \Omega$, $|\Sigma_\alpha \setminus (\Delta_\alpha)h| = |\Omega|$. Then $f = hg_2$. Also, since g_1 and h are both injective and $|\Omega \setminus (\Omega)h| = |\Omega| = |\Omega \setminus (\Omega)g_1|$, there is a permutation $\bar{h} \in S$ such that for all $\alpha \in \Omega$, $(\alpha)g_1\bar{h} = (\alpha)h$. Hence, we have $f = g_1\bar{h}g_2 \in g_1Sg_2$. □

4 The countable case

From now on we will restrict our attention to the case where Ω is countable. It will often be convenient to assume that $\Omega = \omega$, the set of natural numbers. However, we will continue using the symbol Ω; partly in order to distinguish the role of the set as the domain of our maps from its role as an indexing set in some of the proofs, and partly because in Section 7 we will be interested in arbitrary orderings of the set. The symbols $\prec_{\aleph_0,E}$, $\preccurlyeq_{\aleph_0,E}$, and $\approx_{\aleph_0,E}$ will henceforth be written simply as $\prec$, $\preccurlyeq$, and $\approx$, respectively.

We will say that a set A of disjoint nonempty subsets of Ω is a *partition* of Ω if the union of the members of A is Ω. If $U \subseteq E$ ($=$ Self(Ω)) and A is a partition of Ω, let us define $U_{(A)} = \{f \in U : (\Sigma)f \subseteq \Sigma \text{ for all } \Sigma \in A\}$. Also, if $U \subseteq E$ and $\Sigma \subseteq \Omega$, let $U_{(\Sigma)} = \{f \in U : (\alpha)f = \alpha \text{ for all } \alpha \in \Sigma\}$. (The notation $U_{(A)}$ can be considered an extension of the notation $U_{(\Sigma)}$.)

The main aim of this section is to show that given two subgroups $G_1, G_2 \subseteq S$ ($=$ Sym(Ω)) that are closed in the function topology, we have $G_1 \approx G_2$ if and only if G_1 and G_2 are equivalent as subgroups, in the sense of [4]. First, we need two preliminary results.

Proposition 13. *Let A be a partition of Ω. Then $S_{(A)} \approx E_{(A)}$.*

Proof. If A has an infinite member, then $E_{(A)} \approx E$, by Lemma 2, and $S_{(A)} \approx S$, by a similar argument. Hence, the result follows from Theorem 12. Let us, therefore, assume that all members of A are finite, and let us write $A = \{A_i : i \in \omega\}$. Further, let $n_i = |A_i|$, for each $i \in \omega$, and write $A_i = \{a(i,0), a(i,1), \dots, a(i, n_i - 1)\}$. Let $B = \{B_i : i \in \omega\}$ be a partition of Ω such that for each $i \in \omega$, $|B_i| = n_i^2$. By Theorems 13, 15, and 16 of [4], $S_{(A)} \approx S_{(B)}$. We will show that $E_{(A)} \preccurlyeq S_{(B)}$.

For each $i \in \omega$, write $B_i = \{b(i,j,k) : j,k \in \{0,1,\dots,n_i-1\}\}$. Let $g_1 \in E$ be the endomorphism that maps each A_i into B_i via $a(i,j) \mapsto b(i,j,0)$, and let $g_2 \in E$ be the endomorphism that maps each B_i onto A_i via $b(i,j,k) \mapsto a(i,k)$.

Consider any element $h \in E_{(A)}$, and for each $a(i,j) \in \Omega$ write $(a(i,j))h = a(i, c_{ij})$, for some $c_{ij} \in \{0,1,\dots,n_i-1\}$. Let $\bar{h} \in S_{(B)}$ be any permutation such that for each $i \in \omega$ and $j \in \{0,1,\dots,n_i-1\}$, $(b(i,j,0))\bar{h} = b(i,j,c_{ij})$ (e.g., we can define $\bar{h}$ by $b(i,j,k) \mapsto b(i,j,k + c_{ij} (\operatorname{mod} n_i - 1))$). Then, given any $a(i,j) \in \Omega$, we have $(a(i,j))g_1\bar{h}g_2 = (b(i,j,0))\bar{h}g_2 = (b(i,j,c_{ij}))g_2 = a(i,c_{ij}) = (a(i,j))h$, and hence $E_{(A)} \subseteq g_1 S_{(B)} g_2$. □

In the above proof, we called on results from [4] to deduce that $S_{(A)} \approx S_{(B)}$. This was done primarily in the interests of space, since while it is not very difficult to show that $S_{(A)} \approx S_{(B)}$ directly, several different cases would need to be considered.

Lemma 14. *Let A be a partition of Ω into finite sets such that there is no common finite upper bound on the cardinalities of the members of A. Then $E_{(A)} \approx E_{\leq}$. (See Definition 8 for the notation $E_{\leq}$.)*

Proof. To prove that $E_{\leq} \preccurlyeq E_{(A)}$, we will construct $g, h \in E$ such that $E_{\leq} \subseteq gE_{(A)}h$. By our hypotheses on A, we can find $\{B_i \in A : i \in \omega\}$, consisting of disjoint sets, such that each $|B_i| \geq i$. Let us write each B_i as $\{b(i,0), b(i,1), \dots, b(i, m_i - 1)\}$, where $m_i = |B_i|$. Define $g \in E$ by $(i)g = b(i,0)$ for all $i \in \Omega$, and define $h \in E$ by $(b(i,j))h = j$ for all $i \in \omega$ and $j < m_i$ (h can be defined arbitrarily on elements not of the form $b(i,j)$). Then $E_{\leq} \subseteq gE_{(A)}h$.

To prove that $E_{(A)} \preccurlyeq E_{\leq}$, write $A = \{A_i : i \in \omega\}$, and let $g \in E$ be any injective map such that for all $i \in \omega$ and $a \in A_i$, $(a)g$ is greater than all the elements of A_i ($\subseteq \omega$). Now, let $f \in E_{(A)}$ be any element. Then for all $a \in \Omega$, $(a)f < (a)g$, since $a \in A_i$ for some $i \in \omega$. Thus, we can find a map $h \in E_{\leq}$ such that for all $a \in \Omega$, $((a)g)h = (a)f$, since g is injective. Therefore, $E_{(A)} \subseteq gE_{\leq}$. □

Theorem 15. *Let G_1 and G_2 be subgroups of $S = \mathrm{Sym}(\Omega) \subseteq E$ that are closed in the function topology on S. Let us write $G_1 \approx_S G_2$ if G_1 and G_2 are equivalent as groups (i.e., if the group generated by $G_1 \cup U$ is equal to the group generated by $G_2 \cup U$, for some finite set $U \subseteq S$). Then $G_1 \approx_S G_2$ if and only if $G_1 \approx G_2$.*

Proof. To show the forward implication, suppose that $U \subseteq S$ is a finite subset such that the group generated by $G_1 \cup U$ is equal to the group generated by $G_2 \cup U$. Letting U^{-1} be the set consisting of the inverses of the elements of U, we see that $\langle G_1 \cup (U \cup U^{-1})\rangle = \langle G_2 \cup (U \cup U^{-1})\rangle$. Hence $G_1 \approx_S G_2$ implies that $G_1 \approx G_2$.

For the converse, let A be a partition of Ω into finite sets such that there is no common finite upper bound on the cardinalities of the members of A, and let B be a partition of Ω into 2-element sets. By the main results of [4], every closed subgroup of S is $\approx_S$ to exactly one of S, $S_{(A)}$, $S_{(B)}$, or $\{1\}$. We finish the proof by showing that these four groups are $\not\approx$ to each other.

By Proposition 13, $S_{(A)} \approx E_{(A)}$, and by the previous lemma, the latter is $\approx E_{\leq}$. By Theorem 12, $S \approx E$. Part (i) of Theorem 9 then implies that $S_{(A)} \prec S$, and part (ii) of that theorem implies that $S_{(B)} \prec S_{(A)}$. Also, $\{1\} \prec S_{(B)}$, since $S_{(B)}$ is uncountable. Thus, we have $\{1\} \prec S_{(B)} \prec S_{(A)} \prec S$. □

5 An example

The goal of this section is to show that the partial ordering $\preccurlyeq$ of submonoids of $E = \mathrm{Self}(\Omega)$ is not a total ordering, i.e., that there are submonoids $M, M' \subseteq E$ such that $M \not\preccurlyeq M'$ and $M' \not\preccurlyeq M$. In case the reader wishes to skip this section, we note that nothing in subsequent sections will depend on the present discussion.

As in Section 1, upon identifying Ω with ω, let $M_2 \subseteq E$ be the submonoid generated by all maps whose images are contained in $\{0, 1, 2\}$. Let $\Sigma_1, \Sigma_2 \subseteq \Omega$ be disjoint infinite subsets, such that $\{0, 1\} \subseteq \Sigma_1$, $\{2, 3\} \subseteq \Sigma_2$, and $\Sigma_1 \cup \Sigma_2 = \Omega$. Let $M_3' \subseteq E$ be the submonoid generated by all maps that take Σ_1 to $\{0, 1\}$ and Σ_2 to $\{2, 3\}$. We will show that $M_2 \not\preccurlyeq M_3'$ and $M_3' \not\preccurlyeq M_2$.

Using the same argument as in Section 1, it is easy to see that $M_3' \not\preccurlyeq M_2$. (If $g_1 g_2 \dots g_k$ is any word in elements of E, where at least one of the $g_i \in M_2$, then the image of Ω under $g_1 g_2 \dots g_k$ has cardinality at most 3. Hence, if $M_3' \subseteq \langle M_2 \cup U\rangle$ for some finite subset $U \subseteq E$, then all the elements of M_3' whose images have cardinality 4 must be in $\langle U\rangle$. This is impossible, since there are uncountably many such elements.)

Next, let $U \subseteq E$ be a finite set. We will show that $M_2 \not\subseteq \langle M_3' \cup U\rangle$. We begin by characterizing the elements of $\langle M_3' \cup U\rangle$. Let $H \subseteq E$ be the (countable) set of all maps that fix $\Omega \setminus \{0, 1, 2, 3\}$ elementwise. Now, consider any word $f = g_1 g_2 \dots g_k$ in elements of E, such that $g_1 \in M_3'$. Since the image of g_1 is contained in $\{0, 1, 2, 3\}$, f can be written as $g_1 h$, where h is some element of H. Hence, any element $f \in \langle M_3' \cup U\rangle$ can be written as $f = g_1 g_2 h$, where $g_1 \in \langle U\rangle$, $g_2 \in M_3'$, and $h \in H$. (Here we are using that fact that $1 \in \langle U\rangle \cap M_3' \cap H$.)

We note that each $g \in \langle U\rangle$ either takes infinitely many elements of Ω to Σ_1 or takes infinitely many elements to Σ_2, since $\Sigma_1 \cup \Sigma_2 = \Omega$. For each such g, let $\Gamma_g \subseteq \Omega$ denote either the set of those elements that g takes to Σ_1 or the set of those elements that g takes to Σ_2 -

whichever is infinite. Set $F = \{\Gamma_g : g \in \langle U\rangle\}$. Since $\langle U\rangle$ is countable, so is F, and hence we can write it as $F = \{\Delta_i : i \in \omega\}$.

