

TRACES ON SEMIGROUP RINGS AND LEAVITT PATH ALGEBRAS

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ABSTRACT. The trace on matrix rings, along with the augmentation map and Kaplansky trace on group rings, are some of the many examples of linear functions on algebras that vanish on all commutators. We generalize and unify these examples by studying traces on (contracted) semigroup rings over commutative rings. We show that every such ring admits a minimal trace (i.e., one that vanishes *only* on sums of commutators), classify all minimal traces on these rings, and give applications to various classes of semigroup rings and quotients thereof. We then study traces on Leavitt path algebras (which are quotients of contracted semigroup rings), where we describe all linear traces in terms of central maps on graph inverse semigroups and, under mild assumptions, those Leavitt path algebras that admit faithful traces.

Keywords: trace, faithful trace, semigroup ring, graph inverse semigroup, Leavitt path algebra, Cohn path algebra, involution

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1. INTRODUCTION

Perhaps the most familiar example of a linear map t that is central (i.e., satisfies $t(ab) = t(ba)$ for all a, b) is the trace on a matrix ring. Other frequently used central linear functions include the augmentation map and the Kaplansky trace on group rings. Various generalizations of the above maps also have been explored in the literature on ring theory, module theory, and operator algebras.

One of the goals of this paper is to provide a general framework for studying a variety of central linear maps, or *traces*. We begin by characterizing linear traces on (contracted) semigroup rings over commutative rings (Proposition 3), which include both group rings and matrix rings, and hence unify the examples mentioned above. We then show that every (contracted) semigroup ring over a commutative ring admits a minimal linear trace (i.e., one whose kernel consists entirely of sums of commutators and is hence as small as possible), and characterize all minimal linear traces on such rings in terms of an equivalence relation on the underlying semigroups (Theorem 11). This result is motivated by the well-known fact that the usual trace on a matrix ring over a commutative ring is minimal. We then apply our result to matrix rings (Corollary 14), group rings (Corollary 15), and Cohn path algebras (Corollary 21). Along the way we describe all central maps on various classes of semigroups, and in particular on graph inverse semigroups (Proposition 19).

The remainder of the paper is devoted to linear traces on quotients of semigroup rings, especially on Leavitt path algebras. We describe when (minimal) linear traces on (contracted) semigroup rings over commutative rings pass to (minimal) linear traces on quotients thereof (Proposition 16). We then classify all linear traces on an arbitrary Leavitt path algebra in terms of central maps on the underlying graph inverse semigroup (Theorem 28).

Finally, we turn to faithful traces on Leavitt path algebras. (A trace that maps nonzero positive elements, with respect to the involution on the ring, to nonzero positive elements is

said to be *faithful*.) Our interest in faithful traces is motivated by the well-known fact that the usual trace on a matrix ring over a commutative ring with a positive definite involution is faithful. Moreover, Leavitt path algebras (first introduced in [2, 5]) are algebraic analogues of graph C^* -algebras, on which faithful traces have been studied extensively (e.g., [13, 14]). Assuming that K is a field with a positive definite involution and E is a row-finite graph where every infinite path ends either in a sink or in a cycle, we show that the corresponding Leavitt path algebra $L_K(E)$ admits a faithful trace if and only if E has no cycles with exits (Theorem 33). We also give examples to illustrate the necessity of the hypotheses on K and E in this result. Other results about traces on Leavitt path algebras are obtained in [16].

The necessary notions about semigroups, graphs, and Leavitt path algebras are reviewed along the way.

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2. TRACES ON SEMIGROUP RINGS

We begin by explaining our notation and describing the setting for our investigation of traces.

All rings will be assumed to be associative but not necessarily unital. Given a ring R and elements $x, y \in R$, $[x, y]$ will denote the commutator $xy - yx$, and $[R, R]$ will denote the additive subgroup of R generated by the commutators.

Definition 1. *Let R and T be rings. A T -valued trace on R is a map $t : R \rightarrow T$ satisfying $t(x + y) = t(x) + t(y)$ and $t(xy) = t(yx)$ for all $x, y \in R$.*

If R and T are C -algebras, for some commutative ring C , then t is C -linear in case $t(cx) = ct(x)$ for all $x \in R$ and $c \in C$.

We say that t is minimal if for all $x \in R$, $t(x) = 0$ implies that $x \in [R, R]$.

Our usage of “minimal” in the above definition is justified by the fact that $t(x) = 0$ for any trace $t : R \rightarrow T$ and any $x \in [R, R]$.

Given a ring R and a semigroup G (with zero), we denote by RG the corresponding semigroup ring, and by \overline{RG} the corresponding *contracted semigroup ring*, where the zero of G is identified with the zero of RG . That is, $\overline{RG} = RG/I$, where $I = \{x \cdot 0_G \in RG \mid x \in R\}$ is the ideal of RG generated by the zero 0_G of G . We note that if G is a semigroup without zero, and G^0 is the semigroup obtained from G by adjoining a zero element, then $RG \cong \overline{RG^0}$.

For the remainder of this note, “semigroup” will be understood to mean a semigroup with zero. We shall use the notation $\sum_g a_g g$ to denote an arbitrary element of a contracted semigroup ring \overline{RG} , where it is understood that g ranges over all the nonzero elements of G , $a_g \in R$, and all but finitely many of the a_g are zero. Finally, given a map $\delta : G \rightarrow R$ between semigroups, we shall say that δ *preserves zero* if it takes zero to zero.

Definition 2. *Let C be a commutative ring, R a C -algebra, G a semigroup, and $\delta : G \rightarrow R$ a map that preserves zero.*

We denote by t_δ the map $\overline{CG} \rightarrow R$ defined by $t_\delta(\sum_g a_g g) = \sum_g a_g \delta(g)$, for all $a_g \in C$ and $g \in G$.

We say that δ is central if $\delta(gh) = \delta(hg)$ for all $g, h \in G$, and that δ is normalized if R is unital and $\delta(g) \in \{0, 1\}$ for all $g \in G$.

With the help of the above definition we can describe all linear traces on (contracted) semigroup rings over commutative unital rings. This description, while very simple, will be useful throughout the paper.

Proposition 3. *Let C be a commutative ring, R a C -algebra, G a semigroup, and $\delta : G \rightarrow R$ a central map that preserves zero. Then $t_\delta : \overline{CG} \rightarrow R$ is a C -linear trace. Moreover, if C is unital, then every C -linear trace $t : \overline{CG} \rightarrow R$ is of this form.*

Proof. Let $x, y \in \overline{CG}$ be arbitrary elements, and write $x = \sum_g a_g g$, $y = \sum_g b_g g$ for some $a_g, b_g \in C$. Then

$$t_\delta(x + y) = t_\delta\left(\sum_g (a_g + b_g)g\right) = \sum_g (a_g + b_g)\delta(g) = \sum_g a_g\delta(g) + \sum_g b_g\delta(g) = t_\delta(x) + t_\delta(y),$$

and, using the commutativity of C and the fact that δ is central, we have

$$t_\delta(xy) = t_\delta\left(\sum_g \sum_h a_g b_h gh\right) = \sum_g \sum_h a_g b_h \delta(gh) = \sum_h \sum_g b_h a_g \delta(hg) = t_\delta(yx).$$

Thus t_δ is an R -valued trace, and it is clearly C -linear.

For the final claim, suppose that C is unital and $t : \overline{CG} \rightarrow R$ is a C -linear trace. Let $x = \sum_g a_g g \in \overline{CG}$ be any element. Then

$$t(x) = t\left(\sum_g a_g g\right) = \sum_g a_g t(g),$$

since t is C -linear. (Here we identify each $g \in G \setminus \{0\}$ with $1 \cdot g \in \overline{CG}$.) Also, for any $g, h \in G$ we have $t(gh) = t(hg)$, since t is a trace, and hence the restriction of t to G gives a central map, which necessarily preserves zero. Thus t has the desired form. \square

Here are some familiar examples of (normalized) traces of the above form.

Example 4. Let C be a commutative unital ring, G a group, and G^0 the semigroup obtained by adjoining a zero to G . Define $\delta : G^0 \rightarrow C$ by $\delta(e) = 1$, where $e \in G^0$ is the identity element, and $\delta(g) = 0$ for all $g \in G^0 \setminus \{e\}$. Then it is easy to see that δ is central. The map $t_\delta : CG \cong \overline{CG^0} \rightarrow C$ is known as the *Kaplansky trace*.

Example 5. Let C be a commutative unital ring, G a group, and G^0 the semigroup obtained by adjoining a zero to G . Define $\delta : G^0 \rightarrow C$ by $\delta(g) = 1$ for all $g \in G^0 \setminus \{0\}$, and $\delta(0) = 0$. Then δ is clearly central, and $t_\delta : CG \cong \overline{CG^0} \rightarrow C$ is called *the augmentation map*.

Example 6. Let C be a commutative unital ring and $n \geq 1$ an integer. Then the ring $\mathbb{M}_n(C)$ of $n \times n$ matrices over C is isomorphic to the contracted semigroup ring \overline{CG} , where $G = \{e_{ij} \mid 1 \leq i, j \leq n\} \cup \{0\}$, and e_{ij} are the matrix units. Define $\delta : G \rightarrow C$ by $\delta(e_{ij}) = \delta_{ij}$, where δ_{ij} is the Kronecker delta, and $\delta(0) = 0$. Then for all $1 \leq i, j, k, l \leq n$ we have $\delta(e_{ij}e_{kl}) = 1$ if and only if $l = i$, $k = j$ if and only if $\delta(e_{kl}e_{ij}) = 1$. It follows that δ is a central map. It is now easy to see that $t_\delta : \mathbb{M}_n(C) \rightarrow C$ is precisely the usual trace on $\mathbb{M}_n(C)$.

