Polynomials of small degree evaluated on matrices

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Abstract

A celebrated theorem of Shoda states that over any field K (of characteristic 0), every matrix with trace 0 can be expressed as a commutator $AB-BA$, or, equivalently, that the set of values of the polynomial $f(x,y) = xy - yx$ on $\mathbb{M}_n(K)$ contains all matrices with trace 0. We generalize Shoda's theorem by showing that every nonzero multilinear polynomial of degree at most 3, with coefficients in K, has this property. We further conjecture that this holds for every nonzero multilinear polynomial with coefficients in K of degree m, provided that $m - 1 \leq n$.

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1 Introduction

We begin by recalling a theorem, which was originally proved for fields of characteristic 0 by Shoda [12] and later extended to all fields by Albert and Muckenhoupt [1].

Theorem 1 (Shoda/Albert/Muckenhoupt). *Let K be any field, $n \geq 2$ an integer, and $M \in \mathbb{M}_n(K)$. If M has trace 0, then $M = AB - BA$ for some $A, B \in \mathbb{M}_n(K)$.*

For any $A, B \in \mathbb{M}_n(K)$ the traces of AB and BA are equal, and therefore any matrix that can be expressed as a commutator ($AB - BA$) must have trace 0. (This observation is perhaps what prompted Shoda to prove the above result.) Thus, using $[\mathbb{M}_n(K), \mathbb{M}_n(K)]$ to denote the K-subspace of $\mathbb{M}_n(K)$ consisting of the matrices of trace 0, and $f(\mathbb{M}_n(K))$ to denote the set of values of a polynomial f on $\mathbb{M}_n(K)$, the above theorem can be rephrased as follows.

Corollary 2. *Let K be any field, $n \geq 2$ an integer, and $f(x,y) = xy-yx$. Then $f(\mathbb{M}_n(K)) = [\mathbb{M}_n(K), \mathbb{M}_n(K)]$.*

It is therefore natural to ask what other polynomials f have the property $f(\mathbb{M}_n(K)) = [\mathbb{M}_n(K), \mathbb{M}_n(K)]$, or more generally, $f(\mathbb{M}_n(K)) \supseteq [\mathbb{M}_n(K), \mathbb{M}_n(K)]$. In this note we conjecture that this is the case for all nonzero multilinear polynomials f (i.e., polynomials that are linear in each variable), provided that the degree of f is at most $n + 1$. We then prove this conjecture for multilinear polynomials of degree at most 3, over fields with at least n elements, and characterize the polynomials that satisfy $f(\mathbb{M}_n(K)) = [\mathbb{M}_n(K), \mathbb{M}_n(K)]$ in this situation. We also prove that for any unital ring R, the set of values of the polynomial $xzy - xyz + yzx - zyx$ on $\mathbb{M}_n(R)$ contains all matrices with the property that the sums of the elements along the diagonals above (and including) the main diagonal are 0.

Since $[\mathbb{M}_n(K), \mathbb{M}_n(K)]$ is a K-subspace of $\mathbb{M}_n(K)$, our conjecture and result above can be viewed as a partial answer to a more specific version of the following question of Lvov [4, Entry 1.98], which also has been attributed to Kaplansky (see [6]).

Question 3 (Lvov)**.** *Let f be a multilinear polynomial over a field K. Is the set of values of f on the matrix algebra $\mathbb{M}_n(K)$ a vector space?*

Kanel-Belov, Malev, and Rowen [6] have answered this question in the case where $n = 2$ and K is quadratically closed (that is, every non-constant quadratic polynomial over K has a root in K). More specifically, they showed that in this case, the image of a multilinear polynomial must be one of 0, $K \cdot I_2$ (where I_2 is the identity matrix), $[\mathbb{M}_2(K), \mathbb{M}_2(K)]$, or $\mathbb{M}_2(K)$.