Next, let us construct for each $i \in \omega$ a triplet of distinct elements $a_i, b_i, c_i \in \Delta_i$, such that the sets $\{a_i, b_i, c_i\}$ are disjoint. We take $a_0, b_0, c_0 \in \Delta_0$ to be any three distinct elements (which must exist, since Δ_0 is infinite). Let $0 \leq j$ be an integer, and assume that the elements $a_i, b_i, c_i \in \Delta_i$ have been picked for all $i \leq j$. Let $a_{j+1}, b_{j+1}, c_{j+1} \in \Delta_{j+1} \setminus \bigcup_{i\leq j}\{a_i, b_i, c_i\}$ be any three distinct elements. (Again, this is possible, by the fact that Δ_{j+1} is infinite.)

Now, let $f \in M_2$ be an element that takes each set $\{a_i, b_i, c_i\}$ bijectively to $\{0, 1, 2\}$, such that $f \notin \langle U\rangle$. A self-map with these properties exists, since there are uncountably many maps that take each $\{a_i, b_i, c_i\}$ bijectively to $\{0, 1, 2\}$, and $\langle U\rangle$ is countable. We finish the proof by showing that $f \notin \langle M_3' \cup U\rangle$. Suppose, on the contrary, that $f \in \langle M_3' \cup U\rangle$. Then $f = g_1g_2h$, where $g_1 \in \langle U\rangle$, $g_2 \in M_3'$, and $h \in H$, by the above characterization. Since $f \notin \langle U\rangle$, we may assume that $g_2 \neq 1$. Let $\Delta_i \in F$ be the set corresponding to g_1 (i.e., Γ_{g_1}). Then, by the above construction, we can find three distinct elements $a_i, b_i, c_i \in \Delta_i$ such that f takes $\{a_i, b_i, c_i\}$ bijectively to $\{0, 1, 2\}$. On the other hand, by choice of Δ_i, g_1 either takes $\{a_i, b_i, c_i\}$ to Σ_1 or takes $\{a_i, b_i, c_i\}$ to Σ_2. In either case, $|(\{a_i, b_i, c_i\})g_1g_2h| \leq 2$, since g_2 takes each of Σ_1 and Σ_2 to a 2-element set. Hence $f \neq g_1g_2h$; a contradiction. We therefore conclude that $f \notin \langle M_3' \cup U\rangle$.

In summary, we have

Proposition 16. *The partial ordering $\preccurlyeq$ of submonoids of E is not a total ordering.*

6 Four lemmas

The results of this section (except for the first) are close analogs of results in [4]. We will use them in later sections to classify various submonoids of $E = \mathrm{Self}(\Omega)$ into equivalence classes; Lemmas 18-20 will be our main tools for showing that submonoids are $\approx$ to each other. The proofs of these three lemmas are, for the most part, simpler than those of their group-theoretic analogs (namely [4, Lemma 10], [4, Lemma 12], and [4, Lemma 14], respectively).

Lemma 17. *Let $M \subseteq E$ be a submonoid. Then $M \preccurlyeq E_{(A)}$, where $A = \{A_\alpha : \alpha \in \Omega\}$ is any partition of Ω such that for each $\alpha \in \Omega$, $|A_\alpha| = |(\alpha)M|$.*

Proof. Let $g \in E$ be a map such that for all $\alpha \in \Omega$, $(\alpha)g \in A_\alpha$, and let $h \in E$ be a map such that for all $\alpha \in \Omega$, h maps (A_α) onto $(\alpha)M$. Then $M \subseteq gE_{(A)}h$. □

Lemma 18. *Let M be a submonoid of E, and suppose there exist a sequence $(\alpha_i)_{i\in\omega} \in \Omega^\omega$ of distinct elements and a sequence of nonempty subsets $D_i \subseteq \Omega^i$ $(i \in \omega)$, such that*

(i) *For each $i \in \omega$ and $(\beta_0, \dots, \beta_{i-1}) \in D_i$, there exist infinitely many elements $\beta \in \Omega$ such that $(\beta_0, \dots, \beta_{i-1}, \beta) \in D_{i+1}$; and*

(ii) *If $(\beta_i)_{i\in\omega} \in \Omega^\omega$ has the property that $(\beta_0, \dots, \beta_{i-1}) \in D_i$ for each $i \in \omega$, then there exists $g \in M$ such that for all $i \in \omega$, $\beta_i = (\alpha_i)g$, and the elements β_i are all distinct.*

Then there exist $f, h \in E$ such that $E = fMh$. In particular, $M \approx E$.

Proof. For each $i \in \omega$ and $(\beta_0, \ldots, \beta_{i-1}) \in D_i$, let

$$\Gamma(\beta_0, \ldots, \beta_{i-1}) = \{\beta \in \Omega : (\beta_0, \ldots, \beta_{i-1}, \beta) \in D_{i+1}\}. \tag{6}$$

By (i), each $\Gamma(\beta_0, \ldots, \beta_{i-1})$ is an infinite subset of Ω. Since for each $i \in \omega$, Ω^i is countable, $\bigcup_{i\in\omega} \Omega^i$ is countable as well, and therefore, so is $\bigcup_{i\in\omega} D_i \subseteq \bigcup_{i\in\omega} \Omega^i$. Thus, there are only countably many sets of the form $\Gamma(\beta_0, \ldots, \beta_{i-1})$. By a standard inductive construction, we can find a collection $\{\Lambda(\beta_0, \ldots, \beta_{i-1}) : i \in \omega,\ (\beta_0, \ldots, \beta_{i-1}) \in D_i\}$ of disjoint infinite sets, such that each $\Lambda(\beta_0, \ldots, \beta_{i-1}) \subseteq \Gamma(\beta_0, \ldots, \beta_{i-1})$.

Next, we define a map $f \in E$ by

$$(i)f = \alpha_i \text{ for all } i \in \omega\ (= \Omega). \tag{7}$$

Also, let $h \in E$ be any map that takes each $\Lambda(\beta_0, \ldots, \beta_{i-1})$ surjectively to Ω. We will show that $E = fMh$.

Let $g \in E$ be any element. We define recursively a sequence $(\beta_i)_{i\in\omega} \in \Omega^\omega$ as follows: for each $i \in \omega$ let $\beta_i \in \Lambda(\beta_0, \ldots, \beta_{i-1})$ be such that $(\beta_i)h = (i)g$. Since each $\Lambda(\beta_0, \ldots, \beta_{i-1}) \subseteq \Gamma(\beta_0, \ldots, \beta_{i-1})$, our sequence $(\beta_i)_{i\in\omega}$ has the property that $(\beta_0, \ldots, \beta_{i-1}) \in D_i$ for each $i \in \omega$. Thus, by (ii), there exists $\bar{g} \in M$ such that for all $i \in \omega$, $\beta_i = (\alpha_i)\bar{g}$. For each $i \in \omega\ (= \Omega)$, we then have $(i)f\bar{g}h = (\alpha_i)\bar{g}h = (\beta_i)h = (i)g$, and therefore $g = f\bar{g}h$. □

The next argument uses the same basic idea, but it is more complicated.

Lemma 19. *Let M be a submonoid of E, and suppose there exist a sequence $(\alpha_i)_{i\in\omega} \in \Omega^\omega$ of distinct elements, an unbounded sequence of positive integers $(N_i)_{i\in\omega}$, and a sequence of nonempty subsets $D_i \subseteq \Omega^i$ $(i \in \omega)$, such that*

- (i) *For each $i \in \omega$ and $(\beta_0, \ldots, \beta_{i-1}) \in D_i$, there exist at least N_i elements $\beta \in \Omega$ such that $(\beta_0, \ldots, \beta_{i-1}, \beta) \in D_{i+1}$; and*
- (ii) *If $(\beta_i)_{i\in\omega} \in \Omega^\omega$ has the property that $(\beta_0, \ldots, \beta_{i-1}) \in D_i$ for each $i \in \omega$, then there exists $g \in M$ such that for each $i \in \omega$, $\beta_i = (\alpha_i)g$, and the elements β_i are all distinct.*

Then there exist $f, h \in E$ such that $E_\leq \subseteq fMh$. In particular, $E_\leq \preccurlyeq M$.

Proof. We will construct recursively integers $i(-1) < i(0) < \ldots < i(j) < \ldots$, and for each $j \geq 0$ a subset $C_{i(j)} \subseteq D_{i(j)}$.

Set $i(-1) = -1$ and $i(0) = 0$, and let $C_{i(0)} = D_0$ be the singleton consisting of the empty string. Now assume inductively for some $j \geq 1$ that $i(0), \ldots, i(j-1)$ and $C_{i(0)}, \ldots, C_{i(j-1)}$ have been constructed. Let $i(j)$ be an integer such that $N_{i(j)} > j \cdot |C_{i(j-1)}| + \sum_{k=0}^{j-1} |C_{i(k)}|$. Let $C_{i(j)} \subseteq D_{i(j)}$ be a finite subset that for each $(\beta_0, \ldots, \beta_{i(j-1)-1}) \in C_{i(j-1)}$ contains j elements of the form $(\beta_0, \ldots, \beta_{i(j-1)-1}, \ldots, \beta_{i(j)-2}, \beta)$, such that the elements β are distinct from each other and from all elements that occur as last components of elements of $C_{i(0)}, \ldots, C_{i(j-1)}$. Our choice of $i(j)$ and condition (i) make this definition possible. (Actually, it would have sufficed to pick $i(j)$ so that $N_{i(j-1)+1} N_{i(j-1)+2} \ldots N_{i(j)} > j \cdot |C_{i(j-1)}| + \sum_{k=0}^{j-1} |C_{i(k)}|$.)

Once the above integers and subsets have been constructed, let us use the sets $C_{i(j)}$ to construct subsets $F_{i(j)} \subseteq \Omega^j$. Set $F_{i(0)} = C_{i(0)}$. For each element $(\beta_0, \ldots, \beta_{i(j)-1}) \in C_{i(j)}$ with $j \geq 1$, we define a sequence $(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)})$ by setting $\gamma_{i(k)} = \beta_{i(k+1)-1}$ $(0 \leq k \leq j-1)$; i.e., we drop the β_k that do not occur as last components of elements of $C_{i(0)}, \ldots, C_{i(j-1)}$. For each $j \geq 1$, we then let $F_{i(j)}$ consist of the tuples $(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)})$. Also, for each element $(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)}) \in F_{i(j)}$ let

(8) $\quad \Gamma(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)}) = \{\beta \in \Omega : (\gamma_{i(0)}, \ldots, \gamma_{i(j-1)}, \beta) \in F_{i(j+1)}\}.$

By construction, each $|\Gamma(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)})| \geq j+1$; for simplicity, we will assume that this is an equality, after discarding some elements if necessary. Let $h \in E$ be a map such that

(9) $\quad h$ takes each $\Gamma(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)})$ onto $\{0, \ldots, j\}$.

Such a map exists, since the sets $\Gamma(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)})$ are all disjoint. Also, let $f \in E$ be defined by

(10) $\quad (j)f = \alpha_{i(j)}$ for all $j \in \omega$ $(= \Omega)$.

We finish the proof by showing that $E_{\leq} \subseteq fMh$.