Example 7. Let C be a commutative unital ring, κ an infinite cardinal, and $\mathbb{M}_\kappa(C)$ the ring of infinite matrices over C , having rows and columns indexed by κ , with only finitely many nonzero entries. Then $\mathbb{M}_\kappa(C)$ is isomorphic to \overline{CG} , where $G = \{e_{ij} \mid i, j \in \kappa\} \cup \{0\}$, and e_{ij} are the matrix units. Defining $\delta : G \rightarrow C$ as in Example 6 again gives a central map, and hence $t_\delta : \mathbb{M}_\kappa(C) \rightarrow C$ is a C -linear trace.

3. MINIMAL TRACES

It is well known that the linear traces in Examples 6 and 7 are minimal (see e.g., [11, Corollary 17]). In this section we shall generalize this fact by constructing minimal linear traces for all semigroup rings over commutative unital rings and then classifying the minimal linear traces on such rings. The following relation will help us with this task.

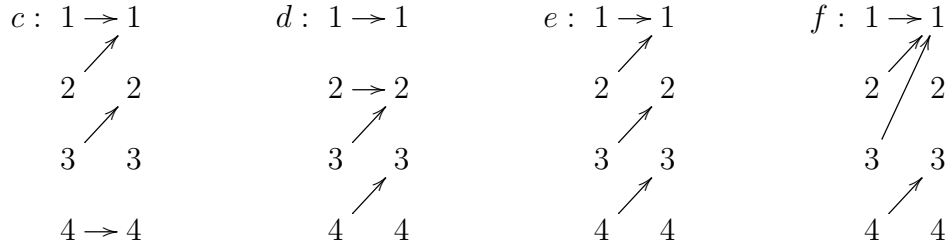
Definition 8. Let G be a semigroup and $g, h \in G$. We shall write $g \sim h$ if either $g = h$ or there are elements $a_1, b_1, a_2, b_2, \dots, a_n, b_n \in G$ such that

$$g = a_1 b_1, \quad b_1 a_1 = a_2 b_2, \quad b_2 a_2 = a_3 b_3, \dots, \quad b_{n-1} a_{n-1} = a_n b_n, \quad b_n a_n = h.$$

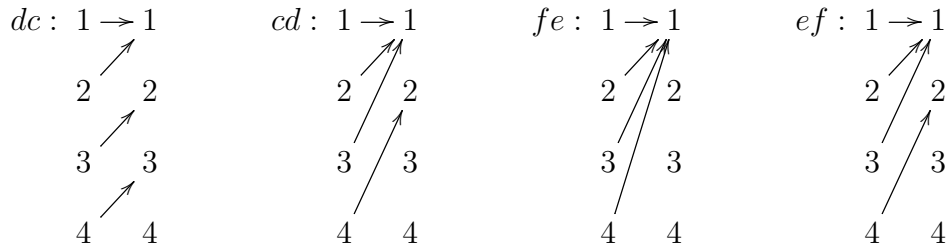
The relation \sim is clearly an equivalence. We shall denote the \sim -equivalence class of $g \in G$ by $[g]$. We note that if $G \setminus \{0\}$ is a group, and $g, h \in G \setminus \{0\}$, then $g \sim h$ if and only if $g = f h f^{-1}$ for some $f \in G \setminus \{0\}$. Thus \sim generalizes the notion of conjugacy from groups to all semigroups.

Before proceeding to our construction, let us give an example of a semigroup G and elements $g, h \in G$ such that $g \sim h$, but where there do not exist $a, b \in G$ satisfying $g = ab$, $ba = h$. This will show that in general \sim does not reduce to the simpler relation where g and h are related if $g = h$ or $g = ab$, $ba = h$ for some $a, b \in G$.

Example 9. Let $S = \{1, 2, 3, 4\}$, and let $G = \text{End}(S)$ be the semigroup of all set maps from S to itself. We define $c, d, e, f \in G$ as follows.



Then, letting elements of G act on S from the left, we have the following composites.



Thus, $cd = ef$, and hence letting $g = dc$ and $h = fe$, we see that $g \sim h$.

Clearly, for all $a, b \in G$ we have $|ab(S)| \leq |a(S)|, |b(S)|$ (where for any set X , $|X|$ denotes the cardinality of X). Hence, if $ab = g$, then $|a(S)|, |b(S)| \geq 3$. But then $|ba(S)| \geq 2$, and therefore $ba \neq h$. Thus, there do not exist $a, b \in G$ such that $g = ab$ and $ba = h$.

Extending a familiar notion from linear algebra, given a commutative ring C and a C -algebra R , we say that a subset S of R is C -linearly independent if for all $x_1, \dots, x_n \in S$ and all $c_1, \dots, c_n \in C$, $c_1 x_1 + \dots + c_n x_n = 0$ implies that $c_1 = \dots = c_n = 0$.

In the following lemma we give our minimal trace construction.

Lemma 10. *Let C be a commutative unital ring, R a C -algebra, G a semigroup, and S the set of all nonzero \sim -equivalence classes of G . Suppose that $\{r_{[g]} \mid [g] \in S\} \subseteq R$ is a C -linearly independent set. Then defining $\delta : G \rightarrow R$ by*

$$\delta(g) = \begin{cases} r_{[g]} & \text{if } g \not\sim 0 \\ 0 & \text{if } g \sim 0 \end{cases},$$

gives a central map, and $t_\delta : \overline{CG} \rightarrow R$ is a minimal C -linear trace.

Proof. For any $g, h \in G$, we have $gh \sim hg$, and hence $\delta(gh) = r_{[gh]} = r_{[hg]} = \delta(hg)$. Thus δ is a central map that preserves zero, and by Proposition 3, $t_\delta : \overline{CG} \rightarrow R$ is a C -linear trace. To show that it is minimal, first note that if $h \sim f$ for some distinct $h, f \in G$, then we can find $a_1, b_1, a_2, b_2, \dots, a_n, b_n \in G$ such that

$$h = a_1 b_1, \quad b_1 a_1 = a_2 b_2, \quad b_2 a_2 = a_3 b_3, \dots, \quad b_{n-1} a_{n-1} = a_n b_n, \quad b_n a_n = f,$$

and therefore (using the fact that C is unital)

$$h - f = a_1 b_1 - b_1 a_1 + a_2 b_2 - b_2 a_2 + \dots + a_n b_n - b_n a_n \in [\overline{CG}, \overline{CG}].$$

In particular, this implies that for all $c_h \in C$,

$$\sum_{h \in [0]} c_h h = \sum_{h \in [0]} c_h (h - 0) \in [\overline{CG}, \overline{CG}].$$

To conclude the proof, let us suppose that $t_\delta(x) = 0$ for some $x \in \overline{CG}$, and show that $x \in [\overline{CG}, \overline{CG}]$. We can write $x = \sum_{[g] \in S \cup \{[0]\}} \sum_{h \in [g]} c_h h$ for some $c_h \in C$. Since

$$0 = t_\delta(x) = \sum_{[g] \in S \cup \{[0]\}} \sum_{h \in [g]} c_h \delta(h) = \sum_{[g] \in S} \sum_{h \in [g]} c_h r_{[g]} = \sum_{[g] \in S} \left(\sum_{h \in [g]} c_h \right) r_{[g]},$$

and since the $r_{[g]}$ are C -linearly independent, we see that $\sum_{h \in [g]} c_h = 0$ for all $[g] \in S$. It is enough to show that $\sum_{h \in [g]} c_h h \in [\overline{CG}, \overline{CG}]$ for each $[g] \in S \cup \{[0]\}$ such that $c_h \neq 0$ for some $h \in [g]$. Further, by the previous paragraph, we may assume that $[g] \neq [0]$. Thus, let $[g] \in S$ and assume that $c_f \neq 0$ for some $f \in [g]$. Then $c_f = -\sum_{h \in [g] \setminus \{f\}} c_h$, and hence

$$\sum_{h \in [g]} c_h h = \sum_{h \in [g] \setminus \{f\}} c_h h - \left(\sum_{h \in [g] \setminus \{f\}} c_h \right) f = \sum_{h \in [g] \setminus \{f\}} c_h (h - f).$$

Since, as noted above, $h - f \in [\overline{CG}, \overline{CG}]$ for each $h \in [g] \setminus \{f\}$, we have $\sum_{h \in [g]} c_h h \in [\overline{CG}, \overline{CG}]$, as desired. \square

Given a ring C and a set S , we denote by $C^{(S)}$ the direct sum of $|S|$ copies of C , indexed by the elements of S . If $S = \emptyset$, then we understand $C^{(S)}$ to be the zero ring.

We are now ready for our main result about minimal traces.

Theorem 11. *Let C be a commutative unital ring, G a semigroup, and S the set of all nonzero \sim -equivalence classes of G . Then the following hold.*

- (1) *There is a minimal C -linear trace $t : \overline{CG} \rightarrow C^{(S)}$.*
- (2) *If R is a C -algebra, and $t : \overline{CG} \rightarrow R$ is a C -linear trace, then t is minimal if and only if for all $g_1, \dots, g_n \in G$ such that $[g_1], \dots, [g_n] \in S$ are distinct, $t(g_1), \dots, t(g_n)$ are C -linearly independent in R .*
- (3) *There is a normalized minimal C -linear trace $t : \overline{CG} \rightarrow R$ for some unital C -algebra R if and only if $|S| \leq 1$.*

Proof. To show (1), define $\delta : G \rightarrow C^{(S)}$ by

$$\delta(g) = \begin{cases} e_{[g]} & \text{if } g \not\sim 0 \\ 0 & \text{if } g \sim 0 \end{cases},$$

where $e_{[g]} \in C^{(S)}$ denotes the element with 1 as the entry in the coordinate indexed by $[g]$ and zeros elsewhere. The set $\{e_{[g]} \mid [g] \in S\}$ is clearly C -linearly independent, and hence, by Lemma 10, $t_\delta : \overline{CG} \rightarrow C^{(S)}$ is a minimal C -linear trace.