Other special cases of Question 3 have been explored elsewhere. For instance, Khurana and Lam [8, Corollary 3.16] showed that the set of values of the multilinear polynomial $xyz - zyx$ on $\mathbb{M}_n(K)$, over an arbitrary field K, is all of $\mathbb{M}_n(K)$, when $n \geq 2$ (along with more general versions of this result for other rings R in place of the field K). Also, Kaplansky [7, Problem 16] asked whether there exists a nonzero multilinear polynomial over a field K which always takes values in the center of $\mathbb{M}_n(K)$, or, equivalently, whether there is such a polynomial whose set of values on $\mathbb{M}_n(K)$ is the center of $\mathbb{M}_n(K)$. (This question arose in the study of polynomial identities on rings.) Such polynomials have indeed been constructed by Formanek [5] and Razmyslov [10].

All the results and questions mentioned above are stated for multilinear (rather than arbitrary) polynomials, since in general, the set of values of a polynomial on $\mathbb{M}_n(K)$ is not a subspace. More specifically, Chuang [3] showed that if K is finite, then every subset S of $\mathbb{M}_n(K)$ that contains 0 and is closed under conjugation is the image of some polynomial. Of course, such subsets S of $\mathbb{M}_n(K)$ are generally not K-subspaces. Let us also give an example of a (non-multilinear) polynomial f and an infinite field K such that $f(\mathbb{M}_n(K))$ is not a K-subspace of $\mathbb{M}_n(K)$.

Example 4. Let K be an algebraically closed field, $n \geq 2$, and $f(x) = x^n$. For all distinct $1 \leq i, j \leq n$ we have $f(E_{ii}) = E_{ii}$ and $f(E_{ii} + E_{ij}) = E_{ii} + E_{ij}$, where E_{ij} are the matrix units. Hence the K-subspace generated by $f(\mathbb{M}_n(K))$ contains all E_{ij}, and therefore must be $\mathbb{M}_n(K)$. Thus, to conclude that $f(\mathbb{M}_n(K))$ is not a subspace, it suffices to show that $f(\mathbb{M}_n(K)) \neq \mathbb{M}_n(K)$. Let $A \in \mathbb{M}_n(K)$ be a nonzero nilpotent matrix. Since 0 is the only eigenvalue of A, the Jordan canonical form of A must be strictly upper-triangular, from which it follows that $A^n = 0$. Thus, if $A = f(B) = B^n$ for some $B \in \mathbb{M}_n(K)$, then B must be nilpotent, and hence $f(B) = B^n = 0$ (by the above argument), contradicting $A \neq 0$. Therefore, $f(\mathbb{M}_n(K))$ contains no nonzero nilpotent matrices, and hence $f(\mathbb{M}_n(K))$ is not a subspace of $\mathbb{M}_n(K)$. □

We conclude the Introduction by mentioning that there has been interest in generalizing Theorem 1 in other directions, particularly, in determining whether the same statement holds over an arbitrary (unital) ring. That is, given a ring R, can every matrix $M \in \mathbb{M}_n(R)$

having trace 0 be expressed as a commutator $M = AB - BA$, for some $A, B \in \mathbb{M}_n(R)$? Intriguingly, this question remained open until 2000, when Rosset and Rosset [11] produced a ring R and a matrix $M \in \mathbb{M}_2(R)$ having trace 0 that cannot be expressed as a commutator. On the other hand, it turns out that over any unital ring R and for any $n \geq 2$, every matrix $M \in \mathbb{M}_n(R)$ having trace 0 can be expressed as a sum of two commutators (see [9, Theorem 15]). To the best of our knowledge, however, there is still no classification of the rings R with the property that every matrix with trace 0 is a commutator in $\mathbb{M}_n(R)$, despite continued attention to the question.

2 Preliminaries

Let us next define multilinear polynomials more rigorously, and then collect some basic facts about them.

Definition 5. *Given a field K and a positive integer m, we denote by $K\langle x_1, \dots, x_m\rangle$ the K-algebra freely generated by the (non-commuting) variables $x_1, \dots, x_m$. A polynomial $f(x_1, \dots, x_m) \in K\langle x_1, \dots, x_m\rangle$ is said to be* multilinear (of degree m) *if it is of the form*

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_m} a_\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(m)},$$

where S_m is the group of all permutations of $\{1, \dots, m\}$ and $a_\sigma \in K$.