Let $g \in E_{\leq}$ be any element. We first construct recursively a sequence $(\gamma_{i(j)})_{j \in \omega}$ such that for each $j \in \omega$, $(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)}) \in F_{i(j)}$. Let $\gamma_{i(0)}$ be the unique element of $\Gamma()$. (We note that $(\gamma_{i(0)})h = 0 = (0)g$, by definition of h.) Assuming that $\gamma_{i(0)}, \ldots, \gamma_{i(j-1)}$ have been defined, let $\gamma_{i(j)} \in \Gamma(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)})$ be such that $(\gamma_{i(j)})h = (j)g$. (Such an element exists, by our definition of h and the fact that for all $k \in \omega$, $(k)g \leq k$.) Since the sequence $(\gamma_{i(j)})_{j \in \omega} \in \Omega^{\omega}$ has the property that $(\gamma_{i(0)}, \ldots, \gamma_{i(j-1)}) \in F_{i(j)}$ for each $j \in \omega$, there exists $\bar{g} \in M$ such that for all $i \in \omega$, $\gamma_{i(j)} = (\alpha_{i(j)})\bar{g}$. This follows from (ii), since $(\gamma_{i(j)})_{j \in \omega}$ is a subsequence of some $(\beta_i)_{i \in \omega}$ as in (ii). Hence, for each $j \in \omega$ $(= \Omega)$, we have $(j)f\bar{g}h = (\alpha_{i(j)})\bar{g}h = (\gamma_{i(j)})h = (j)g$, and therefore $g = f\bar{g}h$. □

Lemma 20. *Let M be a submonoid of E. Suppose there exist three sequences $(\alpha_i)_{i \in \omega}$, $(\beta_i)_{i \in \omega}$, $(\gamma_i)_{i \in \omega} \in \Omega^{\omega}$ of distinct elements, such that $(\beta_i)_{i \in \omega}$ and $(\gamma_i)_{i \in \omega}$ are disjoint, and for every element $(\delta_i)_{i \in \omega} \in \prod_{i \in \omega} \{\beta_i, \gamma_i\} \subseteq \Omega^{\omega}$, there exists $g \in M$ such that for all $i \in \omega$, $\delta_i = (\alpha_i)g$. Then there exist $f, h \in E$ such that $E_{(A)} \subseteq fMh$, where A is a partition of Ω into 2-element sets. In particular $E_{(A)} \preccurlyeq M$.*

Proof. Write $A = \{A_i : i \in \omega\}$, where for each $i \in \omega$, $A_i = \{a_i, b_i\}$. Let $f \in E$ be the map defined by $(a_i)f = \alpha_{2i}$ and $(b_i)f = \alpha_{2i+1}$, and let $h \in E$ be a map that for each $i \in \omega$ takes $\{\beta_{2i}, \beta_{2i+1}\}$ to a_i and $\{\gamma_{2i}, \gamma_{2i+1}\}$ to b_i. Then $E_{(A)} \subseteq fMh$. □

7 Submonoids arising from preorders

Definition 21. *Given a preorder ρ on Ω, let $E(\rho) \subseteq E$ $(= \mathrm{Self}(\Omega))$ denote the subset consisting of all maps f such that for all $\alpha \in \Omega$ one has $(\alpha, (\alpha)f) \in \rho$.*

Clearly, subsets of the form $E(\rho)$ are submonoids. The submonoids $E_{(A)}$ (where A is a partition of Ω) are of this form, as is $E_{\leq}$. The goal of this section is to classify such submonoids into equivalence classes. To facilitate the discussion, let us divide them into five types.

Definition 22. *Let ρ be a preorder on Ω. For each $\alpha \in \Omega$ set $\Delta_{\rho}(\alpha) = \{\beta \in \Omega : (\alpha, \beta) \in \rho\}$ (the "principal up-set generated by α"). We will say that*

The preorder ρ is of type 1 *if there is an infinite subset $\Gamma \subseteq \Omega$ such that for all $\alpha \in \Gamma$, $\Delta_\rho(\alpha)$ is infinite.*

The preorder ρ is of type 2 *if the cardinalities of the sets $\Delta_\rho(\alpha)$ $(\alpha \in \Omega)$ have no common finite upper bound, but $\Delta_\rho(\alpha)$ is infinite for only finitely many α.*

The preorder ρ is of type 3 *if there is a number $n \in \omega$ such that $|\Delta_\rho(\alpha)| \leq n$ for all but finitely many $\alpha \in \Omega$, and there are infinitely many $\alpha \in \Omega$ such that $|\Delta_\rho(\alpha)| > 1$.*

The preorder ρ is of type 4 *if $|\Delta_\rho(\alpha)| = 1$ for all but finitely many $\alpha \in \Omega$.*

Let us further divide preorders of type 3 into two sub-types. We will say that

The preorder ρ is of type 3a *if it is of type 3 and, in addition, there are infinite families $\{\alpha_i\}_{i\in\omega}, \{\beta_i\}_{i\in\omega} \subseteq \Omega$ consisting of distinct elements, such that for each $i \in \omega$, $\alpha_i \neq \beta_i$ and $(\alpha_i, \beta_i) \in \rho$.*

The preorder ρ is of type 3b *if it is of type 3 and, in addition, there is a finite set $\Gamma \subseteq \Omega$ such that for all but finitely many $\alpha \in \Omega$, $\Delta_\rho(\alpha) \subseteq \Gamma \cup \{\alpha\}$.*

It is clear that every preorder on Ω falls into exactly one of the above five types. Further, if ρ is a preorder of type 2 or 3, and $\Sigma = \{\alpha \in \Omega : |\Delta_\rho(\alpha)| = \aleph_0\}$, then $E(\rho) \approx E(\rho)_{(\Sigma)}$. (The notation $E(\rho)_{(\Sigma)}$ is defined at the beginning of Section 4.) For, if $\alpha \in \Omega \setminus \Sigma$, then $\Delta_\rho(\alpha) \cap \Sigma = \emptyset$, since $\Delta_\rho(\alpha)$ is finite. This shows that $(\Omega \setminus \Sigma)E(\rho) \cap \Sigma = \emptyset$. Hence, if $f \in E(\rho)$ is any element, then $f \in E(\rho)_{(\Sigma)}U$, where $U = E_{(\Omega\setminus\Sigma)}$. Therefore, $E(\rho) \subseteq \langle E(\rho)_{(\Sigma)} \cup U \rangle$, and $E(\rho) \approx E(\rho)_{(\Sigma)}$. (Since Σ is finite, U is countable. Theorem 6 then allows us to replace U by a finite set.) We will use this observation a number of times in this section.

Lemma 23. *Let ρ be a preorder on Ω, and let A be a partition of Ω into 2-element sets.*

(i) *If ρ is of type 1, then $E(\rho) \approx E$.*

(ii) *If ρ is of type 2, then $E(\rho) \approx E_\leq$.*

(iii) *If ρ is of type 3a, then $E(\rho) \approx E_{(A)}$.*

Proof. If ρ is of type 1, respectively type 2, then $E(\rho)$ clearly satisfies the hypotheses of Lemma 18, respectively Lemma 19. Hence, $E \approx E(\rho)$, respectively $E_\leq \preccurlyeq E(\rho)$. Further, if ρ is of type 3a, then upon removing some elements if necessary, we may assume that the $\{\alpha_i\}_{i\in\omega}$ and $\{\beta_i\}_{i\in\omega}$ provided by the definition are disjoint. $E(\rho)$ then satisfies the hypotheses of Lemma 20 (with $\alpha_i = \gamma_i$). Hence $E_{(A)} \preccurlyeq E(\rho)$.

To finish the proof, we must show that $E(\rho) \preccurlyeq E_\leq$ if ρ is of type 2, and that $E(\rho) \preccurlyeq E_{(A)}$ if ρ is of type 3a. In either case, we may assume that for all $\alpha \in \Omega$, $\Delta_\rho(\alpha)$ is finite, by the remarks following Definition 22. Now, $E(\rho) \preccurlyeq E_{(B)}$, where B is a partition of Ω into finite sets, by Lemma 17, and in the case where ρ is of type 3a, these finite sets can all be taken to have cardinality n, for some $n \in \omega$. If ρ is of type 2, then we have $E(\rho) \preccurlyeq E_{(B)} \approx E_\leq$, by Lemma 14. If ρ is of type 3a, then $E_{(A)} \approx E_{(B)}$, by Proposition 13 and [4, Theorem 15]. Hence $E(\rho) \preccurlyeq E_{(A)}$. □

Definition 24. *Given $\gamma \in \Omega$, let ρ_γ be the preorder on Ω defined by $(\alpha, \beta) \in \rho_\gamma \Leftrightarrow \beta \in \{\alpha, \gamma\}$.*

It is clear that the preorders ρ_γ are of type $3b$. Further, if $\alpha, \beta \in \Omega$ are any two elements, then $E(\rho_\alpha) \approx E(\rho_\beta)$. More specifically, if $g \in E$ is any permutation of Ω that takes α to β, then $E(\rho_\alpha) = gE(\rho_\beta)g^{-1}$. (Given $\gamma \in \Omega$, an element of $gE(\rho_\beta)g^{-1}$ either fixes γ or takes it to α. Hence $gE(\rho_\beta)g^{-1} \subseteq E(\rho_\alpha)$, and similarly $g^{-1}E(\rho_\alpha)g \subseteq E(\rho_\beta)$. Conjugating the latter expression by g, we obtain $E(\rho_\alpha) \subseteq gE(\rho_\beta)g^{-1}$.)

Let E_{3b} denote the monoid generated by $\{E(\rho_\alpha) : \alpha \in \Omega\}$, and let $g \in E$ be a permutation which is transitive on Ω. Then, by the previous paragraph, $E_{3b} \subseteq \langle E(\rho_\gamma) \cup \{g\} \rangle$ for any $\gamma \in E$. Hence, for all $\gamma \in E$, $E(\rho_\gamma) \approx E_{3b}$. As an aside, we note that E_{3b} is closed under conjugation by permutations of Ω. (Given any permutation g and any $\alpha \in \Omega$, $gE(\rho_\beta)g^{-1} = E(\rho_\alpha)$, where $\beta = (\alpha)g$. Thus $gE_{3b}g^{-1}$ contains all the generators of E_{3b}, and therefore $E_{3b} \subseteq gE_{3b}g^{-1}$. Conjugating by g^{-1}, we obtain $g^{-1}E_{3b}g \subseteq E_{3b}$.)

Lemma 25. *Let ρ and ρ' be preorders on Ω of type $3b$. Then $E(\rho) \approx E(\rho')$.*

Proof. Let ρ be a preorder on Ω of type $3b$. Then there is a $\beta \in \Omega$ and an infinite set $\Sigma \subseteq \Omega$ such that all $\alpha \in \Sigma$, we have $(\alpha, \beta) \in \rho$. Let $f_1 \in E$ be a map that takes Ω bijectively to Σ, while fixing β (which, we may assume, is an element of Σ), and let $f_2 \in E$ be a right inverse of f_1. Then $E(\rho_\beta) \subseteq f_1E(\rho)f_2$, and hence $E_{3b} \preccurlyeq E(\rho)$. We conclude the proof by showing that $E(\rho) \preccurlyeq E_{3b}$.