For (2), let $t : \overline{CG} \rightarrow R$ be a C -linear trace. Suppose that $g_1, \dots, g_n \in G$ are such that $[g_1], \dots, [g_n] \in S$ are distinct, but $c_1 t(g_1) + \dots + c_n t(g_n) = 0$ for some $c_1, \dots, c_n \in C$, not all of which are zero. Then $t(c_1 g_1 + \dots + c_n g_n) = 0$, but

$$t_\delta(c_1 g_1 + \dots + c_n g_n) = c_1 e_{[g_1]} + \dots + c_n e_{[g_n]} \neq 0$$

(where t_δ is the map from the previous paragraph), and hence $c_1 g_1 + \dots + c_n g_n \notin [\overline{CG}, \overline{CG}]$, showing that t is not minimal.

Conversely, suppose that for all $g_1, \dots, g_n \in G$ such that $[g_1], \dots, [g_n] \in S$ are distinct, $t(g_1), \dots, t(g_n)$ are C -linearly independent in R . By Proposition 3, we have $t = t_\delta$, where $\delta : G \rightarrow R$ is the restriction of t to G . Set $r_{[g]} := \delta([g])$ for all $g \in G$ such that $g \not\sim 0$. (Since δ is central, it must agree on all elements of a \sim -equivalence class.) Then $\{r_{[g]} \mid [g] \in S\}$ is a C -linearly independent set, by hypothesis, and therefore it follows from Lemma 10 that t is minimal.

For (3), we note that if $t : \overline{CG} \rightarrow R$ is a minimal C -linear trace that is normalized on G , then $|t(G)| \leq 2$. Hence, by (2), there can be at most one nonzero \sim -equivalence class in G (i.e., $|S| \leq 1$). Conversely, if $|S| \leq 1$, then defining $\delta : G \rightarrow C$ by

$$\delta(g) = \begin{cases} 1 & \text{if } g \not\sim 0 \\ 0 & \text{if } g \sim 0 \end{cases}$$

gives a minimal C -linear trace $t_\delta : \overline{CG} \rightarrow C$, by Lemma 10, which is clearly normalized. \square

Let us next give several consequences of the above theorem, along with related observations.

Corollary 12. *Let C be a commutative unital ring, R a nonzero C -algebra, and G a semi-group. Then the following are equivalent.*

- (1) *The only central map $\delta : G \rightarrow R$ that preserves zero is the zero map.*
- (2) *For every $f \in G$ we have $f \sim 0$.*
- (3) $\overline{CG} = [\overline{CG}, \overline{CG}]$.
- (4) *The only trace $t : \overline{CG} \rightarrow R$ is the zero map.*
- (5) *The only C -linear trace $t : \overline{CG} \rightarrow R$ is the zero map.*

Proof. (1) \Rightarrow (2) Let $X \subseteq G$ consist of all the elements $f \in G$ such that $f \not\sim 0$, and let $r \in R \setminus \{0\}$. If $X \neq \emptyset$, then

$$\delta(f) = \begin{cases} r & \text{if } f \in X \\ 0 & \text{if } f \notin X \end{cases}$$

defines a nonzero map $\delta : G \rightarrow R$ that preserves zero. Moreover, δ is central, since for all $g, h \in G$ we have $gh \sim hg$, and hence

$$\delta(gh) = 0 \iff gh \sim 0 \iff hg \sim 0 \iff \delta(hg) = 0.$$

Thus, if (1) holds, then $X = \emptyset$, and therefore (2) must also hold.

(2) \Rightarrow (3) If for every $f \in G$ we have $f \sim 0$, then by Theorem 11(1), there is a minimal trace from \overline{CG} to the zero ring, which can only happen if $\overline{CG} = [\overline{CG}, \overline{CG}]$.

(3) \Rightarrow (4) Clearly, if every element of \overline{CG} is a sum of commutators, then any trace on \overline{CG} must send every element to zero.

(4) \Rightarrow (5) This is a tautology.

(5) \Rightarrow (1) This follows from Proposition 3. \square

In the following example we construct a nonzero semigroup satisfying statement (2) of Corollary 12. We obtain it by modifying the semigroups G in Examples 6 and 7.

Example 13. Let $G = \{e_{ij} \mid i, j \in \mathbb{R}, i < j\} \cup \{0\}$, where \mathbb{R} is the set of the real numbers (though, any dense well-ordered set would do in place of \mathbb{R}). Define multiplication on G by

$$e_{ij} \cdot e_{kl} = \begin{cases} e_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

and $0 \cdot e_{ij} = 0 = e_{ij} \cdot 0$ for all $i, j, k, l \in \mathbb{R}$. It is easy to see that this operation is associative, and hence that (G, \cdot) is a semigroup. Now, let $e_{ij} \in G \setminus \{0\}$ be any element, and let $k \in \mathbb{R}$ be any number such that $i < k < j$. Then $e_{ij} = e_{ik}e_{kj}$, but $e_{kj}e_{ik} = 0$. Hence $e_{ij} \sim 0$, implying that all $f \in G$ satisfy $f \sim 0$.

By Corollary 12, if C is any commutative unital ring and G is the semigroup constructed above, then $\overline{CG} = [\overline{CG}, \overline{CG}]$. For other examples of rings R satisfying $R = [R, R]$ see [11, 12].

Using Theorem 11 we can strengthen the well-known fact that the usual trace on a matrix ring is minimal. See Example 7 for the definition of $\mathbb{M}_\kappa(C)$, when κ is infinite.

Corollary 14. *Let C be a commutative unital ring, κ a nonzero cardinal, and $\text{tr} : \mathbb{M}_\kappa(C) \rightarrow C$ the usual trace. Then a C -linear map $t : \mathbb{M}_\kappa(C) \rightarrow C$ is a trace if and only if there exists $c \in C$ such that $t(M) = c \cdot \text{tr}(M)$ for all $M \in \mathbb{M}_\kappa(C)$. In this case, t is minimal if and only if c is not a zero divisor.*

Proof. Let $G = \{e_{ij} \mid i, j \in \kappa\} \cup \{0\}$, where e_{ij} are the matrix units. Then $\mathbb{M}_\kappa(C)$ is isomorphic to the contracted semigroup ring \overline{CG} , as noted in Examples 6 and 7. Letting $i, j \in \kappa$, we see that if $i \neq j$, then $e_{ij} = e_{ij}e_{jj}$ and $e_{jj}e_{ij} = 0$, while $e_{ii} = e_{ij}e_{ji}$ and $e_{ji}e_{ij} = e_{jj}$. It follows that G has only one nonzero \sim -equivalence class, namely $\{e_{ii} \mid i \in \kappa\}$. Thus, for any central map $\delta : G \rightarrow C$ that preserves zero there must be some $c \in C$ such that

$$\delta(e_{ij}) = \begin{cases} c & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

Noting that for such δ we have $t_\delta = c \cdot \text{tr}$, the first claim now follows from Proposition 3.

The second claim follows from Theorem 11(2), since $\{t(e_{ii})\} = \{c\}$ is C -linearly independent if and only if $ac \neq 0$ for all $a \in C \setminus \{0\}$, i.e., c is not a zero divisor. \square

Next, let us turn to traces on group rings. By Theorem 11, for any group G and commutative unital ring C , CG admits a minimal C -linear trace. However, the usual traces on CG (see Examples 4 and 5) do not have this property, as the next result shows.

Corollary 15. *Let C be a commutative unital ring and G a nontrivial group. Then there are no minimal C -linear traces $t : CG \rightarrow C$.*

Proof. As noted in Section 2, we may identify CG with $\overline{CG^0}$, where G^0 is the semigroup obtained from G by adjoining a zero element. By Theorem 11(2), if there is a minimal

C -linear trace $t : \overline{CG}^0 \rightarrow C$, then there must be only one \sim -equivalence class of elements of G . But, the \sim -equivalence class of the identity e consists of just one element, and hence this can happen only if $G = \{e\}$. \square

We conclude this section by noting that in certain situations our theory of minimal traces on semigroup rings extends to quotients of such rings.

Proposition 16. *Let C be a commutative ring, R a C -algebra, G a semigroup, $\delta : G \rightarrow R$ a central map that preserves zero, and $I \subseteq \overline{CG}$ an ideal. If $t_\delta(I) = 0$, then $\bar{t}_\delta(r + I) = t_\delta(r)$ defines a C -linear trace $\bar{t}_\delta : \overline{CG}/I \rightarrow R$. Moreover, t_δ is minimal if and only if \bar{t}_δ is minimal and $I \subseteq [\overline{CG}, \overline{CG}]$.*

Proof. Since $t_\delta(I) = 0$, \bar{t}_δ is well-defined, and it is routine to verify that \bar{t}_δ is a C -linear trace. For instance, for all $x, y \in \overline{CG}$ we have

$$\bar{t}_\delta((x + I)(y + I)) = \bar{t}_\delta(xy + I) = t_\delta(xy) = t_\delta(yx) = \bar{t}_\delta(yx + I) = \bar{t}_\delta((y + I)(x + I)).$$

Now, suppose that t_δ is minimal. Since $t_\delta(I) = 0$, we must have $I \subseteq [\overline{CG}, \overline{CG}]$. Also, if $\bar{t}_\delta(x + I) = 0$ for some $x \in \overline{CG}$, then $t_\delta(x) = 0$, and hence $x \in [\overline{CG}, \overline{CG}]$. Thus, $x + I \in [\overline{CG}/I, \overline{CG}/I]$, implying that \bar{t}_δ is minimal. Conversely, suppose that \bar{t}_δ is minimal and $I \subseteq [\overline{CG}, \overline{CG}]$. If $t_\delta(x) = 0$ for some $x \in \overline{CG}$, then $\bar{t}_\delta(x + I) = 0$, and hence $x + I \in [\overline{CG}/I, \overline{CG}/I]$. Since $I \subseteq [\overline{CG}, \overline{CG}]$, this implies that $x \in [\overline{CG}, \overline{CG}]$. Therefore, t_δ is minimal. \square

4. GRAPH INVERSE SEMIGROUPS

In this section we describe the central maps and \sim -equivalence classes in certain semigroups arising from graphs. In addition to supplying further examples to which Proposition 3 and Theorem 11 can be applied, this will help us study traces on Leavitt path algebras in subsequent sections. We begin by recalling some notions pertaining to semigroups and graphs.