Some authors call these polynomials *homogeneous multilinear*. The motivation for calling such polynomials "multilinear" is that given any K-algebra A, any element $f(x_1, \dots, x_m) \in K\langle x_1, \dots, x_m\rangle$ can be viewed as a map $f : A \times \dots \times A \to A$ by evaluating the x_i on elements of A, and it is easy to see that the polynomials of the above form are precisely the ones that give rise to maps $f : A \times \dots \times A \to A$ that are linear in each variable (for every K-algebra A). That is, maps f such that for all $a, b \in K$, $j \in \{1, \dots, m\}$, and $r_1, \dots, r_m, r'_j \in A$, we have

$$f(r_1, \dots, r_{j-1}, ar_j + br'_j, r_{j+1}, \dots, r_m)$$
$$= af(r_1, \dots, r_{j-1}, r_j, r_{j+1}, \dots, r_m) + bf(r_1, \dots, r_{j-1}, r'_j, r_{j+1}, \dots, r_m).$$

Given a polynomial $f(x_1, \dots, x_m) \in K\langle x_1, \dots, x_m\rangle$ and K-algebra A, we set $f(A) = \{f(r_1, \dots, r_m) \mid r_1, \dots, r_m \in A\}$. Also, for any associative ring R we shall denote by $[r, p]$ the commutator $rp - pr$ of $r, p \in R$, denote by $[R, R]$ the additive subgroup of R generated by the commutators, and let $[r, R] = \{[r, p] \mid p \in R\}$ for $r \in R$.

In the next lemma we record a couple of basic facts about multilinear polynomials for later reference. Both claims follow immediately from Definition 5.

Lemma 6. *Let K be a field, A a K-algebra, m a positive integer, and $f(x_1, \dots, x_m) \in K\langle x_1, \dots, x_m\rangle$ a multilinear polynomial. Also, let $r_1, \dots, r_m \in A$ be arbitrary elements. Then the following hold.*

(1) *For any $a \in K$ we have $af(r_1, \dots, r_m) = f(ar_1, \dots, r_m)$.*

(2) *For any invertible $p \in A$ we have $pf(r_1, \dots, r_m)p^{-1} = f(pr_1p^{-1}, \dots, pr_mp^{-1})$.*

The next lemma, which is an easy consequence of Theorem 1, describes the sets of values of multilinear polynomials of degree at most 2 on matrix algebras.

Lemma 7. *Let K be a field, m a positive integer, and $f(x_1, \dots, x_m) \in K\langle x_1, \dots, x_m\rangle$ a multilinear polynomial.*

(1) *If $m = 1$, then $f(\mathbb{M}_n(K)) \in \{0, \mathbb{M}_n(K)\}$.*

(2) *If $m = 2$, then $f(\mathbb{M}_n(K)) \in \{0, [\mathbb{M}_n(K), \mathbb{M}_n(K)], \mathbb{M}_n(K)\}$.*

Proof. If $m = 1$, then f must be of the form $f(x) = ax$ for some $a \in K$, from which (1) follows.

If $m = 2$, then f must be of the form $f(x, y) = axy + byx$ for some $a, b \in K$. If $a + b \neq 0$, then $f(I_n, Y) = (a + b)Y$ for all $Y \in \mathbb{M}_n(K)$, which implies that $f(\mathbb{M}_n(K)) = \mathbb{M}_n(K)$. While, if $a + b = 0$, then $f(x, y) = a(xy - yx)$, which has the same image as the polynomial $xy - yx$, as long as $a \neq 0$, by Lemma 6(1). Statement (2) now follows from Corollary 2. □

The following fact will be useful in subsequent arguments. For a polynomial $f(x_1, \dots, x_m)$ in $K\langle x_1, \dots, x_m\rangle$ and an integer $1 \leq l \leq m$ we view $f(x_1, \dots, x_l, 1, \dots, 1)$ as a polynomial in $K\langle x_1, \dots, x_l\rangle$.