By the remarks following Definition 22, we may assume that for all $\alpha \in \Omega$, $\Delta_\rho(\alpha)$ is finite. Let $\Gamma \subseteq \Omega$ be a finite set such that for all $\alpha \in \Omega$, $\Delta_\rho(\alpha) \subseteq \Gamma \cup \{\alpha\}$, and write $\Gamma = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$. We will first show that $E(\rho)_{(\Gamma)} \subseteq E_{3b}$.

Let $h \in E(\rho)_{(\Gamma)}$ be any element. Then we can write Ω as a disjoint union $\Lambda_0 \cup \Lambda_1 \cup \dots \cup \Lambda_n \cup \Lambda$, where for all $\beta \in \Lambda_i$, $(\beta)h = \alpha_i$, and h acts as the identity on Λ. For each $i \in \{0, 1, \dots, n\}$, let $g_i \in E(\rho_{\alpha_i})$ be the map that takes $\Lambda_i \setminus \{\alpha_0, \alpha_1, \dots, \alpha_n\}$ to α_i and fixes all other elements. Then $h = g_0g_1 \dots g_n \in E_{3b}$, and hence $E(\rho)_{(\Gamma)} \subseteq E_{3b}$.

For each $g \in \mathrm{Self}(\Gamma)$ let $f_g \in E$ be such that f_g acts as g on Γ and as the identity elsewhere, and set $V = \{f_g : g \in \mathrm{Self}(\Gamma)\}$. Now, let $h \in E(\rho)$ be any element, and define $\bar{h} \in E(\rho)_{(\Gamma)}$ by $(\alpha)\bar{h} = (\alpha)h$ for all $\alpha \notin \Gamma$. Noting that, by definition of Γ, $(\Gamma)h \subseteq \Gamma$, there is an element $f_g \in V$ that agrees with h on Γ. Then $h = f_g\bar{h} \in VE(\rho)_{(\Gamma)} \subseteq VE_{3b}$, and hence $E(\rho) \preccurlyeq E_{3b}$. □

We will say that a map $f \in E$ is *finitely-many-to-one* (or, more succinctly, *fm-to-one*) if the preimage of each element of Ω under f is finite.

Lemma 26. *Let ρ and ρ' be preorders on Ω of types $3a$ and $3b$, respectively. Then $E(\rho') \prec E(\rho)$.*

Proof. Let A be a partition of Ω into 2-element sets, and let us fix an element $\gamma \in \Omega$. By the previous two lemmas, it suffices to show that $E(\rho_\gamma) \prec E_{(A)}$. We begin by proving that $E_{(A)} \not\preccurlyeq E(\rho_\gamma)$.

For each finite subset $\Sigma \subseteq \Omega$ let $f_\Sigma \in E$ be the map defined by

$$(11) \qquad (\alpha)f_\Sigma = \begin{cases} \gamma & \text{if } \alpha \in \Sigma \\ \alpha & \text{if } \alpha \notin \Sigma. \end{cases}$$

Now, let $U \subseteq E$ be a finite subset, and consider a monoid word $g = g_0 g_1 \dots g_{i-1} g_i g_{i+1} \dots g_n$ in elements of $E(\rho_\gamma) \cup U$. Suppose that g is fm-to-one and that the element g_i is in $E(\rho_\gamma)$. Let $\Gamma \subseteq \Omega$ be the preimage of γ under g_i. Then $\Sigma := ((\Omega)g_0 \dots g_{i-1}) \cap \Gamma$ must be finite, and so we have $g = g_0 \dots g_{i-1} f_\Sigma g_{i+1} \dots g_n$. In a similar fashion, assuming that g is fm-to-one, we can replace every element of $E(\rho_\gamma)$ occurring in the word g by an element of the form f_Σ, for some finite $\Sigma \subseteq \Omega$. Considering that all elements of $E_{(A)}$ are fm-to-one, we conclude that if $h \in E_{(A)} \cap \langle E(\rho_\gamma) \cup U \rangle$, then $h \in \langle \{f_\Sigma : \Sigma \subseteq \Omega \text{ finite}\} \cup U \rangle$. But, the latter set is countable and hence cannot contain all of $E_{(A)}$. Therefore $E_{(A)} \not\preccurlyeq E(\rho_\gamma)$.

It remains to show that $E(\rho_\gamma) \preccurlyeq E_{(A)}$. Write $A = \{A_i : i \in \Omega\}$, and for each $i \in \Omega$ set $A_i = \{\alpha_{i1}, \alpha_{i2}\}$. Let $g_1 \in E$ be the map defined by $(i)g_1 = \alpha_{i1}$ $(i \in \Omega)$, and let $g_2 \in E$ be defined by $(\alpha_{i1})g_2 = \alpha_{i1}$ and $(\alpha_{i2})g_2 = \gamma$. Then $E(\rho_\gamma) \subseteq g_1 E_{(A)} g_2$. □

Theorem 27. *Let ρ and ρ' be preorders on Ω. Then $E(\rho) \approx E(\rho')$ if and only if ρ and ρ' are of the same type.*

Proof. The above lemmas, in conjunction with Theorem 9, give the desired conclusion if ρ and ρ' are each of type 1, 2, 3a, or 3b. The result then follows from the fact that ρ is of type 4 if and only if $E(\rho)$ is countable. □

8 The function topology

From now on we will be concerned with submonoids that are closed in the function topology on E, so let us recall some facts about this topology.

Regarding the infinite set Ω as a discrete topological space, the monoid $E = \mathrm{Self}(\Omega)$, viewed as the set of all functions from Ω to Ω, becomes a topological space under the function topology. A subbasis of open sets in this topology is given by the sets $\{f \in E : (\alpha)f = \beta\}$ $(\alpha, \beta \in \Omega)$. The closure of a set $U \subseteq E$ consists of all maps f such that, for every finite subset $\Gamma \subseteq \Omega$, there exists an element of U agreeing with f at all members of Γ. It is easy to see that composition of maps is continuous in this topology. Given a subset $U \subseteq E$, we will write $\mathrm{cl}_E(U)$ for the closure of U in E.

The following lemma will be useful later on. It is an analog of [4, Lemma 8], with monoids in place of groups and forward orbits in place of orbits. (Given an element $\alpha \in \Omega$ and a subset $U \subseteq E$, we will refer to the set $(\alpha)U$ as the *forward orbit* of α under U.) The proof is carried over from [4] almost verbatim.

Lemma 28. *Let $M \subseteq E$ be a submonoid. Then*

(i) $\mathrm{cl}_E(M)$ *is also a submonoid of* E.

(ii) M *and* $\mathrm{cl}_E(M)$ *have the same forward orbits in* Ω.

(iii) *If* Γ *is a finite subset of* Ω, *then* $\mathrm{cl}_E(M)_{(\Gamma)} = \mathrm{cl}_E(M_{(\Gamma)})$.

Proof. Statement (i) follows from the fact that composition of maps is continuous.

From the characterization of the closure of a set in our topology, we see that for $\alpha, \beta \in \Omega$, the set $\mathrm{cl}_E(M)$ will contain elements taking α to β if and only if M does, establishing (ii).

Given any subset $\Gamma \subseteq \Omega$, the elements of $\mathrm{cl}_E(M_{(\Gamma)})$ fix Γ elementwise, by (ii). Hence, $\mathrm{cl}_E(M)_{(\Gamma)} \supseteq \mathrm{cl}_E(M_{(\Gamma)})$. To show $\mathrm{cl}_E(M)_{(\Gamma)} \subseteq \mathrm{cl}_E(M_{(\Gamma)})$, assume that $\Gamma \subseteq \Omega$ is finite, and let $f \in \mathrm{cl}_E(M)_{(\Gamma)}$. Then every neighborhood of f contains elements of M, since $f \in \mathrm{cl}_E(M)$. But, since f fixes all points of the finite set Γ, every sufficiently small neighborhood of f consists of elements which do the same. Hence, every such neighborhood contains elements of $M_{(\Gamma)}$, and so $f \in \mathrm{cl}_E(M_{(\Gamma)})$. □

9 Large stabilizers

Let us say that a submonoid M of $E = \mathrm{Self}(\Omega)$ has *large stabilizers* if for each finite subset $\Sigma \subseteq \Omega$, $M_{(\Sigma)} \approx M$. For example, all subgroups of S have large stabilizers, by [4, Lemma 2], as do submonoids of the form $E_{(A)}$, where A is a partition of Ω. More generally, we have

Proposition 29. *Let ρ be a preorder on Ω. Then $E(\rho)$ has large stabilizers.*

Proof. Let $\Sigma \subseteq \Omega$ be finite. Then $E(\rho)_{(\Sigma)}$ is still a submonoid of the form $E(\rho')$, where ρ' is a preorder on Ω of the same type as ρ. The desired conclusion then follows from Theorem 27. □

The following lemma gives another class of submonoids that have large stabilizers. The proof is similar to the one for [4, Lemma 2]

Lemma 30. *Let $G \subseteq \mathrm{Sym}(\Omega) \subseteq E$ be a subgroup. Then $\mathrm{cl}_E(G)$ has large stabilizers.*

Proof. Let $\Sigma \subseteq \Omega$ be finite, and take $f \in \mathrm{cl}_E(G)$. Then there is a sequence of elements of G that has limit f. Eventually, the elements of this sequence must agree on the members of Σ, and hence, must lie in some right coset of $G_{(\Sigma)}$, say the right coset represented by $g \in G$. Then $f \in \mathrm{cl}_E(G_{(\Sigma)})g = \mathrm{cl}_E(G)_{(\Sigma)}g$, by Lemma 28. Letting $R \subseteq G$ be a set of representatives of the right cosets of $G_{(\Sigma)}$, we have $\mathrm{cl}_E(G) \subseteq \langle \mathrm{cl}_E(G)_{(\Sigma)} \cup R\rangle$. Now, R is countable, since Ω is countable and Σ is finite, and hence $\mathrm{cl}_E(G) \approx \mathrm{cl}_E(G)_{(\Sigma)}$, by Theorem 6. □

Not all submonoids of E, however, have large stabilizers. Given a natural number n, the monoid M_n (generated by all maps whose images are contained in $\{0, 1, \ldots, n\}$) mentioned in Section 1 is an easy example of this. (For any finite $\Sigma \subseteq \Omega$ $(= \omega)$ such that $\Sigma \cap \{0, 1, \ldots, n\} = \emptyset$, $(M_n)_{(\Sigma)} = \{1\} \not\approx M_n$.)

Definition 31. *Let us say that a map $f \in E$ is* a.e. injective *if there exists some finite set $\Gamma \subseteq \Omega$ such that f is injective on $\Omega \setminus \Gamma$.*

In much of the sequel we will be concerned with submonoids where the a.e. injective maps form dense subsets (viewing E as topological space under the function topology). A submonoid $M \subseteq E$ has this property if and only if for every finite $\Sigma \subseteq \Omega$ and every $f \in M$, there is an a.e. injective map $g \in M$ that agrees with f on Σ. Clearly, a.e. injective maps form dense subsets in submonoids of E that consist of injective maps, such as subgroups of $\mathrm{Sym}(\Omega)$ and their closures in the function topology. It is easy to see that this is also the case for the submonoids $E(\rho)$, since given any $g \in E(\rho)$ and any finite $\Sigma \subseteq \Omega$, we can find an $f \in E(\rho)$ that agrees with g on Σ and acts as the identity elsewhere. From now on we will abuse language by referring to submonoids in which the a.e. injective maps form dense subsets as submonoids having dense a.e. injective maps.