A semigroup S is an *inverse semigroup* if for each $x \in S$ there is a unique element $x^{-1} \in S$, called the *semigroup inverse* of x , satisfying $x = xx^{-1}x$ and $x^{-1} = x^{-1}xx^{-1}$.

A *directed graph* $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ consists of two sets E^0, E^1 (the elements of which are called *vertices* and *edges*, respectively), together with functions $\mathbf{s}, \mathbf{r} : E^1 \rightarrow E^0$, called *source* and *range*, respectively. We shall refer to directed graphs as simply “graphs” from now on. A *path* p in E is a finite sequence $e_1 \cdots e_n$ of (not necessarily distinct) edges $e_1, \dots, e_n \in E^1$ such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $i \in \{1, \dots, n-1\}$. Here we define $\mathbf{s}(p) := \mathbf{s}(e_1)$ to be the *source* of p , $\mathbf{r}(p) := \mathbf{r}(e_n)$ to be the *range* of p , and n to be the *length* of p . We view the elements of E^0 as paths of length 0, and denote by $\text{Path}(E)$ the set of all paths in E . A path $p = e_1 \cdots e_n$ is said to be *closed* if $\mathbf{s}(p) = \mathbf{r}(p)$. Such a path is said to be a *cycle* if in addition $\mathbf{s}(e_i) \neq \mathbf{s}(e_j)$ for every $i \neq j$. A cycle consisting of just one edge is called a *loop*. We denote the set of all closed paths in $\text{Path}(E)$ by $\text{ClPath}(E)$. A graph which contains no cycles (other than vertices) is called *acyclic*. An edge $e \in E^1$ is an *exit* for a path $p = e_1 \cdots e_n$ if $\mathbf{s}(e) = \mathbf{s}(e_i)$ and $e \neq e_i$ for some $i \in \{1, \dots, n\}$. A graph where no cycle has an exit is called a *no-exit graph*.

A vertex $v \in E^0$ for which the set $\mathbf{s}^{-1}(v) = \{e \in E^1 \mid \mathbf{s}(e) = v\}$ is finite is said to have *finite out-degree*. A graph E is said to have *finite out-degree*, or to be *row-finite*, if every vertex of E has finite out-degree. A vertex $v \in E^0$ which is the source of no edges of E is called a *sink*, while a vertex having finite out-degree which is not a sink is called *regular*.

A graph E where both E^0 and E^1 are finite, respectively countable, sets is called a *finite*, respectively *countable*, graph.

Definition 17. Given a graph $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$, the graph inverse semigroup G_E of E is the semigroup with zero generated by the sets E^0 and E^1 , together with a set of variables $\{e^* \mid e \in E^1\}$, satisfying the following relations for all $v, w \in E^0$ and $e, f \in E^1$:

- (V) $vw = \delta_{v,w}v$,
- (E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,
- (E2) $\mathbf{r}(e)e^* = e^*\mathbf{s}(e) = e^*$,
- (CK1) $e^*f = \delta_{e,f}\mathbf{r}(e)$.

For all $v \in E^0$ we define $v^* := v$, and for all paths $p = e_1 \cdots e_n$ ($e_1, \dots, e_n \in E^1$) we set $p^* := e_n^* \cdots e_1^*$, $\mathbf{r}(p^*) := \mathbf{s}(p)$, and $\mathbf{s}(p^*) := \mathbf{r}(p)$. With this notation, every nonzero element of G_E can be written uniquely as pq^* for some $p, q \in \text{Path}(E)$, by the CK1 relation, and multiplication of two such elements pq^* and rs^* is given by

$$pq^*rs^* = \begin{cases} pts^* & \text{if } r = qt \text{ for some } t \in \text{Path}(E) \\ pt^*s^* & \text{if } q = rt \text{ for some } t \in \text{Path}(E) \\ 0 & \text{otherwise} \end{cases}.$$

It is easy to verify that G_E is indeed an inverse semigroup, where the semigroup inverse of a nonzero element pq^* is qp^* . We also note that for any graph E , extending \mathbf{s} and \mathbf{r} to $E^1 \cup \{e^* \mid e \in E^1\}$ as above turns $(E^0, E^1 \cup \{e^* \mid e \in E^1\}, \mathbf{s}, \mathbf{r})$ into a (directed) graph, which is called the *extended graph of E* .

Graph inverse semigroups were first introduced in [7], in order to show that every partially ordered set can be realized as the partially ordered set of nonzero \mathcal{J} -classes of an inverse semigroup. (Two elements in a semigroup are \mathcal{J} -equivalent if they generate the same ideal.) This class of semigroups has since been studied in its own right (e.g., [9, 10, 15]).

The following notation will help us classify all central maps on graph inverse semigroups that preserve zero (and hence also all linear traces on contracted semigroup rings arising from them, by Proposition 3).

Definition 18. Let E be a graph and $p, q \in \text{ClPath}(E)$. We write $p \approx q$ if there exist paths $x, y \in \text{Path}(E)$ such that $p = xy$ and $yx = q$.

It is shown in [12, Lemma 12] that \approx is an equivalence relation. Observe that if $q = e_1 \cdots e_n \in \text{ClPath}(E) \setminus E^0$ for some $e_1, \dots, e_n \in E^1$, then the \approx -equivalence class of q consists of $e_1 \cdots e_n, e_2 \cdots e_n e_1, \dots, e_n e_1 \cdots e_{n-1}$. The \approx -equivalence class of a vertex consists of just one element. We shall discuss the connection between \approx and the relation \sim on G_E from Definition 8 in Proposition 20 below.

Proposition 19. Let R be a ring, E a graph, and $\delta : G_E \rightarrow R$ a map that preserves zero. Then δ is central if and only if the following conditions hold.

- (1) If $\delta(x) \neq 0$ for some $x \in G_E$, then either $x = pqp^*$ or $x = pq^*p^*$ for some $p \in \text{Path}(E)$ and $q \in \text{ClPath}(E)$.
- (2) For all $p \in \text{Path}(E)$ and $q \in \text{ClPath}(E)$, we have $\delta(pqp^*) = \delta(q)$ and $\delta(pq^*p^*) = \delta(q^*)$.
- (3) For all $p, q \in \text{ClPath}(E)$ such that $p \approx q$, we have $\delta(p) = \delta(q)$ and $\delta(p^*) = \delta(q^*)$.

Proof. Suppose that δ is central, and let $p \in \text{Path}(E)$ be such that $\mathbf{s}(p) \neq \mathbf{r}(p)$. Then $\mathbf{s}(p)p = p$, $p\mathbf{s}(p) = 0$, $p^*\mathbf{s}(p) = p^*$, and $\mathbf{s}(p)p^* = 0$. We therefore must have $\delta(p) = 0 = \delta(p^*)$.

Now, let $p, s \in \text{Path}(E)$ be such that $\delta(ps^*) \neq 0$. Then $s^*p \neq 0$, and hence either $s = pq$ or $p = sq$ for some $q \in \text{Path}(E)$. If $s = pq$, then

$$0 \neq \delta(ps^*) = \delta(pq^*p^*) = \delta(q^*p^*p) = \delta(q^*),$$

and hence $\mathbf{s}(q) = \mathbf{r}(q)$, by the previous paragraph. If $p = sq$, then

$$0 \neq \delta(ps^*) = \delta(sqs^*) = \delta(s^*sq) = \delta(q),$$

and hence $\mathbf{s}(q) = \mathbf{r}(q)$, as before. In either case, q is a closed path, and therefore $x = ps^*$ has the form described in (1). Clearly, δ being central implies (2) and (3).

Conversely, suppose that (1), (2), and (3) hold. Let $p, q, r, s \in \text{Path}(E)$, and suppose that $\delta(pq^*rs^*) \neq 0$. We wish to show that $\delta(pq^*rs^*) = \delta(rs^*pq^*)$. By (1), $pq^*rs^* = ghg^*$ or $pq^*rs^* = gh^*g^*$ for some $g \in \text{Path}(E)$ and $h \in \text{ClPath}(E)$. Let us assume that $pq^*rs^* = ghg^*$, since the other situation can be handled analogously. Thus, there must be some $v \in \text{Path}(E)$ such that either $q = rv$ or $r = qv$. In the first case,

$$ghg^* = pq^*rs^* = pv^*r^*rs^* = pv^*s^*,$$

which implies that $g = sv$ and $p = gh = svh$. Hence using (2), we have

$$\begin{aligned} \delta(pq^*rs^*) &= \delta(svhv^*r^*rs^*) = \delta(svhv^*s^*) = \delta(h) \\ &= \delta(rvhv^*r^*) = \delta(rs^*svhv^*r^*) = \delta(rs^*pq^*). \end{aligned}$$

We may therefore suppose that $r = qv$. Then

$$ghg^* = pq^*rs^* = pq^*qvs^* = pvs^*,$$

which implies that $s = g$ and $pv = gh = sh$. Thus there must be some $u \in \text{Path}(E)$ such that either $p = su$ or $s = pu$. In the first case $h = uv$, and hence, using (2) and (3), we have

$$\begin{aligned} \delta(pq^*rs^*) &= \delta(suq^*qvs^*) = \delta(suvs^*) = \delta(uv) \\ &= \delta(vu) = \delta(qvuq^*) = \delta(qvs^*suq^*) = \delta(rs^*pq^*). \end{aligned}$$

Let us therefore assume that $r = qv$ and $s = pu$. Then

$$ghg^* = pq^*rs^* = pq^*qvu^*p^* = pvu^*p^*,$$

which implies that $g = pu$ and therefore $v = uh$. Hence, again using (2), we have

$$\begin{aligned} \delta(pq^*rs^*) &= \delta(pvu^*p^*) = \delta(puhu^*p^*) = \delta(h) \\ &= \delta(quhu^*q^*) = \delta(qvu^*q^*) = \delta(qvu^*p^*pq^*) = \delta(rs^*pq^*). \end{aligned}$$