Corollary 8. *Let K be a field, m a positive integer, and*

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_m} a_\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(m)} \in K\langle x_1, \dots, x_m\rangle$$

a multilinear polynomial.

(1) *If $\sum_{\sigma \in S_m} a_\sigma \neq 0$, then $f(\mathbb{M}_n(K)) = \mathbb{M}_n(K)$.*

(2) *If $m \geq 2$ and $f(x_1, x_2, 1, \dots, 1) \neq 0$, then $f(\mathbb{M}_n(K)) \in \{[\mathbb{M}_n(K), \mathbb{M}_n(K)], \mathbb{M}_n(K)\}$.*

Proof. This follows from Lemma 7 upon noting that $f(x_1, 1, \dots, 1) = (\sum_{\sigma \in S_m} a_\sigma)x_1$, and $f(x_1, x_2, 1, \dots, 1) = bx_1x_2 + cx_2x_1$, for some $b, c \in K$. □

3 A conjecture

The main goal of this section is to justify a conjecture regarding the sets of values of certain multilinear polynomials. We shall first require the following result of Amitsur and Rowen [2, Proposition 1.8] to prove a fact about the linear spans of the images of our polynomials.

Proposition 9 (Amitsur/Rowen)**.** *Let D be a division ring, $n \geq 2$ an integer, and $A \in \mathbb{M}_n(D)$. Then A is similar to a matrix in $\mathbb{M}_n(D)$ with at most one nonzero entry on the main diagonal. In particular, if A has trace zero, then it is similar to a matrix in $\mathbb{M}_n(D)$ with only zeros on the main diagonal.*

Proposition 10. *Let K be a field, $n \geq 2$ and $m \geq 1$ integers, and $f(x_1, \dots, x_m)$ a nonzero multilinear polynomial in $K\langle x_1, \dots, x_m\rangle$. If $n \geq m - 1$, then the K-subspace $\langle f(\mathbb{M}_n(K))\rangle$ of $\mathbb{M}_n(K)$ generated by $f(\mathbb{M}_n(K))$ contains $[\mathbb{M}_n(K), \mathbb{M}_n(K)]$.*

Proof. Write

$$f(x_1, \dots, x_m) = \sum_{\sigma \in S_m} a_\sigma x_{\sigma(1)} x_{\sigma(2)} \dots x_{\sigma(m)},$$

for some $a_\sigma \in K$. Since f is nonzero, upon relabeling the variables if necessary, we may assume that $a_1 \neq 0$. Furthermore, by Lemma 7, if $m \leq 2$, then $[\mathbb{M}_n(K), \mathbb{M}_n(K)] \subseteq f(\mathbb{M}_n(K))$ for any $n \geq 2$. We therefore may assume that $m \geq 3$.

Let $i, j \in \{1, \dots, n\}$ be distinct. Since $n \geq m - 1 \geq 2$, we can find distinct elements $l_1, \dots, l_{m-3} \in \{1, \dots, n\} \setminus \{i, j\}$. Letting E_{kl} denote the matrix units, we then have

$$f(E_{ii}, E_{ij}, E_{jl_1}, E_{l_1 l_2}, \dots, E_{l_{m-4} l_{m-3}}, E_{l_{m-3} j})$$

$$= a_1 E_{ii} E_{ij} E_{jl_1} E_{l_1 l_2} \dots E_{l_{m-4} l_{m-3}} E_{l_{m-3} j} + 0 = a_1 E_{ij},$$

since multiplying $E_{ii}, E_{ij}, E_{jl_1}, E_{l_1 l_2}, \dots, E_{l_{m-4} l_{m-3}}, E_{l_{m-3} j}$ in any other order yields 0. (If $m = 3$, then we interpret the above equation as $f(E_{ii}, E_{ij}, E_{jj}) = a_1 E_{ij}$.) Therefore, $E_{ij} \in f(\mathbb{M}_n(K))$ for all distinct i and j, and hence $\langle f(\mathbb{M}_n(K)) \rangle$ contains all matrices with zeros on the main diagonal. Now, Lemma 6 implies that $\langle f(\mathbb{M}_n(K)) \rangle$ is closed under conjugation. Hence, by Proposition 9, we have $[\mathbb{M}_n(K), \mathbb{M}_n(K)] \subseteq \langle f(\mathbb{M}_n(K)) \rangle$. □