Our goal will be to classify into equivalence classes submonoids of E that are closed in the function topology, and have large stabilizers and dense a.e. injective maps. By the remarks above, examples of such submonoids include closures in the function topology in E of subgroups of $\mathrm{Sym}(\Omega)$ and submonoids of the form $E(\rho)$. These equivalence classes will be shown to be represented by E, $E_{\leq}$, $E_{(A)}$, $E(\rho_\gamma)$, and $\{1\}$, respectively (where A is a partition of Ω into 2-element sets, and $\gamma \in \Omega$). Most of the work will go into showing that a given monoid from the above list is $\preccurlyeq M$, for a monoid M satisfying an appropriate condition. In the first three cases (and to some extent in the fourth) our arguments will follow a similar pattern. We will construct sequences of the form $g_0, g_0g_1, g_0g_1g_2, \ldots$, and use closure in the function topology to conclude that such sequences have limits in the appropriate monoid M. The subsets consisting of these limits will then satisfy the hypotheses of one of the Lemmas 18 -20, giving us the desired conclusion.

Proposition 32. *Let $M \subseteq E$ be a submonoid that is closed in the function topology and has dense fm-to-one (and hence a.e. injective) maps. If $M_{(\Sigma)}$ has an infinite forward orbit for every finite $\Sigma \subseteq \Omega$, then $M \approx E$.*

Proof. This proof closely follows that of [4, Theorem 11]. We begin by recursively constructing for each $j \geq 0$ an element $\alpha_j \in \Omega$ and a finite subset $K_j \subseteq M$, consisting of fm-to-one maps ("fm-to-one" is defined directly preceding Lemma 26), indexed

$$K_j = \{g(k_0, k_1, \ldots, k_{r-1}) : k_0, k_1, \ldots, k_{r-1}, r \in \omega,\ r + k_0 + \ldots + k_{r-1} = j\}. \tag{12}$$

Let α_0 be any element that has an infinite forward orbit under M. Assume inductively that $\alpha_0, \ldots, \alpha_{j-1}$ have been defined, and let $\Gamma_j = \{\alpha_0, \ldots, \alpha_{j-1}\} \cup \{0, \ldots, j-1\}$. We then take $\alpha_j \in \Omega$ to be any element that has an infinite forward orbit under $M_{(\Gamma_j)}$.

Now we construct the sets K_j. If $j = 0$, we have only one element to choose, $g()$, and we take this to be the identity element $1 \in M$. Assume inductively that the sets K_i have been defined for all nonnegative $i < j$. Let us fix arbitrarily an order in which the elements of K_j are to be constructed. When it is time to define $g(k_0, k_1, \ldots, k_{r-1})$, let us write $g' = g(k_0, k_1, \ldots, k_{r-2})$, noting that this is a member of $K_{j-k_{r-1}-1}$ and hence already defined (and fm-to-one). We set $g(k_0, k_1, \ldots, k_{r-1}) = hg'$, where $h \in M_{(\Gamma_{r-1})}$ is chosen so that the image of α_{r-1} under hg' is distinct from the images of $\alpha_0, \ldots, \alpha_{r-2}$ under the finitely many elements of $K_0 \cup \ldots \cup K_{j-1}$, and also under the elements of K_j that have been constructed so far. This is possible, since $(\alpha_{r-1})M_{(\Gamma_{r-1})}$ is infinite, by the choice of α_{r-1}, and $(\alpha_{r-1})M_{(\Gamma_{r-1})}g'$ is infinite, since g' is fm-to-one. Further, h can be chosen to be fm-to-one, by our hypothesis on M, making $g(k_0, k_1, \ldots, k_{r-1})$ fm-to-one as well. We note that the images of $\alpha_0, \ldots, \alpha_{r-2}$ and $0, \ldots, r-2$ under hg' will be the same as their images under g', since elements of $M_{(\Gamma_{r-1})}$ fix $\{\alpha_0, \ldots, \alpha_{r-2}\} \cup \{0, \ldots, r-2\} = \Gamma_{r-1}$.

Once the elements of each set K_j are constructed, we have monoid elements $g(k_0, \ldots, k_{i-1})$ for all $i,\ k_0, \ldots, k_{i-1} \in \omega$. We can thus define, for each $i \in \omega$,

$$D_i = \{((\alpha_0)g, \ldots, (\alpha_{i-1})g) : g = g(k_0, \ldots, k_{i-1}) \text{ for some } k_0, \ldots, k_{i-1} \in \omega\}. \tag{13}$$

Any two elements of the form $g(k_0, \ldots, k_i)$ with indices $k_0, \ldots, k_{i-1}$ the same, but different last indices k_i, act differently on α_i, so the sets D_i satisfy condition (i) of Lemma 18. Suppose that $(\beta_i) \in \Omega^\omega$ has the property that for every i the sequence $(\beta_0, \ldots, \beta_{i-1})$ is in D_i. By construction, the elements β_i are all distinct. Also, we see inductively that successive strings $(), (\beta_0), \ldots, (\beta_0, \ldots, \beta_{i-1}), \ldots$ must arise from unique elements of the forms $g()$, $g(k_0)$, $\ldots$, $g(k_0, \ldots, k_{i-1})$, $\ldots$. By including $\{0, \ldots, r-1\}$ in Γ_{r-1} above, we have ensured that the elements of this sequence agree on larger and larger subsets of $\omega = \Omega$, which have union Ω. Thus, this sequence converges to a map $g \in E$, which necessarily sends (α_i) to (β_i). Since M is closed, $g \in M$; which establishes condition (ii) of Lemma 18. Hence, that lemma tells us that $M \approx E$. □

In the above proposition, the hypothesis that $M_{(\Sigma)}$ has an infinite forward orbit for every finite $\Sigma \subseteq \Omega$ is not necessary for $M \approx E$. For example, let $E_> \subseteq E$ denote the submonoid generated by maps that are strictly increasing with respect to the usual ordering of $\omega = \Omega$ (i.e., maps $f \in E$ such that for all $\alpha \in \Omega$, $(\alpha)f > \alpha$). This submonoid is closed in the function topology, and its a.e. injective maps form a dense subset. It is also easy to see that the submonoid $E_>$ satisfies the hypotheses of Lemma 18, and hence is $\approx E$, but given any finite $\Sigma \subseteq \Omega$, $(E_>)_{(\Sigma)} = \{1\}$. However, for submonoids $M \subseteq E$ that have large stabilizers, the condition that $M_{(\Sigma)}$ has an infinite forward orbit for every finite $\Sigma \subseteq \Omega$ is necessary for $M \approx E$.

Corollary 33. *Let $M \subseteq E$ be a submonoid that is closed in the function topology, and has large stabilizers and dense fm-to-one maps. Then $M \approx E$ if and only if $M_{(\Sigma)}$ has an infinite forward orbit for every finite $\Sigma \subseteq \Omega$.*

Proof. If there is a finite set $\Sigma \subseteq \Omega$ such that $M_{(\Sigma)}$ has no infinite forward orbits, then $M_{(\Sigma)} \prec E$, by Theorem 9(i). Since M has large stabilizers, this implies that $M \prec E$. The converse follows from the previous proposition. □

Let us now turn to submonoids whose stabilizers have finite forward orbits. We will say that a fm-to-one map $f \in E$ is *boundedly finitely-many-to-one* (abbreviated *bfm-to-one*) if there is a common finite upper bound on the cardinalities of the preimages of elements of Ω under f.

Proposition 34. *Let $M \subseteq E$ be a submonoid that is closed in the function topology and has dense bfm-to-one (and hence a.e. injective) maps. If for every finite $\Sigma \subseteq \Omega$ the cardinalities of the forward orbits of $M_{(\Sigma)}$ have no common finite bound, then $E_\leq \preccurlyeq M$.*

Proof. Again, this proof closely follows that of [4, Theorem 13]. Let us fix an unbounded sequence of positive integers $(N_i)_{i\in\omega}$. We begin by recursively constructing for each $j \geq 0$ an element $\alpha_j \in \Omega$ and a finite subset $K_j \subseteq M$, consisting of bfm-to-one maps, indexed

(14) $$K_j = \{g(k_0, k_1, \ldots, k_{j-1}) : 0 \leq k_i < N_i \ (0 \leq i < j)\}.$$

Let the 1-element set $K_0 = \{g()\}$ consist of the identity map $1 \in M$. Assume inductively that for some $j \geq 0$ the elements $\alpha_0, \ldots, \alpha_{j-1}$ and the sets $K_0, \ldots, K_j$ have been defined, and let $\Gamma_j = \{\alpha_0, \ldots, \alpha_{j-1}\} \cup \{0, \ldots, j-1\}$. Now, let us choose $\alpha_j \in \Omega$ so that for each $g = g(k_0, k_1, \ldots, k_{j-1}) \in K_j$, the forward orbit of α_j under $M_{(\Gamma_j)}g$ has cardinality at least

$$(15) \qquad (j+1)\cdot(\sum_{i=0}^{j+1}|K_i|) = (j+1)\cdot(\sum_{i=0}^{j}|K_i| + N_0N_1\dots N_j).$$

This is possible, since each such g is bfm-to-one, and since K_j is finite.

Next, let us fix arbitrarily an order in which the elements of K_{j+1} are to be constructed. When it is time to construct $g(k_0, k_1, \dots, k_j)$, let us write $g' = g(k_0, k_1, \dots, k_{j-1}) \in K_j$. We define $g(k_0, k_1, \dots, k_j) = hg'$, where $h \in M_{(\Gamma_j)}$ is chosen so that the image of α_j under hg' is distinct from the images of $\alpha_0, \dots, \alpha_{j-1}$ under the elements of $K_0 \cup \dots \cup K_j$ and also under the elements of K_{j+1} that have been constructed so far. Our choice of α_j makes this possible. Further, h can be chosen to be bfm-to-one, by our hypothesis on M, making $g(k_0, k_1, \dots, k_j)$ bfm-to-one as well. We note that the images of $\alpha_0, \dots, \alpha_{j-1}$ and $0, \dots, j-1$ under hg' will be the same as their images under g', since elements of $M_{(\Gamma_j)}$ fix $\{\alpha_0, \dots, \alpha_{j-1}\} \cup \{0, \dots, j-1\} = \Gamma_j$.