Thus if $\delta(pq^*rs^*) \neq 0$, then $\delta(pq^*rs^*) = \delta(rs^*pq^*)$. By symmetry, it also follows that if $\delta(rs^*pq^*) \neq 0$, then $\delta(rs^*pq^*) = \delta(pq^*rs^*)$. Hence, $\delta(pq^*rs^*) = \delta(rs^*pq^*)$ for all values of $\delta(pq^*rs^*)$ and $\delta(rs^*pq^*)$, and therefore δ is central. \square

Proposition 20. *Let E be a graph, and for each $q \in \text{ClPath}(E)$ set*

$$EQ(q) := \{ptp^* \mid p \in \text{Path}(E), t \in \text{ClPath}(E), \mathbf{r}(p) = \mathbf{s}(t), t \approx q\} \text{ and}$$

$$EQ(q^*) := \{pt^*p^* \mid p \in \text{Path}(E), t \in \text{ClPath}(E), \mathbf{r}(p) = \mathbf{r}(t), t \approx q\}.$$

Then every nonzero \sim -equivalence class of G_E (in the sense of Definition 8) is of the form $EQ(q)$ or $EQ(q^)$ for some $q \in \text{ClPath}(E)$.*

In particular, for all $q_1, q_2 \in \text{ClPath}(E)$ we have $EQ(q_1) \cap EQ(q_2) \neq \emptyset$ if and only if $q_1 \approx q_2$ if and only if $EQ(q_1^) \cap EQ(q_2^*) \neq \emptyset$, and $EQ(q_1) \cap EQ(q_2^*) \neq \emptyset$ if and only if $q_1 = q_2 \in E^0$.*

Proof. Let $q \in \text{ClPath}(E)$ and $p_1 t_1 p_1^*, p_2 t_2 p_2^* \in EQ(q)$ be any elements. Since $t_1 \approx q \approx t_2$, there exist $x, y \in \text{Path}(E)$ such that $t_1 = xy$ and $t_2 = yx$. Hence

$$p_1 t_1 p_1^* \sim p_1^* p_1 t_1 = t_1 = xy \sim yx = t_2 = p_2^* p_2 t_2 \sim p_2 t_2 p_2^*,$$

and therefore all elements of $EQ(q)$ are \sim -equivalent. A similar argument shows the analogous statement for $EQ(q^*)$.

Next, let $x \in G_E$ be any element such that $x \not\sim 0$. We wish to show that $x \in EQ(q) \cup EQ(q^*)$ for some $q \in \text{ClPath}(E)$. Let C be any commutative unital ring. Then, by Theorem 11(1), there is a minimal C -linear trace $t : \overline{CG_E} \rightarrow R$, for some C -algebra R . By Proposition 3, $t = t_\delta$ for some central map $\delta : G_E \rightarrow R$ that preserves zero. Since t is minimal and $x \not\sim 0$, by Theorem 11(2), $\delta(x) \neq 0$. Hence, by Proposition 19, $x = pqp^*$ or $x = pq^*p^*$ for some $p \in \text{Path}(E)$ and $q \in \text{ClPath}(E)$. Therefore either $x \in EQ(q)$ or $x \in EQ(q^*)$.

To conclude the proof of the first claim it now suffices to show that for any $q \in \text{ClPath}(E)$ and $x \in G_E$, $x \sim q$ implies that $x \in EQ(q)$, while $x \sim q^*$ (and $q \notin E^0$) implies that $x \in EQ(q^*)$. Let C be any commutative unital ring, let S be the set of \approx -equivalence classes of $\text{ClPath}(E)$, and let S^* be another copy of S . Define $\delta : G_E \rightarrow C^{(S \cup S^*)}$ by

$$\delta(y) = \begin{cases} \epsilon_{(q)} & \text{if } y \in EQ(q) \text{ for some } q \in \text{ClPath}(E) \\ \epsilon_{(q)^*} & \text{if } y \in EQ(q^*) \text{ for some } q \in \text{ClPath}(E) \setminus E^0 \\ 0 & \text{otherwise} \end{cases},$$

where $\epsilon_{(q)} \in C^{(S \cup S^*)}$ denotes the element with $1 \in C$ in the coordinate indexed by the \approx -equivalence class of q in S and zeros elsewhere, while $\epsilon_{(q)^*} \in C^{(S \cup S^*)}$ denotes the element with $1 \in C$ in the coordinate indexed by the \approx -equivalence class of q in S^* and zeros elsewhere. We shall show below that δ is well defined, but assuming for the moment that this is established, by Proposition 19, δ is a central map. It follows that if $x \sim q$, then $\delta(x) = \delta(q) = \epsilon_{(q)}$, and hence $x \in EQ(q)$. Similarly, if $x \sim q^*$ and $q \notin E^0$, then $\delta(x) = \delta(q^*) = \epsilon_{(q)^*}$, and hence $x \in EQ(q^*)$.

To show that the map δ is well defined, suppose that $y \in G_E$ belongs to $EQ(q_1)$ and to $EQ(q_2)$ for some $q_1, q_2 \in \text{ClPath}(E)$. Then $y = ptp^*$ for some $p \in \text{Path}(E)$ and $t \in \text{ClPath}(E)$, where $t \approx q_1$ and $t \approx q_2$. It follows that $q_1 \approx q_2$, and therefore $EQ(q_1) = EQ(q_2)$, showing that the assignment $\delta(y) = \epsilon_{(q_1)}$ is unambiguous. Similarly, if $y \in G_E$ belongs to $EQ(q_1^*)$ and to $EQ(q_2^*)$ for some $q_1, q_2 \in \text{ClPath}(E)$, then $q_1 \approx q_2$, and therefore $\delta(y) = \epsilon_{(q_1)^*}$ is well defined. Finally, if $y \in G_E$ belongs to $EQ(q_1)$ and to $EQ(q_2^*)$ for some $q_1, q_2 \in \text{ClPath}(E)$, then it must be the case that $q_1 = q_2$ is a vertex. We have excluded E^0 in the second case of our definition of δ for this reason, and hence again, $\delta(y) = \epsilon_{(q_1)}$ is well defined.

In the course of showing that δ is well defined we have also proved one direction of each equivalence in the final claim. The other direction of each equivalence is trivial. \square

Using Theorem 11 and Proposition 20 we obtain a complete description of the minimal linear traces on the contracted semigroup ring $\overline{CG_E}$, for any commutative unital ring C and graph E . In the case where C is a field, $\overline{CG_E}$ is known as a *Cohn path algebra*.

Corollary 21. *Let C be a commutative unital ring, R a C -algebra, E a graph, and $t : \overline{CG_E} \rightarrow R$ a C -linear trace. Then t is minimal if and only if for all $q_1, \dots, q_n \in \text{ClPath}(E)$ and $p_1, \dots, p_m \in \text{ClPath}(E) \setminus E^0$, such that $q_i \not\approx q_j$ and $p_i \not\approx p_j$ for $i \neq j$, the elements $t(q_1), \dots, t(q_n), t(p_1^*), \dots, t(p_m^*)$ are C -linearly independent in R .*

Proof. We shall show that the following two statements are equivalent, from which the desired conclusion will follow, by Theorem 11(2).

- (1) For all $g_1, \dots, g_l \in G_E$, such that $[g_1], \dots, [g_l]$ are distinct nonzero \sim -equivalence classes of G_E , the elements $t(g_1), \dots, t(g_l)$ are C -linearly independent in R .
- (2) For all $q_1, \dots, q_n \in \text{ClPath}(E)$ and $p_1, \dots, p_m \in \text{ClPath}(E) \setminus E^0$, such that $q_i \not\approx q_j$ and $p_i \not\approx p_j$ for $i \neq j$, the elements $t(q_1), \dots, t(q_n), t(p_1^*), \dots, t(p_m^*)$ are C -linearly independent in R .

Suppose that (1) holds, and let $q_1, \dots, q_n, p_1, \dots, p_m$ be as in (2). By Proposition 20, the elements $q_1, \dots, q_n, p_1^*, \dots, p_m^*$ fall into distinct nonzero \sim -equivalence classes of G_E . Hence $t(q_1), \dots, t(q_n), t(p_1^*), \dots, t(p_m^*)$ must be C -linearly independent, by (1), and therefore (2) holds.

Conversely, suppose that (2) holds, and let $g_1, \dots, g_l \in G_E$ be such that $[g_1], \dots, [g_l]$ are distinct and nonzero. By Proposition 20, for each g_i we can find some $q \in \text{ClPath}(E)$ such that $g_i \sim q$, or some $p \in \text{ClPath}(E) \setminus E^0$ such that $g_i \sim p^*$. Let us index the elements q and p selected as $q_1, \dots, q_n, p_1, \dots, p_m$. Since $[g_1], \dots, [g_l]$ are distinct, we must have $q_i \not\approx q_j$ and $p_i \not\approx p_j$ for $i \neq j$. Then, by (2), the elements $t(q_1), \dots, t(q_n), t(p_1^*), \dots, t(p_m^*)$ are C -linearly independent. Since t is central, it must take the same value on all elements of a \sim -equivalence class, and hence $t(g_1), \dots, t(g_l)$ must be C -linearly independent as well, proving (1). \square

5. INVOLUTIONS AND FAITHFUL TRACES

For the rest of this paper we shall be interested in traces on rings with involutions. Let us recall the relevant notions.

Definition 22. *Let R be a ring. A map $*$: $R \rightarrow R$ (written using superscript notation) is an involution if $(x^*)^* = x$, $(x + y)^* = x^* + y^*$, and $(xy)^* = y^*x^*$ for all $x, y \in R$. If R has an involution, then it is said to be a $*$ -ring.*

If R is a $$ -ring and $x \in R$ is a sum of elements of the form yy^* ($y \in R$), then x is referred to as a positive element, and this is denoted by $x \geq 0$, or by $x > 0$, if $x \neq 0$.*

An involution $$ on R is positive definite if for all $x_1, \dots, x_n \in R$, $\sum_{i=1}^n x_i x_i^* = 0$ implies that each $x_i = 0$.*

If $f : R \rightarrow T$ is a homomorphism of $$ -rings, then f is a $*$ -homomorphism or an involution homomorphism provided that $f(x^*) = f(x)^*$ for all $x \in R$. A $*$ -isomorphism, or an involution isomorphism, is defined analogously.*

It is well known (see e.g., [8, Section 50]) and easy to see that extending \geq to all elements of R via

$$x \geq y \text{ if and only if } x - y \geq 0$$

gives a reflexive and transitive relation.