The claim in the above proposition does not in general hold when $n < m-1$. For example, it is known that the set of values of the polynomial $(xy - yx)^2$ on $\mathbb{M}_2(K)$ is contained in the center $K \cdot I_2$ (see [7]). The same is true of the linearization

$$f(x_1, x_2, y_1, y_2) = [x_1, y_1][x_2, y_2] + [x_1, y_2][x_2, y_1] + [x_2, y_1][x_1, y_2] + [x_2, y_2][x_1, y_1]$$

of this polynomial. Thus, $\langle f(\mathbb{M}_2(K)) \rangle = K \cdot I_2 \not\supseteq [\mathbb{M}_2(K), \mathbb{M}_2(K)]$.

An affirmative answer to Question 3 would imply that in the above proposition, if $n \geq m - 1$, then $f(\mathbb{M}_n(K))$, and not just $\langle f(\mathbb{M}_n(K)) \rangle$, contains $[\mathbb{M}_n(K), \mathbb{M}_n(K)]$. Since it is suspected that the question does have an affirmative answer (e.g., see [6]), we make the following conjecture.

Conjecture 11. *Let K be a field, $n \geq 2$ and $m \geq 1$ integers, and $f(x_1, \dots, x_m)$ a nonzero multilinear polynomial in $K\langle x_1, \dots, x_m \rangle$. If $n \geq m - 1$, then $f(\mathbb{M}_n(K)) \supseteq [\mathbb{M}_n(K), \mathbb{M}_n(K)]$.*

Lemma 7 shows that this conjecture holds for $m < 3$. In the next section we shall show that it holds for $m = 3$ as well, if K has at least n elements.

4 The three-variable case

We shall require another fact proved by Amitsur and Rowen [2, Lemma 1.2].

Lemma 12 (Amitsur/Rowen). *Let K be a field and $n \geq 2$ an integer. Suppose that $A = (a_{ij}) \in \mathbb{M}_n(K)$ is a diagonal matrix with $a_{ii} \neq a_{jj}$ for $i \neq j$. Then $[A, \mathbb{M}_n(K)]$ consists of all the matrices with only zeros on the main diagonal.*

We are now ready for our main result.

Theorem 13. *Let $n \geq 2$ be an integer, K a field with at least n elements, and $f \in K\langle x, y, z\rangle$ any nonzero multilinear polynomial. Then $f(\mathbb{M}_n(K))$ contains every matrix having trace 0.*

Proof. If f has degree at most 2, then this follows from Lemma 7. Thus, let us assume that the degree of f is 3. We can then write

$$f(x, y, z) = axyz + bxzy + cyxz + dyzx + ezxy + gzyx \ (a, b, c, d, e, g \in K).$$

If $a + b + c + d + e + g \neq 0$, then $[\mathbb{M}_n(K), \mathbb{M}_n(K)] \subseteq f(\mathbb{M}_n(K))$, Corollary 8(1). Let us therefore suppose that $a + b + c + d + e + g = 0$. In this case

$$f(x, y, z) = a(xyz - zyx) + b(xzy - zyx) + c(yxz - zyx) + d(yzx - zyx) + e(zxy - zyx).$$