Once the sets K_j are constructed, for each $i \in \omega$, let

$$(16) \qquad D_i = \{((\alpha_0)g, (\alpha_1)g, \dots, (\alpha_{i-1})g) : g \in K_i\}.$$

Any two elements of the form $g(k_0, \dots, k_i)$ with indices $k_0, \dots, k_{i-1}$ the same but different last indices k_i act differently on α_i, so the sets D_i satisfy condition (i) of Lemma 19. Suppose that $(\beta_i) \in \Omega^\omega$ has the property that for every i the sequence $(\beta_0, \dots, \beta_{i-1})$ is in D_i. By construction, the elements β_i are all distinct. Also, we see inductively that successive strings $(), (\beta_0), \dots, (\beta_0, \dots, \beta_{i-1}), \dots$ must arise from unique elements of the forms $g()$, $g(k_0)$, $\dots$, $g(k_0, \dots, k_{i-1})$, $\dots$. The elements of the above sequence agree on the successive sets $\{0, \dots, j-1\}$ (having union $\omega = \Omega$), and hence the sequence converges to a map $g \in E$, which must send (α_i) to (β_i). Since M is closed, we have $g \in M$, establishing condition (ii) of Lemma 19. Hence, that lemma tells us that $E_{\leq} \preccurlyeq M$. □

As with Proposition 32, the hypothesis that for every finite $\Sigma \subseteq \Omega$ the cardinalities of the forward orbits of $M_{(\Sigma)}$ have no common finite bound is not necessary for $E_{\leq} \preccurlyeq M$. For instance, the submonoid $E_{<} \subseteq E$ generated by maps that are strictly decreasing with respect to the usual ordering of $\omega = \Omega$ (i.e., maps $f \in E$ such that for all $\alpha \in \Omega \setminus \{0\}$, $(\alpha)f < \alpha$, and $(0)f = 0$) is closed in the function topology, has dense a.e. injective maps, and satisfies the hypotheses of Lemma 19. Therefore $E_{\leq} \preccurlyeq E_{<}$, but given any finite $\Sigma \subseteq \Omega$, $(E_{<})_{(\Sigma)} = \{1\}$.

Corollary 35. *Let $M \subseteq E$ be a submonoid that is closed in the function topology, and has large stabilizers and dense bfm-to-one maps. Then $M \approx E_{\leq}$ if and only if for every finite $\Sigma \subseteq \Omega$ the cardinalities of the forward orbits of $M_{(\Sigma)}$ have no common finite bound, and there exists a finite set $\Sigma \subseteq \Omega$ such that all forward orbits of $M_{(\Sigma)}$ are finite.*

Proof. If $M \approx E_{\leq}$, then for every finite $\Sigma \subseteq \Omega$, the cardinalities of the forward orbits of $M_{(\Sigma)}$ have no common finite bound. For, otherwise, there would be some finite $\Sigma \subseteq \Omega$ for which $M_{(\Sigma)}$ would satisfy the hypotheses of Theorem 9(ii), implying that $M \approx M_{(\Sigma)} \prec E_{\leq}$. Also, there must be a finite set $\Sigma \subseteq \Omega$ such that all forward orbits of $M_{(\Sigma)}$ are finite, since otherwise we would have $M \approx E$, by Proposition 32, contradicting Theorem 9(i).

Conversely, if for every finite $\Sigma \subseteq \Omega$, the cardinalities of the forward orbits of $M_{(\Sigma)}$ have no common finite bound, then $E_{\leq} \preccurlyeq M$, by the previous proposition. Suppose that, in addition, there exists a finite set $\Sigma \subseteq \Omega$ such that all forward orbits of $M_{(\Sigma)}$ are finite.

Then $M_{(\Sigma)} \preccurlyeq E_{(A)}$, where A is a partition of Ω into finite sets, by Lemma 17. By the large stabilizer hypothesis and Lemma 14, we have $M \approx M_{(\Sigma)} \preccurlyeq E_{(A)} \approx E_{\leq}$. Hence $M \approx E_{\leq}$. $\square$

The rest of the section is devoted to the more intricate case of submonoids with stabilizers whose forward orbits have a common finite bound.

Lemma 36. *Let $M \subseteq E$ be a submonoid that has large stabilizers, and let A be a partition of Ω into 2-element sets. If there exists a finite set $\Sigma \subseteq \Omega$ and a positive integer n such that the cardinalities of all forward orbits of $M_{(\Sigma)}$ are bounded by n, then $M \preccurlyeq E_{(A)}$.*

Proof. Given a finite set $\Sigma \subseteq \Omega$ and a positive integer n as above, $M_{(\Sigma)} \preccurlyeq E_{(B)}$, where B is a partition of Ω into sets of cardinality n, by Lemma 17. But, $E_{(A)} \approx E_{(B)}$, by Proposition 13 and [4, Theorem 15]. Hence $M \approx M_{(\Sigma)} \preccurlyeq E_{(A)}$. $\square$

The following proof is based on that of [4, Theorem 15], but it is more complicated.

Lemma 37. *Let $M \subseteq E$ be a submonoid that is closed in the function topology and has dense a.e. injective maps. If for all finite $\Gamma, \Delta \subseteq \Omega$ there exists $\alpha \in \Omega$ such that $(\alpha)M_{(\Delta)} \not\subseteq \Gamma \cup \{\alpha\}$, then $E_{(A)} \preccurlyeq M$ (where A is a partition of Ω into 2-element sets).*

Proof. We may assume that there exist finite sets $\Gamma, \Delta \subseteq \Omega$ and a positive integer n such that for all $\alpha \in \Omega$, $|(\alpha)M_{(\Delta)} \setminus \Gamma| \leq n$. For, otherwise $E_{\leq} \preccurlyeq M$, by Proposition 34, and $E_{(A)} \prec E_{\leq}$, by Theorem 9(ii).

Let $m > 1$ be the largest integer such that for all finite $\Gamma, \Delta \subseteq \Omega$, there exists $\alpha \in \Omega$ such that $|(\alpha)M_{(\Delta)} \setminus \Gamma| \geq m$. Then there exist finite sets $\Gamma', \Delta' \subseteq \Omega$ such that for all $\alpha \in \Omega$, $|(\alpha)M_{(\Delta')} \setminus \Gamma'| \leq m$. Now, $M' = M_{(\Delta')}$ has the property that for all finite $\Gamma, \Delta \subseteq \Omega$ there exists $\alpha \in \Omega$ such that $(\alpha)M'_{(\Delta)} \not\subseteq \Gamma \cup \{\alpha\}$, and for all finite $\Gamma, \Delta \subseteq \Omega$, with $\Gamma' \subseteq \Gamma$, the maximum of the cardinalities of the sets $(\alpha)M'_{(\Delta)} \setminus \Gamma$ ($\alpha \in \Omega$) is m. From now on we will be working with M' in place of M.

A consequence of the above considerations is that if for some $\alpha \in \Omega$ and finite $\Delta, \Gamma \subseteq \Omega$, with $\Gamma' \subseteq \Gamma$, we have $|(\alpha)M'_{(\Delta)} \setminus \Gamma| = m$, then $(\alpha)M'_{(\Delta)} \setminus \Gamma = (\alpha)M' \setminus \Gamma'$, since $(\alpha)M' \setminus \Gamma'$ cannot have cardinality larger than m. Thus

(17) If $\Gamma, \Delta \subseteq \Omega$ are finite subsets, with $\Gamma' \subseteq \Gamma$, and $\alpha \in \Omega$ has the property that $|(\alpha)M'_{(\Delta)} \setminus \Gamma| = m$, then for every $g \in M'$ that embeds $(\alpha)M'_{(\Delta)} \setminus \Gamma$ in $\Omega \setminus \Gamma'$, we have $((\alpha)M'_{(\Delta)} \setminus \Gamma)g = (\alpha)M'_{(\Delta)} \setminus \Gamma$.

(This is because for such an element g, $((\alpha)M'_{(\Delta)} \setminus \Gamma)g \subseteq (\alpha)M' \setminus \Gamma'$, and the latter set is equal to $(\alpha)M'_{(\Delta)} \setminus \Gamma$.)

We now construct recursively, for each $j \geq 0$, elements $\alpha_j, \beta_j, \gamma_j \in \Omega$ and a finite subset $K_j \subseteq M'$, consisting of a.e. injective maps, indexed

(18) $K_j = \{g(k_0, k_1, \ldots, k_{j-1}) : (k_0, k_1, \ldots, k_{j-1}) \in \{0,1\}^j\}.$

Let the 1-element set $K_0 = \{g()\}$ consist of the identity map $1 \in M'$. Assume inductively that for some $j \geq 0$ the elements $\alpha_0, \ldots, \alpha_{j-1}, \beta_0, \ldots, \beta_{j-1}, \gamma_0, \ldots, \gamma_{j-1}$ and the sets $K_0, \ldots, K_j$ have been defined. Let $\Delta_j = \{\alpha_0, \ldots, \alpha_{j-1}\} \cup \{1, \ldots, j-1\}$, and let $\Gamma_j \subseteq \Omega$ be a finite set, containing $\Gamma' \cup \{\beta_0, \ldots, \beta_{j-1}, \gamma_0, \ldots, \gamma_{j-1}\}$, such that all elements of $K_0 \cup \cdots \cup K_j$

embed $\Omega \setminus \Gamma_j$ in $\Omega \setminus \Gamma'$. (Since $K_0 \cup \cdots \cup K_j$ is finite and consists of elements that are a.e. injective, we can find a finite set Γ_j, containing $\Gamma' \cup \{\beta_0, \ldots, \beta_{j-1}, \gamma_0, \ldots, \gamma_{j-1}\}$, such that all elements of $K_0 \cup \cdots \cup K_j$ are injective on $\Omega \setminus \Gamma_j$. We can then enlarge this Γ_j to include the finitely many elements that are mapped to Γ' by $K_0 \cup \cdots \cup K_j$.) Now, let us choose $\alpha_j \in \Omega$ so that $|(\alpha_j)M'_{(\Delta_j)} \setminus \Gamma_j| = m$, and let $\beta_j, \gamma_j \in (\alpha_j)M'_{(\Delta_j)} \setminus \Gamma_j$ be two distinct elements. We note that β_j and γ_j are distinct from $\beta_0, \ldots, \beta_{j-1}, \gamma_0, \ldots, \gamma_{j-1}$, since the latter are elements of Γ_j.

Next, let us construct the elements $g(k_0, \ldots, k_{j-1}, 0), g(k_0, \ldots, k_{j-1}, 1) \in K_{j+1}$. Writing $g' = g(k_0, k_1, \ldots, k_{j-1}) \in K_j$, we define $g(k_0, \ldots, k_{j-1}, 0) = hg'$ and $g(k_0, \ldots, k_{j-1}, 1) = fg'$, where $h, f \in M'_{(\Delta_j)}$ are chosen so that $(\alpha_j)hg' = \beta_j$ and $(\alpha_j)fg' = \gamma_j$. It is possible to find such h and f, by (17), using the fact that g' embeds $\Omega \setminus \Gamma_j$ in $\Omega \setminus \Gamma'$. By the hypothesis that a.e. injective maps form a dense subset in M (and hence in M'), we may further assume that f and g are a.e. injective. As before, the images of $\alpha_0, \ldots, \alpha_{j-1}$ and $1, \ldots, j-1$ under hg' and fg' will be the same as their images under g', and these two maps will be a.e. injective (as composites of a.e. injective maps).

Given an infinite string $(k_i)_{i\in\omega} \in \{0,1\}^\omega$, the elements of the sequence $g()$, $g(k_0)$, $\ldots$, $g(k_0, \ldots, k_{i-1})$, $\ldots$ agree on successive sets $\{1, \ldots, j-1\}$, and hence the sequence converges in E. Since M is closed, the limit belongs to M (in fact, M'). Considering our definition of the elements α_i, β_i, γ_i, the submonoid M satisfies the hypotheses of Lemma 20, and hence $E_{(A)} \preccurlyeq M$. □

In the previous lemma we were concerned with monoids under which elements of Ω, for the most part, had disjoint forward orbits. Next, we give a similar argument, but adjusted for monoids under which forward orbits can fall together.