Definition 23. *Let R and T be $*$ -rings, and let $t : R \rightarrow T$ be a trace. We say that t is positive in case $t(x) \geq 0$ for all $x \in R$ with $x \geq 0$, and that t is faithful in case $t(x) > 0$ for all $x \in R$ with $x > 0$.*

Our primary interest henceforth will be in Leavitt path algebras, which constitute a particular class of quotients of semigroup rings. However, we conclude this section with several preliminary results of a more general nature, which will be required later. These are all easy to show and are mostly part of the folklore. We therefore only sketch the proofs.

Given $*$ -rings T_i ($i \in I$), we understand the direct sum $\bigoplus_{i \in I} T_i$ to have the corresponding component-wise involution, where $(a_i)_{i \in I}^* = (a_i^*)_{i \in I}$ for every $(a_i)_{i \in I} \in \bigoplus_{i \in I} T_i$.

Lemma 24. *Let C be a commutative ring, and let P, R, S be C -algebras with involutions.*

- (1) *If $t : P \rightarrow R$ and $u : R \rightarrow S$ are C -linear positive (respectively, faithful) traces, then so is $u \circ t : P \rightarrow S$.*
- (2) *If $\{T_i\}_{i \in I}$ is a collection of commutative C -subalgebras of R , and the involution $*$ on R is positive definite, then $t((x_i)_{i \in I}) = \sum_{i \in I} x_i$ defines a faithful C -linear trace $t : \bigoplus_{i \in I} T_i \rightarrow R$.*

Proof. The verification of (1) is routine.

Given t as in (2), it is clear that t is an additive C -linear map, and since each T_i is commutative, t is a trace. It is also positive, since for any $(x_i)_{i \in I} \in \bigoplus_{i \in I} T_i$, we have

$$t((x_i)_{i \in I}(x_i)_{i \in I}^*) = t((x_i x_i^*)_{i \in I}) = \sum_{i \in I} x_i x_i^*.$$

That t is faithful follows from the fact that the involution on R is positive definite. \square

Lemma 25. *Let C be a commutative ring with involution $*$, κ an arbitrary nonzero cardinal, and $\text{tr} : \mathbb{M}_\kappa(C) \rightarrow C$ the usual trace. Then $(a_{ij})^* := (a_{ji}^*)$ ($(a_{ij}) \in \mathbb{M}_\kappa(C)$) defines an involution on $\mathbb{M}_\kappa(C)$, with respect to which tr is positive. Also, if $*$ is positive definite on C , then tr is faithful.*

Proof. The fact that $*$ extends to an involution on $\mathbb{M}_\kappa(C)$ follows immediately from properties of $*$ on C and the transpose operation on $\mathbb{M}_\kappa(C)$. Also, for any $A = (a_{ij}) \in \mathbb{M}_\kappa(C)$ we have $\text{tr}(AA^*) = \sum_i \sum_j a_{ij} a_{ij}^*$, from which we conclude that tr is positive. If $*$ is positive definite on C , then this formula also implies that tr is faithful. \square

Lemma 26. *Let C be a commutative ring with involution $*$.*

- (1) *Defining $(\sum_i a_i x^i)^* := \sum_i a_i^* x^{-i}$ ($a_i \in C$) gives an involution on $C[x, x^{-1}]$.*
- (2) *If $*$ is positive definite on C , then its extension to $C[x, x^{-1}]$ in (1) is positive definite as well.*
- (3) *If $*$ is positive definite on C , then defining $t : C[x, x^{-1}] \rightarrow C$ by $t(\sum_i a_i x^i) = a_0$ ($a_i \in C$) gives a faithful C -linear trace.*

Proof. Checking that $*$ extends to an involution on $C[x, x^{-1}]$ as in (1) is routine.

To show (2), suppose that $*$ is positive definite on C , let $r_1, \dots, r_n \in C[x, x^{-1}]$, and suppose that $\sum_{i=1}^n r_i r_i^* = 0$. Write each r_i as $r_i = \sum_j a_{ij} x^j$ for some $a_{ij} \in C$. Then

$$0 = \sum_{i=1}^n r_i r_i^* = \sum_{i=1}^n \left(\sum_j a_{ij} x^j \right) \left(\sum_j a_{ij}^* x^{-j} \right) = \sum_{i=1}^n \left(\sum_j a_{ij} a_{ij}^* + f_i(x) \right),$$

where each $f_i(x) \in C[x, x^{-1}]$ is a polynomial with 0 as the degree-zero term. Thus, $0 = \sum_{i=1}^n \sum_j a_{ij} a_{ij}^*$, which implies that each $a_{ij} = 0$, since $*$ is positive definite on C . Therefore, each $r_i = 0$, showing that $*$ is positive definite on $C[x, x^{-1}]$.

For (3), suppose that $*$ is positive definite on C , and let t be as in the statement. Then clearly, t is a C -linear map, and since $C[x, x^{-1}]$ is commutative, it is a trace. The map t is also positive, since for any $r = \sum_i a_i x^i \in C[x, x^{-1}]$ we have

$$t(rr^*) = t\left(\left(\sum_i a_i x^i\right)\left(\sum_i a_i^* x^{-i}\right)\right) = t\left(\sum_i \sum_j a_i a_j^* x^{i-j}\right) = \sum_{i-j=0} a_i a_j^* = \sum_i a_i a_i^*.$$

The faithfulness of t follows from $*$ being positive definite on C . \square

6. TRACES ON LEAVITT PATH ALGEBRAS

In this section we describe all linear traces on Leavitt path algebras, and, under mild assumptions, the Leavitt path algebras that admit faithful traces. We begin by recalling the relevant definitions.

Definition 27. *Let K be a field, E a graph, and G_E the corresponding graph inverse semigroup. Then the contracted semigroup ring $\overline{KG_E}$ is known as the Cohn path K -algebra of E , and is usually denoted by $C_K(E)$.*

Let N be the ideal of $C_K(E)$ generated by all elements of the form $v - \sum_{e \in \mathbf{s}^{-1}(v)} ee^$, where $v \in E^0$ is a regular vertex. Then $C_K(E)/N$ is called the Leavitt path K -algebra of E , and is denoted by $L_K(E)$.*

We note that Leavitt path algebras are typically defined without reference to Cohn path algebras or graph inverse semigroups, as the K -algebras generated by the sets $\{v \mid v \in E^0\}$ and $\{e, e^* \mid e \in E^1\}$ (arising from a directed graph $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$), which satisfy the conditions (V), (E1), (E2), and (CK1) from Definition 17, along with $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$ for all regular $v \in E^0$ (e.g., [2, 3, 4, 5, 6]).

From our description of G_E in Section 4, it follows that every element of $C_K(E)$ and $L_K(E)$ (by slightly abusing notation) can be expressed in the form $\sum_{i=1}^n a_i p_i q_i^*$ for some $a_i \in K$ and $p_i, q_i \in \text{Path}(E)$. It is known (see [1, Theorem 1.5.17] or [16, Lemma 4.8]) that every Cohn path K -algebra is isomorphic to some Leavitt path K -algebra. We shall therefore restrict our attention to Leavitt path algebras from now on.

If K has an involution $*$, then $L_K(E)$ acquires one as well via $(\sum_{i=1}^n a_i p_i q_i^*)^* = \sum_{i=1}^n a_i^* q_i p_i^*$. It is shown in [6, Proposition 2.4] that if the involution on K is positive definite, then the same is true of the induced involution on $L_K(E)$.

Next, let us describe the linear traces on Leavitt path algebras. To do so we shall require [4, Lemma 9], which says that letting $N \subseteq C_K(E)$ be as in Definition 27, for every regular $v \in E^0$ and every $p \in \text{Path}(E) \setminus E^0$ we have

$$\left(v - \sum_{e \in \mathbf{s}^{-1}(v)} ee^*\right)p = 0 = p^* \left(v - \sum_{e \in \mathbf{s}^{-1}(v)} ee^*\right).$$

Theorem 28. *Let K be a field, R a K -algebra, and E a graph. Then there is a one-to-one correspondence between the K -linear traces $t : L_K(E) \rightarrow R$ and the maps $\delta : G_E \rightarrow R$ that preserve zero and satisfy the following conditions.*

- (1) *If $\delta(x) \neq 0$ for some $x \in G_E$, then either $x = pqp^*$ or $x = pq^*p^*$ for some $p \in \text{Path}(E)$ and $q \in \text{ClPath}(E)$.*
- (2) *For all $p \in \text{Path}(E)$ and $q \in \text{ClPath}(E)$, we have $\delta(pqp^*) = \delta(q)$ and $\delta(pq^*p^*) = \delta(q^*)$.*
- (3) *For all $p, q \in \text{ClPath}(E)$ such that $p \approx q$, we have $\delta(p) = \delta(q)$ and $\delta(p^*) = \delta(q^*)$. (See Definition 18 for the relation \approx .)*
- (4) *For all regular $v \in E^0$ we have $\delta(v) = \sum_{e \in \mathbf{s}^{-1}(v)} \delta(\mathbf{r}(e))$.*

The one-to-one correspondence maps K -linear traces on $L_K(E)$ to their restrictions to G_E .

Proof. Let $t : L_K(E) \rightarrow R$ be a K -linear trace, and let $\delta = t|_{G_E}$ be the restriction of t to G_E . Then δ clearly must preserve zero. Since t is central, δ must be central as well, and

hence must satisfy conditions (1)–(3), by Proposition 19. Also, if $v \in E^0$ is regular, then, by Definition 27, $v = \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$. Hence

$$\delta(v) = t\left(\sum_{e \in \mathbf{s}^{-1}(v)} ee^*\right) = \sum_{e \in \mathbf{s}^{-1}(v)} t(ee^*) = \sum_{e \in \mathbf{s}^{-1}(v)} \delta(e^*e) = \sum_{e \in \mathbf{s}^{-1}(v)} \delta(\mathbf{r}(e)),$$

and therefore, δ satisfies (4) as well. Thus, $t \mapsto t|_{G_E}$ gives a map from the set of all K -linear traces $t : L_K(E) \rightarrow R$ to the set of all maps $\delta : G_E \rightarrow R$ that preserve zero and satisfy conditions (1)–(4).