Moreover, if any of $f(1, y, z)$, $f(x, 1, z)$, or $f(x, y, 1)$ are nonzero, then by Corollary 8(2), $[\mathbb{M}_n(K), \mathbb{M}_n(K)] \subseteq f(\mathbb{M}_n(K))$. Thus, let us assume that

$$0 = f(1, y, z) = a(yz - zy) + c(yz - zy) + d(yz - zy) = (a + c + d)(yz - zy),$$

which implies that $0 = a + c + d$. Setting $0 = f(x, 1, z)$ and $0 = f(x, y, 1)$ we similarly get $0 = a+b+c$ and $0 = a+b+e$. Solving the resulting system of equations gives $b = d$, $c = e$, and $a = -b - c$. Therefore,

$$f(x, y, z) = (-b - c)(xyz - zyx) + b(xzy - zyx + yzx - zyx) + c(yxz - zyx + zxy - zyx)$$

$$= b(xzy - zyx + yzx - xyz) + c(yxz - zyx + zxy - xyz)$$
$$= b(x[z, y] - [z, y]x) + c(z[x, y] - [x, y]z) = b[x, [z, y]] + c[z, [x, y]].$$

Now, since $f(x, y, z) \neq 0$, renaming the variables, if necessary, we may assume that $b \neq 0$. Then, by Lemma 6(1), $f(x, y, z)$ and $b^{-1}f(x, y, z)$ have the same set of values. Therefore, we may assume that $f(x, y, z) = [x, [z, y]] + b[z, [x, y]]$ for some $b \in K$.

Let $A \in \mathbb{M}_n(K)$ be a matrix having trace 0. We wish to show that $A \in f(\mathbb{M}_n(K))$. By Proposition 9, A is conjugate to a matrix $A' \in \mathbb{M}_n(K)$ with only zeros on the main diagonal. Hence, by Lemma 6(2), to conclude the proof it is enough to show that that $A' \in f(\mathbb{M}_n(K))$. By our assumption on K, we can find a diagonal matrix $M \in \mathbb{M}_n(K)$ with distinct elements of K on its main diagonal. By Lemma 12, there is some $B \in \mathbb{M}_n(K)$ such that $A' = [M, B]$. Now, write $B = C + D$, where C has only zeros on the main diagonal and D is diagonal. Then

$$A' = [M, B] = [M, C + D] = [M, C] + [M, D] = [M, C],$$

since M commutes with all diagonal matrices. Using Lemma 12 once again, we can find a matrix $E \in \mathbb{M}_n(K)$ such that $C = [E, M]$. Finally, we have

$$f(M, M, E) = [M, [E, M]] + b[E, [M, M]] = [M, [E, M]] = [M, C] = A',$$

as desired. □

Let us next describe the degree-three multilinear polynomials f satisfying $f(\mathbb{M}_n(K)) = [\mathbb{M}_n(K), \mathbb{M}_n(K)]$.

Lemma 14. *Let $n \geq 2$ be an integer, K a field, and*

$$f(x_1, x_2, x_3) = \sum_{\sigma \in S_3} a_\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \in K\langle x_1, x_2, x_3 \rangle$$

a multilinear polynomial of degree 3. Then $f(\mathbb{M}_n(K)) \subseteq [\mathbb{M}_n(K), \mathbb{M}_n(K)]$ if and only if $\sum_{\sigma \in S_3} a_\sigma = 0 = \sum_{\sigma \in A_3} a_\sigma$, where $A_3 \subseteq S_3$ is the alternating subgroup.

Proof. By Corollary 8(1), we may assume that $\sum_{\sigma \in S_3} a_\sigma = 0$. Now, suppose that $\sum_{\sigma \in A_3} a_\sigma = 0$. Then we must also have $\sum_{\sigma \in S_3 \setminus A_3} a_\sigma = \sum_{\sigma \in S_3} a_\sigma - \sum_{\sigma \in A_3} a_\sigma = 0$. Therefore,

$$f(x_1, x_2, x_3) = \sum_{\sigma \in A_3} a_\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} + \sum_{\sigma \in S_3 \setminus A_3} a_\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}$$