Lemma 38. *Let $M \subseteq E$ be a submonoid that is closed in the function topology and has dense a.e. injective maps. If for every finite $\Sigma \subseteq \Omega$, $M_{(\Sigma)} \neq \{1\}$, then $E(\rho_\gamma) \preccurlyeq M$ for some $\gamma \in \Omega$.*

Proof. We may assume that there exist finite sets $\Gamma, \Delta \subseteq \Omega$ such that for all $\alpha \in \Omega$, $(\alpha)M_{(\Delta)} \subseteq \Gamma \cup \{\alpha\}$. For, otherwise, the previous lemma implies that $E_{(A)} \preccurlyeq M$, where A is a partition of Ω into 2-element sets (and $E(\rho_\gamma) \preccurlyeq E_{(A)}$ for all $\gamma \in \Omega$, by Lemmas 23, 25, and 26). Now, for every finite $\Sigma \subseteq \Omega$, we have $(M_{(\Delta)})_{(\Sigma)} = M_{(\Delta\cup\Sigma)} \neq \{1\}$. Also, $M_{(\Delta)}$ is closed in the function topology, and a.e. injective maps form a dense subset in $M_{(\Delta)}$. Hence, we may replace M with $M_{(\Delta)}$ and thus assume that

(19) For every $\alpha \in \Omega$, $(\alpha)M \subseteq \Gamma \cup \{\alpha\}$.

We now construct recursively for each $j \geq 0$ an element $\alpha_j \in \Omega$ and a finite subset $K_j \subseteq M$, consisting of a.e. injective maps, indexed

(20) $K_j = \{g(k_0, k_1, \ldots, k_{j-1}) : (k_0, k_1, \ldots, k_{j-1}) \in \{0,1\}^j\}$.

Let the 1-element set $K_0 = \{g()\}$ consist of the identity map $1 \in M$. Assume inductively that for some $j \geq 0$ the elements $\alpha_0, \ldots, \alpha_{j-1}$ and the sets $K_0, \ldots, K_j$ have been defined. Since these sets are finite and consist of a.e. injective maps, there is a finite set $\Delta_j \subseteq \Omega$, such that all members of $K_0 \cup \cdots \cup K_j$ are injective on $\Omega \setminus \Delta_j$. We note that every $h \in K_0 \cup \cdots \cup K_j$, since it is injective on $\Omega \setminus \Delta_j$, must act as the identity on all but finitely many elements of $\Omega \setminus \Delta_j$, by (19). Let $\Gamma_j \subseteq \Omega$ be the finite set consisting of $\Delta_j \cup \Gamma \cup \{\alpha_0, \ldots, \alpha_{j-1}\} \cup \{1, \ldots, j-1\}$ and those elements of $\Omega \setminus \Delta_j$ that are taken to Γ by members of $K_0 \cup \cdots \cup K_j$. We then take $\alpha_j \in \Omega \setminus \Gamma_j$ to be any element such that $|(\alpha_j)M_{(\Gamma_j)}| > 1$.

Next, let us construct the elements $g(k_0, \ldots, k_{j-1}, 0), g(k_0, \ldots, k_{j-1}, 1) \in K_{j+1}$. Writing $g' = g(k_0, k_1, \ldots, k_{j-1}) \in K_j$, we define $g(k_0, \ldots, k_{j-1}, 0) = g'$, noting that $(\alpha_j)g' = \alpha_j$, by definition of Γ_j. Also, let us define $g(k_0, \ldots, k_{j-1}, 1) = fg'$, where $f \in M_{(\Gamma_j)}$ is chosen so that $(\alpha_j)fg' \in \Gamma$ and f is a.e. injective. (It is possible to find an a.e. injective $f \in M_{(\Gamma_j)}$ such that $(\alpha_j)f \in \Gamma$, by our hypotheses that $|(\alpha_j)M_{(\Gamma_j)}| > 1$, that for every $\alpha \in \Omega$, $(\alpha)M \subseteq \Gamma \cup \{\alpha\}$, and that a.e. maps form a dense subset in M. Then $(\alpha_j)fg' \in \Gamma$, by (19).) We note that the images of $\alpha_0, \ldots, \alpha_{j-1}$ and $1, \ldots, j-1$ under fg' will be the same as their images under g', and that fg' will be a.e. injective.

Given an infinite string $(k_i)_{i \in \omega} \in \{0, 1\}^\omega$, the elements of the sequence $g()$, $g(k_0)$, $\ldots$, $g(k_0, \ldots, k_{i-1})$, $\ldots$ agree on successive sets $\{1, \ldots, j-1\}$, and hence the sequence converges in E. Since M is closed, the limit belongs to M. Now, let us fix some $\gamma \in \Gamma$. Also, let $h_1 \in E$ be a map that takes Ω bijectively to $\{\alpha_i\}_{i \in \omega}$, let $h_2 \in E$ be the map that takes Γ to γ and fixes all other elements of Ω, and let $h_3 \in E$ be a right inverse of h_1 that fixes γ. Then $E(\rho_\gamma) \subseteq h_1 M h_2 h_3$. □

Lemma 39. *Let $M \subseteq E$ be a submonoid that has large stabilizers. Assume that there exist finite sets $\Gamma, \Delta \subseteq \Omega$ such that for all $\alpha \in \Omega$, $(\alpha)M_{(\Delta)} \subseteq \Gamma \cup \{\alpha\}$. Then $M \preccurlyeq E(\rho_\gamma)$ (where $\gamma \in \Omega$ is any fixed element).*

Proof. Since M has large stabilizers, we may replace M with $M_{(\Delta)}$ and assume that for all $\alpha \in \Omega$, $(\alpha)M \subseteq \Gamma \cup \{\alpha\}$. Let us define a preorder ρ on Ω by $(\alpha, \beta) \in \rho$ if and only if $\beta \in (\alpha)M$. Then $M \subseteq E(\rho)$, and ρ is of type 3b. Hence, by Lemma 25, $M \preccurlyeq E(\rho_\gamma)$. □

The following proposition summarizes the previous four lemmas.

Proposition 40. *Let $M \subseteq E$ be a submonoid that is closed in the function topology, and has large stabilizers and dense a.e. injective maps. Assume that there exists a finite set $\Sigma \subseteq \Omega$ and a positive integer n such that the cardinalities of all forward orbits of $M_{(\Sigma)}$ are bounded by n, but for every finite $\Sigma \subseteq \Omega$, $M_{(\Sigma)} \neq \{1\}$. Let A be a partition of Ω into 2-element sets and $\gamma \in \Omega$. Then*

(i) *$M \approx E_{(A)}$ if and only if for all finite $\Gamma, \Delta \subseteq \Omega$ there exists $\alpha \in \Omega$ such that $(\alpha)M_{(\Delta)} \nsubseteq \Gamma \cup \{\alpha\}$, and*

(ii) *$M \approx E(\rho_\gamma)$ if and only if there exist finite sets $\Gamma, \Delta \subseteq \Omega$ such that for all $\alpha \in \Omega$, $(\alpha)M_{(\Delta)} \subseteq \Gamma \cup \{\alpha\}$.*

10 Main results

Combining the results of the previous section, we obtain our desired classification.

Theorem 41. *Let $M \subseteq \mathrm{Self}(\Omega)$ be a submonoid that is closed in the function topology, and has large stabilizers and dense a.e. injective maps. Then M falls into exactly one of five possible equivalence classes with respect to $\approx$, depending on which of the following conditions M satisfies:*

(i) *$M_{(\Sigma)}$ has an infinite forward orbit for every finite $\Sigma \subseteq \Omega$.*

(ii) *There exists a finite set $\Sigma \subseteq \Omega$ such that all forward orbits of $M_{(\Sigma)}$ are finite, but for every finite $\Sigma \subseteq \Omega$ the cardinalities of the forward orbits of $M_{(\Sigma)}$ have no common finite bound.*

(iii) *There exists a finite set $\Sigma \subseteq \Omega$ and a positive integer n such that the cardinalities of all forward orbits of $M_{(\Sigma)}$ are bounded by n, but for all finite $\Gamma, \Delta \subseteq \Omega$ there exists $\alpha \in \Omega$ such that $(\alpha)M_{(\Delta)} \not\subseteq \Gamma \cup \{\alpha\}$.*

(iv) *There exist finite sets $\Gamma, \Delta \subseteq \Omega$ such that for all $\alpha \in \Omega$, $(\alpha)M_{(\Delta)} \subseteq \Gamma \cup \{\alpha\}$, but for every finite $\Sigma \subseteq \Omega$, $M_{(\Sigma)} \neq \{1\}$.*

(v) *There exists a finite set $\Sigma \subseteq \Omega$ such that $M_{(\Sigma)} = \{1\}$.*

Proof. Every submonoid of $E = \mathrm{Self}(\Omega)$ clearly satisfies exactly one of the above five conditions. By Corollary 33, Corollary 35, Proposition 40, and Theorem 9, the first four of these conditions describe disjoint $\approx$-equivalence classes, when considering submonoids that are closed in the function topology, and have large stabilizers and dense a.e. injective maps.

Now, let $M, M' \subseteq E$ be submonoids that satisfy (v). Then we can find finite sets $\Sigma, \Gamma \subseteq \Omega$ such that $M_{(\Sigma)} = \{1\}$ and $M'_{(\Gamma)} = \{1\}$. If M and M' have large stabilizers, this implies that $M \approx \{1\} \approx M'$. Finally, for submonoids of E that satisfy the hypotheses of the theorem, those that are described by condition (v) are countable (since they are $\approx \{1\}$) and hence are $\prec$ those submonoids that satisfy one of the other four conditions. □

One can recover Theorem 27 from this by noting that when restricting to submonoids of the form $E(\rho)$, the above five cases exactly correspond to the five types of preorders on Ω identified in Definition 22.

If we instead restrict our attention to submonoids $M \subseteq E$ consisting of injective maps, then case (iv) of the above theorem cannot occur, and case (iii) can be stated more simply.

Corollary 42. *Let $M \subseteq \mathrm{Self}(\Omega)$ be a submonoid that has large stabilizers, is closed in the function topology, and consists of injective maps. Then M falls into exactly one of four possible equivalence classes with respect to $\approx$, depending on which of the following conditions M satisfies:*

(i) *$M_{(\Sigma)}$ has an infinite forward orbit for every finite $\Sigma \subseteq \Omega$.*

(ii) *There exists a finite set $\Sigma \subseteq \Omega$ such that all forward orbits of $M_{(\Sigma)}$ are finite, but for every finite $\Sigma \subseteq \Omega$ the cardinalities of the forward orbits of $M_{(\Sigma)}$ have no common finite bound.*

(iii) *There exists a finite set $\Sigma \subseteq \Omega$ and a positive integer n such that the cardinalities of all forward orbits of $M_{(\Sigma)}$ are bounded by n, but for every finite $\Sigma \subseteq \Omega$, $M_{(\Sigma)} \neq \{1\}$.*

(iv) *There exists a finite set* $\Sigma \subseteq \Omega$ *such that* $M_{(\Sigma)} = \{1\}$.