Next, suppose that $\delta : G_E \rightarrow R$ is a map that preserves zero and satisfies conditions (1)–(4). Then δ is central, by Proposition 19. Let $t_\delta : C_K(E) = \overline{KG_E} \rightarrow R$ be as in Definition 2. Then t_δ is a K -linear trace, by Proposition 3. We wish to show that $t_\delta(N) = 0$, where $N \subseteq C_K(E)$ is as in Definition 27. To do so, it suffices to prove that for every generator $x = v - \sum_{e \in \mathbf{s}^{-1}(v)} ee^*$ of N and any two elements $y, z \in C_K(E)$, we have $t_\delta(yxz) = 0$. But since t_δ is a trace, $t_\delta(yxz) = t_\delta(zyx)$, and hence we only need to show that $t_\delta(yx) = 0$ for all $y \in C_K(E)$. As t_δ is K -linear, we may further assume that $y = pq^*$ for some $p, q \in \text{Path}(E)$. Again, since t_δ is a trace, we have $t_\delta(yx) = t_\delta(pq^*x) = t_\delta(q^*xp)$. But by [4, Lemma 9], the expression q^*xp is zero unless $q^* = v = p$, in which case $q^*xp = x$. Thus it suffices to show that $t_\delta(x) = 0$. Now

$$t_\delta(x) = t_\delta\left(v - \sum_{e \in \mathbf{s}^{-1}(v)} ee^*\right) = \delta(v) - \sum_{e \in \mathbf{s}^{-1}(v)} \delta(ee^*) = \delta(v) - \sum_{e \in \mathbf{s}^{-1}(v)} \delta(\mathbf{r}(e)) = 0,$$

since δ satisfies condition (4), and hence $t_\delta(N) = 0$.

Thus, defining $\bar{t}_\delta : L_K(E) = C_K(E)/N \rightarrow R$ via $\bar{t}_\delta(x + N) = t_\delta(x)$ gives a K -linear trace, by Proposition 16. Thus, $\delta \mapsto \bar{t}_\delta$ gives a function from the set of all maps $\delta : G_E \rightarrow R$ that preserve zero and satisfy conditions (1)–(4) to the set of all K -linear traces $t : L_K(E) \rightarrow R$. Note that the restriction of \bar{t}_δ to G_E is δ , from which it is easy to see that the two maps $t \mapsto t|_{G_E}$ and $\delta \mapsto \bar{t}_\delta$ defined above are inverses of each other (in either order). Thus we have the desired one-to-one correspondence. \square

Our final goal is to describe Leavitt path algebras that admit faithful traces. We begin with a preliminary result.

Proposition 29. *Let K be a field with involution $*$, R a $*$ -ring, E a graph, and $t : L_K(E) \rightarrow R$ a trace. If t is positive, then the following hold.*

- (1) *For all $v \in E^0$ we have $t(v) \geq 0$.*
- (2) *For all $v, w \in E^0$, if there is a path $p \in \text{Path}(E)$ such that $\mathbf{s}(p) = v$ and $\mathbf{r}(p) = w$, then $t(v) \geq t(w)$.*
- (3) *For all $v \in E^0$ and distinct $e_1, \dots, e_n \in E^1$ with v as the source, $t(v) \geq \sum_{i=1}^n t(\mathbf{r}(e_i))$.*

Moreover, if t is faithful, then the following hold.

- (4) *For all $v \in E^0$ we have $t(v) > 0$.*
- (5) *For any $x \in L_K(E)$ and any idempotent $u \in L_K(E)$, satisfying $xu = x = ux$ and $x^*u = x^* = ux^*$, we have $xx^* = u$ if and only if $x^*x = u$.*
- (6) *E is a no-exit graph.*

Proof. Suppose that t is positive. To show (1), we note that $0 \leq t(vv^*) = t(v)$ for all $v \in E^0$. Next, let v, w , and p be as in (2), and set $x = v - pp^*$. Then $xx^* = v - pp^*$, and therefore

$$0 \leq t(xx^*) = t(v) - t(pp^*) = t(v) - t(p^*p) = t(v) - t(w).$$

For (3), let $e_1, \dots, e_n \in E^1$ be distinct edges with source $v \in E^0$, and set $x = v - \sum_{i=1}^n e_i e_i^*$. Then $xx^* = v - \sum_{i=1}^n e_i e_i^*$, and therefore

$$0 \leq t(xx^*) = t(v) - \sum_{i=1}^n t(e_i e_i^*) = t(v) - \sum_{i=1}^n t(e_i^* e_i) = t(v) - \sum_{i=1}^n t(\mathbf{r}(e_i)).$$

Now suppose that t is faithful. Then, $0 < t(vv^*) = t(v)$ for all $v \in E^0$, which proves (4). Next, let x and u be as in (5). If $xx^* = u$, then $u = u^*$ and hence

$$0 \leq (u - x^*x)(u - x^*x)^* = u - x^*x - x^*x + x^*ux = u - x^*x.$$

Since t is faithful,

$$t(u - x^*x) = t(u) - t(x^*x) = t(u) - t(xx^*) = 0$$

implies that $x^*x = u$. The converse follows by symmetry.

To show (6), first suppose that E has at least two vertices, and that $p \in \text{Path}(E)$ is a cycle with an exit $e \in E^1$. We may assume that $\mathbf{s}(p) = \mathbf{r}(p) = \mathbf{s}(e)$, and we shall denote this vertex by v . Let $w \in E^0 \setminus \{v\}$ be any vertex, and let $x = p + w$. Then

$$x^*x = (p^* + w)(p + w) = p^*p + w = v + w.$$

Now, $v + w$ is an idempotent which clearly satisfies $x(v + w) = x = (v + w)x$ and $x^*(v + w) = x^* = (v + w)x^*$. Therefore, by (5) we have

$$v + w = xx^* = (p + w)(p^* + w) = pp^* + w,$$

and hence, $v = pp^*$. But then $0 = e^*pp^* = e^*v = e^*$ contradicts our choice of e , implying that there cannot be a cycle with an exit.

Next, suppose that $E^0 = \{v\}$, and there is an edge $e \in E^1$ (necessarily a loop). Then $e^*e = v = 1$ implies that $ee^* = v$, by (5). It follows that there cannot be another edge in E , for if $f \in E^1 \setminus \{e\}$, then $0 = f^*ee^* = f^*v = f^*$, a contradiction. Thus, in all cases, no cycle in E can have an exit. \square

Let us pause to give an example showing that satisfying conditions (1)–(3) above is generally not sufficient for a trace t to be positive, and satisfying conditions (1)–(6) is generally not sufficient for t to be faithful.

Example 30. Let $K = \mathbb{C}$ be the field of the complex numbers, with the usual involution $*$, defined by $(a + bi)^* = a - bi$. Then the positive elements here are precisely the nonnegative real numbers. Also let E be the one-loop graph, with $E^0 = \{v\}$ and $E^1 = \{e\}$, pictured below.



Using the fact that $e^*e = v = ee^*$, it is easy to see (and well-known) that

$$\{v\} \cup \{e^n \mid n \geq 1\} \cup \{(e^*)^n \mid n \geq 1\}$$

is a basis for $L_{\mathbb{C}}(E)$ as a \mathbb{C} -vector space. In particular, $L_{\mathbb{C}}(E)$ is commutative. Now define $t : L_{\mathbb{C}}(E) \rightarrow \mathbb{C}$ by $t(v) = 1$, $t(e^n) = i^n$, $t((e^*)^n) = i^n$, and extend \mathbb{C} -linearly to all of $L_{\mathbb{C}}(E)$. Since $L_{\mathbb{C}}(E)$ is commutative, t is a \mathbb{C} -linear trace. Since $t(v) = 1$ and v is the only vertex in E , conditions (1)–(6) of Proposition 29 are clearly satisfied. However, letting $x = v + e$, we see that

$$t(xx^*) = t((v + e)(v + e^*)) = t(v + e + e^* + v) = 2 + 2i.$$

Since $2 + 2i$ is not a nonnegative real number, t is not positive, and consequently not faithful either.

We require one more ingredient to prove our main result about faithful traces on Leavitt path algebras; it is part of [3, Theorem 3.7]. In order to state it we first recall some terminology from [3]. Given a graph E and an infinite path $p = e_1e_2 \dots$ ($e_i \in E^1$), we say that p is an *infinite sink* if it has neither cycles nor exits. We also say that p *ends in a sink* if there is a subpath $e_n e_{n+1} \dots$ ($n \geq 1$) which is an infinite sink, and that p *ends in a cycle* if there is subpath $e_n e_{n+1} \dots$ ($n \geq 1$) and a cycle $c \in \text{Path}(E) \setminus E^0$ such that $e_n e_{n+1} \dots = cc \dots$

Theorem 31 (Abrams/Aranda Pino/Perera/Siles Molina). *Let K be a field and E a countable row-finite graph. Then the following are equivalent.*

- (1) E is a no-exit graph where every infinite path ends either in a sink or in a cycle.
- (2) $L_K(E) \cong \bigoplus_{i \in I} \mathbb{M}_{\kappa_i}(K) \oplus \bigoplus_{i \in J} \mathbb{M}_{\kappa_i}(K[x, x^{-1}])$, where I and J are countable sets, and each κ_i is a countable cardinal.

Assuming that K has an involution, the isomorphism

$$f : L_K(E) \cong \bigoplus_{i \in I} \mathbb{M}_{\kappa_i}(K) \oplus \bigoplus_{i \in J} \mathbb{M}_{\kappa_i}(K[x, x^{-1}])$$

constructed in the proof of (1) \Rightarrow (2) of the above result is actually a K -linear $*$ -isomorphism, viewing each $\mathbb{M}_{\kappa_i}(K)$ and $\mathbb{M}_{\kappa_i}(K[x, x^{-1}])$ as a $*$ -ring via Lemmas 25 and 26.