$$= \sum_{\sigma \in A_3 \setminus \{1\}} a_\sigma (x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} - x_1 x_2 x_3) + \sum_{\sigma \in S_3 \setminus (A_3 \cup \{(12)\})} a_\sigma (x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} - x_2 x_1 x_3)$$

$$= a_{(123)}(x_2 x_3 x_1 - x_1 x_2 x_3) + a_{(132)}(x_3 x_1 x_2 - x_1 x_2 x_3)$$

$$+ a_{(13)}(x_3 x_2 x_1 - x_2 x_1 x_3) + a_{(23)}(x_1 x_3 x_2 - x_2 x_1 x_3)$$

$$= a_{(123)}[x_2 x_3, x_1] + a_{(132)}[x_3, x_1 x_2] + a_{(13)}[x_3, x_2 x_1] + a_{(23)}[x_1 x_3, x_2].$$

Thus, $f(\mathbb{M}_n(K)) \subseteq [\mathbb{M}_n(K), \mathbb{M}_n(K)]$.

On the other hand, $f(E_{11}, E_{12}, E_{21}) = a_1 E_{11} + a_{(123)} E_{11} + a_{(132)} E_{22}$ has trace $\sum_{\sigma \in A_3} a_\sigma$. Hence, if this sum is not zero, then $f(\mathbb{M}_n(K)) \not\subseteq [\mathbb{M}_n(K), \mathbb{M}_n(K)]$. □

Applying Theorem 13 to this lemma we obtain the following.

Corollary 15. *Let $n \geq 2$ be an integer, K a field with at least n elements, and*

$$f(x_1, x_2, x_3) = \sum_{\sigma \in S_3} a_\sigma x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} \in K\langle x_1, x_2, x_3 \rangle$$

a nonzero multilinear polynomial of degree 3. Then $f(\mathbb{M}_n(K)) = [\mathbb{M}_n(K), \mathbb{M}_n(K)]$ if and only if $\sum_{\sigma \in S_3} a_\sigma = 0 = \sum_{\sigma \in A_3} a_\sigma$, where $A_3 \subseteq S_3$ is the alternating subgroup.

We note that in general the condition $\sum_{\sigma \in S_m} a_\sigma = 0 = \sum_{\sigma \in A_m} a_\sigma$ does not characterize the nonzero multilinear polynomials $f(x_1, \dots, x_m)$ satisfying $f(\mathbb{M}_n(K)) = [\mathbb{M}_n(K), \mathbb{M}_n(K)]$. For example, if $m = 2$, then $\sum_{\sigma \in S_m} a_\sigma = 0 = \sum_{\sigma \in A_m} a_\sigma$ implies that $f(x_1, x_2) = 0$. Also, $f(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 x_4 - x_4 x_3 x_2 x_1$ satisfies $\sum_{\sigma \in S_4} a_\sigma = 0 = \sum_{\sigma \in A_4} a_\sigma$, but $f(\mathbb{M}_n(K)) \not\subseteq [\mathbb{M}_n(K), \mathbb{M}_n(K)]$ for any $n \geq 2$, since $f(E_{11}, E_{12}, E_{22}, E_{21}) = E_{11} \notin [\mathbb{M}_n(K), \mathbb{M}_n(K)]$.

Let us conclude with a fact about the image of the degree-three multilinear polynomial $[x, [y, z]]$ on matrices over an arbitrary ring. We first require the following lemma.

Lemma 16. *Let R be a unital ring, let $n \geq 2$ be an integer, and let $A = (a_{ij}) \in \mathbb{M}_n(R)$ be such that for each $j \in \{0, 1, \dots, n-1\}$ we have $\sum_{i=1}^{n-j} a_{i,i+j} = 0$ (i.e., sums of the elements along the diagonals above and including the main diagonal are 0). Then $A = DX - XD$ for some $D \in \mathbb{M}_n(R)$ and $X = \sum_{i=1}^{n-1} E_{i+1,i}$, where E_{ij} are the matrix units.*