Let $E^{\text{inj}} \subseteq E$ be the submonoid consisting of injective maps, and write $\approx_{\text{inj}}$ to denote $\approx_{\aleph_0, E^{\text{inj}}}$. One might wonder whether the above four $\approx$-classes are also $\approx_{\text{inj}}$-classes. They are not. For example, let $E_{\geq} \subseteq E$ denote the submonoid of maps that are increasing with respect to the usual ordering of $\omega = \Omega$ (i.e., maps $f \in E$ such that for all $\alpha \in \Omega$, $(\alpha)f \geq \alpha$), and set $E_{\geq}^{\text{inj}} = E^{\text{inj}} \cap E_{\geq}$. Then E^{inj} and $E_{\geq}^{\text{inj}}$ are both closed in the function topology (in E), since limits of sequences of injective (respectively, increasing) maps are injective (respectively, increasing). Also, these two submonoids have large stabilizers. For, let $\Sigma \subseteq \Omega$ be any finite set. Upon enlarging Σ, if necessary, we may assume that $\Sigma = \{0, 1, \dots, n-1\}$ for some $n \in \omega$. Let $f \in E$ be defined by $(i)f = i + n$ for all $i \in \omega$, and let $g \in E$ be defined by $(i)g = i - n$ for $i \geq n$ (and arbitrarily on Σ). Then $E^{\text{inj}} \subseteq f E_{(\Sigma)}^{\text{inj}} g$ and $E_{\geq}^{\text{inj}} \subseteq f (E_{\geq}^{\text{inj}})_{(\Sigma)} g$. (For, given $h \in E^{\text{inj}}$, respectively $E_{\geq}^{\text{inj}}$, we can find $\bar{h} \in E_{(\Sigma)}^{\text{inj}}$, respectively $(E_{\geq}^{\text{inj}})_{(\Sigma)}$, such that $(i+n)\bar{h} = (i)h + n$ for all $i \in \omega$. Then $h = f\bar{h}g$.) This shows that E^{inj} and $E_{\geq}^{\text{inj}}$ satisfy the hypotheses of the above corollary. They both clearly satisfy condition (i), and hence $E^{\text{inj}} \approx E_{\geq}^{\text{inj}}$. On the other hand, $E^{\text{inj}} \not\approx_{\text{inj}} E_{\geq}^{\text{inj}}$. For, suppose that $E^{\text{inj}} = \langle E_{\geq}^{\text{inj}} \cup U \rangle$ for some finite $U \subseteq E^{\text{inj}}$. Then every element of E^{inj} can be written as a word $f_0 f_1 \dots f_m$ in elements of $E_{\geq}^{\text{inj}} \cup U$. Let $f = f_0 f_1 \dots f_{i-1} f_i f_{i+1} \dots f_m$ be such a word, and suppose that $f_i \in E_{\geq}^{\text{inj}} \setminus \{1\}$. As an increasing injective non-identity element, f_i is necessarily not surjective (e.g., if $j \in \omega$ is the least element that is not fixed by f_i, then $j \notin (\Omega)f_i$, since only elements that are $\leq j$ can be mapped to j by f_i). Since $f_{i+1} \dots f_m$ is injective, this implies that f cannot be surjective either. Thus, if $f \in \langle E_{\geq}^{\text{inj}} \cup U \rangle$ is surjective, then $f \in \langle U \rangle$, which is absurd, since U is finite.

We can further show that E^{inj} and $E_{\geq}^{\text{inj}}$ satisfy the analog of the large stabilizer condition in E^{inj} (i.e., for each finite subset $\Sigma \subseteq \Omega$, $E_{(\Sigma)}^{\text{inj}} \approx_{\text{inj}} E^{\text{inj}}$ and $(E_{\geq}^{\text{inj}})_{(\Sigma)} \approx_{\text{inj}} E_{\geq}^{\text{inj}}$), demonstrating that the obvious analog of the previous corollary for E^{inj} in place of E is not true. Let $\Sigma \subseteq \Omega$ be a finite subset. For each injective map $f : \Sigma \to \Omega$, let us pick a permutation $g_f \in E^{\text{inj}}$ that agrees with f on Σ, and let U be the (countable) set consisting of these g_f. Now, let $f \in E^{\text{inj}}$ be any map. Then $f g_f^{-1} \in E_{(\Sigma)}^{\text{inj}}$, which implies that $f \in \langle E_{(\Sigma)}^{\text{inj}} \cup U \rangle$. Now, as a countable set of permutations, U can be embedded in a subgroup of $\text{Sym}(\Omega)$ generated by two permutations, by [6, Theorem 5.7], and hence in a submonoid of $\text{Sym}(\Omega) \subseteq E^{\text{inj}}$ generated by four elements. Therefore $E^{\text{inj}} \approx_{\text{inj}} E_{(\Sigma)}^{\text{inj}}$. To show that $E_{\geq}^{\text{inj}} \approx_{\text{inj}} (E_{\geq}^{\text{inj}})_{(\Sigma)}$, we will employ a similar method, though now we will assume, for simplicity, that $\Sigma = \{0, 1, \dots, n-1\}$ for some $n \in \Omega$, and require a more specific definition of the elements g_f. For each injective $f : \Sigma \to \Omega$, let us define a map $g_f \in E^{\text{inj}}$ to agree with f on Σ, fix all elements not in $\Sigma \cup (\Sigma)f$, and act on $(\Sigma)f$ in any way that turns g_f into a permutation of Ω. As before, let U be the set consisting of these elements g_f. Now, let $f \in E_{\geq}^{\text{inj}}$ be any map, and let $h \in (E_{\geq}^{\text{inj}})_{(\Sigma)}$ be such that h agrees with f on $\Omega \setminus \Sigma$. We note that for any $\alpha \in \Omega \setminus \Sigma$, $(\alpha)h \notin \Sigma \cup (\Sigma)f$, since f is injective and increasing. Hence $f = h g_f$, and therefore $f \in \langle (E_{\geq}^{\text{inj}})_{(\Sigma)} \cup U \rangle$. As before, this implies that $E_{\geq}^{\text{inj}} \approx_{\text{inj}} (E_{\geq}^{\text{inj}})_{(\Sigma)}$.

11 Groups

We recall that the group $S = \mathrm{Sym}(\Omega)$ inherits from $E = \mathrm{Self}(\Omega)$ the function topology but is not closed in E in this topology. For instance, using cycle notation for permutations of $\Omega = \omega$, we see that the sequence $(0,1), (0,1,2), \dots, (0,\dots,n), \dots$ converges to the map $i \mapsto i+1$, which is not surjective. Thus we need a different notation for the closure of a set of permutations of Ω in S; given a subset $U \subseteq S$, let $\mathrm{cl}_S(U) = \mathrm{cl}_E(U) \cap S$. It is easy to see that given a subset $U \subseteq S$, $\mathrm{cl}_E(\mathrm{cl}_S(U)) = \mathrm{cl}_E(U)$.

Let $G \subseteq S$ be a subgroup, and let $\Sigma \subseteq \Omega$ be finite. Then the forward orbits of $\mathrm{cl}_E(G)_{(\Sigma)} = \mathrm{cl}_E(G_{(\Sigma)})$ coincide, by Lemma 28, with the forward orbits of $G_{(\Sigma)}$, which are simply the (group-theoretic) orbits of $G_{(\Sigma)}$ in Ω. In particular, by the above remark, the forward orbits of $\mathrm{cl}_E(G)_{(\Sigma)} = \mathrm{cl}_E(\mathrm{cl}_S(G))_{(\Sigma)}$ coincide with the orbits of $\mathrm{cl}_S(G)_{(\Sigma)}$.

We are now ready to prove that while $\mathrm{cl}_S(G)$ and $\mathrm{cl}_E(G)$ may be different for a given group G, they are $\approx$-equivalent.

Proposition 43. *Let G be a subgroup of $S \subseteq E$. Then $\mathrm{cl}_S(G) \approx \mathrm{cl}_E(G)$.*

Proof. Let A, B, C, and D be partitions of Ω such that A consists of only one set, B consists of finite sets such that there is no common finite upper bound on their cardinalities, C consists of 2-element sets, and D consists of 1-element sets. By the main results of [4], $\mathrm{cl}_S(G)$ is $\approx_S$ to exactly one of $S = S_{(A)}$, $S_{(B)}$, $S_{(C)}$, or $\{1\} = S_{(D)}$ (see Theorem 15 for the notation $\approx_S$), and by Corollary 42, $\mathrm{cl}_E(G)$ is $\approx$ to exactly one of $E = E_{(A)}$, $E_{(B)}$, $E_{(C)}$, or $\{1\} = E_{(D)}$. Moreover, by the above remarks about (forward) orbits, $\mathrm{cl}_S(G) \approx_S S_{(X)}$ for some $X \in \{A, B, C, D\}$ if and only if $\mathrm{cl}_E(G) \approx E_{(X)}$. But, $S_{(X)} \approx E_{(X)}$, by Proposition 13, and $\mathrm{cl}_S(G) \approx_S S_{(X)}$ if and only if $\mathrm{cl}_S(G) \approx S_{(X)}$, by Theorem 15. Hence $\mathrm{cl}_S(G) \approx \mathrm{cl}_E(G)$. □

We have referred to the main results of Bergman and Shelah in [4] while proving our results. However, it is interesting to note that the Bergman-Shelah theorems can be recovered from them.

Theorem 44 (Bergman and Shelah). *Let $G \subseteq \mathrm{Sym}(\Omega)$ be a subgroup that is closed in the function topology on S. Then G falls into exactly one of four possible equivalence classes with respect to $\approx_S$ (as defined in Theorem 15), depending on which of the following conditions G satisfies:*

(i) *$G_{(\Sigma)}$ has an infinite orbit for every finite $\Sigma \subseteq \Omega$.*

(ii) *There exists a finite set $\Sigma \subseteq \Omega$ such that all orbits of $G_{(\Sigma)}$ are finite, but for every finite $\Sigma \subseteq \Omega$ the cardinalities of the orbits of $G_{(\Sigma)}$ have no common finite bound.*

(iii) *There exists a finite set $\Sigma \subseteq \Omega$ and a positive integer n such that the cardinalities of all orbits of $G_{(\Sigma)}$ are bounded by n, but for every finite $\Sigma \subseteq \Omega$, $G_{(\Sigma)} \neq \{1\}$.*

(iv) *There exists a finite set $\Sigma \subseteq \Omega$ such that $G_{(\Sigma)} = \{1\}$.*

Proof. Let $G_1, G_2 \subseteq \mathrm{Sym}(\Omega)$ be two subgroups that are closed in the function topology. By the remarks at the beginning of the section, G_1 and G_2 fall into the same one of the four classes above if and only if $\mathrm{cl}_E(G_1)$ and $\mathrm{cl}_E(G_2)$ fall into the same one of the four classes in Corollary 42 if and only if $\mathrm{cl}_E(G_1) \approx \mathrm{cl}_E(G_2)$. By the previous proposition, $\mathrm{cl}_E(G_1) \approx \mathrm{cl}_E(G_2)$ if and only if $G_1 \approx G_2$, which, by Theorem 15, occurs if and only if $G_1 \approx_S G_2$. □

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