Proof. Letting $Z = \sum_{i=1}^{n-1} E_{i,i+1}$ we have $ZX = \sum_{i=1}^{n-1} E_{ii} = I - E_{nn}$. For any $l \in \{0, 1, \dots, n-1\}$ and $k \in \{1, 2, \dots, n\}$ we then have

$$E_{kk}X^lAZ^lE_{nn} = E_{kk}\left(\sum_{i=1}^{n-l} E_{i+l,i}\right)A\left(\sum_{i=1}^{n-l} E_{i,i+l}\right)E_{nn} = E_{k,k-l}AE_{n-l,n} = a_{k-l,n-l}E_{kn}$$

if $l < k$, and $E_{kk}X^lAZ^lE_{nn} = 0 \cdot AZ^lE_{nn} = 0$ if $l \geq k$.

Letting $D = \sum_{i=0}^{n-2} X^iAZ^{i+1}$, we have

$$DX - XD = \left(\sum_{i=0}^{n-2} X^iAZ^i\right)ZX - \sum_{i=0}^{n-2} X^{i+1}AZ^{i+1}$$

$$= \left(A + \sum_{i=1}^{n-2} X^iAZ^i\right)(I - E_{nn}) - \sum_{i=1}^{n-1} X^iAZ^i = A - \left(\sum_{i=0}^{n-2} X^iAZ^i\right)E_{nn} - X^{n-1}AZ^{n-1}$$

$$= A - \left(\sum_{i=0}^{n-2} X^iAZ^i\right)E_{nn} - X^{n-1}AE_{1n} = A - \left(\sum_{i=0}^{n-1} X^iAZ^i\right)E_{nn}.$$

Now, for every $k \in \{1, 2, \dots, n\}$ we have

$$E_{kk}\left(\sum_{i=0}^{n-1} X^iAZ^i\right)E_{nn} = \sum_{i=0}^{k-1} E_{kk}X^iAZ^iE_{nn} = \left(\sum_{i=0}^{k-1} a_{k-i,n-i}\right)E_{kn} = \left(\sum_{i=1}^{k} a_{i,i+(n-k)}\right)E_{kn},$$

by the computation in the first paragraph. Finally, the last sum is 0, by hypothesis on A, and hence $(\sum_{i=0}^{n-1} X^iAZ^i)E_{nn} = 0$, showing that $DX - XD = A$. □

Theorem 17. *Let R be a unital ring, let $n \geq 2$ be an integer, let $f(x, y, z) = [x, [z, y]]$, and let $A = (a_{ij}) \in \mathbb{M}_n(R)$ be such that for each $j \in \{0, 1, \dots, n-1\}$ we have $\sum_{i=1}^{n-j} a_{i,i+j} = 0$. Then $A = f(D, X, Y)$ for some $D \in \mathbb{M}_n(R)$, $X = \sum_{i=1}^{n-1} E_{i+1,i}$, and $Y = \sum_{i=1}^{n}(i-1)E_{ii}$.*

Proof. For any matrix $M = (m_{ij}) \in \mathbb{M}_n(K)$ we have

$$[Y, M] = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ m_{21} & m_{22} & m_{23} & \dots & m_{2n} \\ 2m_{31} & 2m_{32} & 2m_{33} & \dots & 2m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} - \begin{pmatrix} 0 & m_{12} & 2m_{13} & \dots & (n-1)m_{1n} \\ 0 & m_{22} & 2m_{23} & \dots & (n-1)m_{2n} \\ 0 & m_{32} & 2m_{33} & \dots & (n-1)m_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

$$= ((i-1)m_{ij}) - ((j-1)m_{ij}) = ((i-j)m_{ij}).$$

Hence, in particular, $[Y, X] = X$.

Now, by Lemma 16, $A = [D, X]$ for some $D \in \mathbb{M}_n(R)$. We therefore have

$$f(D, X, Y) = [D, [Y, X]] = [D, X] = A,$$

proving the statement. □

This argument extends easily to all polynomials of the form $[x_1, [x_2, \dots, [x_{m-1}, x_m]] \dots]$.

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