More specifically, let $\{c_i\}_{i \in J}$ be all the cycles in $\text{Path}(E) \setminus E^0$, let $\{s_i\}_{i \in I_1}$ be all the sinks in E^0 , and let $\{u_i\}_{i \in I_2}$ be all the infinite sinks in E (where distinct u_i have no edges in common, and where the starting vertex of each u_i is fixed, though it can be chosen arbitrarily). Also, for each $i \in I_1$ let $\{p_{ij}\}_{j \in L_i}$ be all the paths that end in s_i , for each $i \in I_2$ let $\{q_{ij}\}_{j \in M_i}$ be all the paths that end in u_i (such that $\mathbf{r}(q_{ij})$ is the only vertex lying on both q_{ij} and u_i), and for each $i \in J$ let $\{r_{ij}\}_{j \in N_i}$ be all the paths that end in $\mathbf{s}(c_i)$ but do not contain c_i (where $\mathbf{s}(c_i)$ is a fixed, though arbitrary, vertex on c_i). It can be shown that

$$\bigcup_{i \in I_1} \{p_{ij} p_{il}^* \mid j, l \in L_i\} \cup \bigcup_{i \in I_2} \{q_{ij} x_{ijl} q_{il}^* \mid j, l \in M_i\} \cup \bigcup_{i \in J} \{r_{ij} c_i^k r_{il}^* \mid j, l \in N_i, k \in \mathbb{N}\}$$

is a basis for $L_K(E)$, where c_i^k is understood to be $\mathbf{s}(c_i)$ when $k = 0$ and $(c_i^*)^{-k}$ when $k < 0$, and where, writing each u_i as $u_i = e_{i1} e_{i2} \dots$ ($e_{ij} \in E^1$), we define

$$x_{ijl} = \begin{cases} e_{ik} e_{i,k+1} \dots e_{im} & \text{if } \mathbf{r}(q_{ij}) = \mathbf{s}(e_{ik}), \mathbf{r}(q_{il}) = \mathbf{r}(e_{im}), \text{ and } k \leq m \\ e_{ik}^* e_{i,k-1}^* \dots e_{im}^* & \text{if } \mathbf{r}(q_{ij}) = \mathbf{r}(e_{ik}), \mathbf{r}(q_{il}) = \mathbf{s}(e_{im}), \text{ and } k \geq m \\ \mathbf{r}(q_{ij}) & \text{if } \mathbf{r}(q_{ij}) = \mathbf{r}(q_{il}) \end{cases}.$$

Then f maps $L_K(E)$ to the ring

$$\bigoplus_{i \in I_1} \mathbb{M}_{|L_i|}(K) \oplus \bigoplus_{i \in I_2} \mathbb{M}_{|M_i|}(K) \oplus \bigoplus_{i \in J} \mathbb{M}_{|N_i|}(K[x, x^{-1}]),$$

via setting $f(p_{ij} p_{il}^*) = e_{jl} \in \mathbb{M}_{|L_i|}(K)$, $f(q_{ij} x_{ijl} q_{il}^*) = e_{jl} \in \mathbb{M}_{|M_i|}(K)$, and $f(r_{ij} c_i^k r_{il}^*) = x^k e_{jl} \in \mathbb{M}_{|N_i|}(K[x, x^{-1}])$, and then extending K -linearly to all of $L_K(E)$. Note that for all such elements we have

$$f((p_{ij} p_{il}^*)^*) = f(p_{il} p_{ij}^*) = e_{lj} = e_{jl}^* = f(p_{ij} p_{il}^*)^*,$$

$$f((q_{ij} x_{ijl} q_{il}^*)^*) = f(q_{il} x_{ijl}^* q_{ij}^*) = e_{lj} = e_{jl}^* = f(q_{ij} x_{ijl} q_{il}^*)^*, \text{ and}$$

$$f((r_{ij} c_i^k r_{il}^*)^*) = f(r_{il} c_i^{-k} r_{ij}^*) = x^{-k} e_{lj} = (x^k e_{jl})^* = f(r_{ij} c_i^k r_{il}^*)^*,$$

from which it follows that $f(y^*) = f(y)^*$ for all $y \in L_K(E)$.

We also observe that the proof of Theorem 31 does not rely on E being countable to work. Thus, with trivial modifications, the proof actually shows the following.

Corollary 32. *Let K be a field with an involution, and let E be a row-finite graph. Then the following are equivalent.*

- (1) E is a no-exit graph where every infinite path ends either in a sink or in a cycle.
- (2) $L_K(E) \cong \bigoplus_{i \in I} \mathbb{M}_{\kappa_i}(K) \oplus \bigoplus_{i \in J} \mathbb{M}_{\kappa_i}(K[x, x^{-1}])$, for some sets I, J and cardinals κ_i .

Moreover, the isomorphism in (2) can be taken to be a K -linear $*$ -isomorphism.

We are now ready to prove our main result about faithful traces on Leavitt path algebras.

Theorem 33. *Let K be a field with a positive definite involution, and let E be a row-finite graph where every infinite path ends either in a sink or in a cycle. Then the following are equivalent.*

- (1) There is a faithful K -linear trace $t : L_K(E) \rightarrow K$.
- (2) There is a faithful K -linear trace $t : L_K(E) \rightarrow R$ for some K -algebra R with an involution.
- (3) There is a faithful K -linear minimal trace $t : L_K(E) \rightarrow R$ for some K -algebra R with an involution.
- (4) There is a faithful minimal trace $t : L_K(E) \rightarrow R$ for some $*$ -ring R .
- (5) There is a faithful trace $t : L_K(E) \rightarrow R$ for some $*$ -ring R .
- (6) E is a no-exit graph.

Proof. By Proposition 29(6), (5) implies (6). The other implications in the following diagram are tautological.

$$\begin{array}{ccc}
 (1) \Rightarrow (2) & & \\
 & \searrow & \\
 & & (5) \Rightarrow (6) \\
 & \nearrow & \\
 (3) \Rightarrow (4) & &
 \end{array}$$

We shall conclude the proof by showing that (6) implies (1) and (3).

Suppose that (6) holds. Since every infinite path in E ends either in a sink or in a cycle, and E is a row-finite no-exit graph, by Corollary 32, $L_K(E)$ is isomorphic to $\bigoplus_{i \in I} R_i \oplus \bigoplus_{i \in J} S_i$ via a K -linear $*$ -isomorphism, where each R_i is of the form $\mathbb{M}_{\kappa}(K)$ (for some cardinal κ) and each S_i is of the form $\mathbb{M}_{\kappa}(K[x, x^{-1}])$ (for some cardinal κ).

By Lemma 25, each R_i admits a faithful K -linear trace $t_i : R_i \rightarrow K$, which is in addition minimal, by Corollary 14. Also, extending the involution from K to $K[x, x^{-1}]$, as in Lemma 26(1), and then to $\mathbb{M}_{\kappa}(K[x, x^{-1}])$, as in Lemma 25 (for any nonzero cardinal κ), makes the usual trace $\text{tr} : \mathbb{M}_{\kappa}(K[x, x^{-1}]) \rightarrow K[x, x^{-1}]$ faithful, since, by Lemma 26(2), the above involution on $K[x, x^{-1}]$ is positive definite. Thus, each S_i admits a faithful K -linear trace $u_i : S_i \rightarrow K[x, x^{-1}]$, which is again minimal, by Corollary 14. Let

$$f : \bigoplus_{i \in I} R_i \oplus \bigoplus_{i \in J} S_i \rightarrow \bigoplus_{i \in I} K \oplus \bigoplus_{i \in J} K[x, x^{-1}]$$

act on each direct summand as the corresponding map t_i or u_i . Then it is easy to see that f gives a faithful K -linear minimal trace on $\bigoplus_{i \in I} R_i \oplus \bigoplus_{i \in J} S_i$, and hence on $L_K(E)$. Thus (6) implies (3).

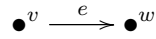
Now, letting $t : K[x, x^{-1}] \rightarrow K$ be as in Lemma 26(3), we see that for each $i \in J$, $t \circ u_i : S_i \rightarrow K$ is a faithful K -linear trace, by Lemma 24(1). Therefore, setting $t_i = t \circ u_i$ for each $i \in J$, and letting

$$h : \bigoplus_{i \in I} R_i \oplus \bigoplus_{i \in J} S_i \rightarrow \bigoplus_{i \in I \cup J} K$$

act on each direct summand as the corresponding map t_i , again gives a faithful K -linear trace on $\bigoplus_{i \in I} R_i \oplus \bigoplus_{i \in J} S_i$, and hence on $L_K(E)$. Further, by Lemma 24(2), there is a faithful K -linear trace $\bigoplus_{i \in I \cup J} K \rightarrow K$. Composing this trace with h (and the aforementioned $*$ -isomorphism) then gives a faithful K -linear trace $L_K(E) \rightarrow K$, by Lemma 24(1). Therefore (6) implies (1). \square

We conclude with three examples showing the necessity of the assumptions on K and E in the previous theorem.

Example 34. Let E be the graph consisting of two vertices v, w and one edge e , having source v and range w , as pictured below.



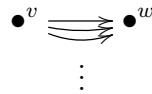
Then E is a no-exit row-finite graph with no infinite paths. Also, let $K = \mathbb{C}$ be the field of the complex numbers, endowed with the identity involution $*$. Since $1 \cdot 1^* + i \cdot i^* = 0$, the involution is not positive definite. We shall show that there are no faithful \mathbb{C} -linear traces $t : L_{\mathbb{C}}(E) \rightarrow \mathbb{C}$, and hence that here condition (6) of Theorem 33 is satisfied, but not condition (1).

Let $t : L_{\mathbb{C}}(E) \rightarrow \mathbb{C}$ be any trace. Then

$$t((v + iw)(v + iw)^*) = t(v - w) = t(ee^*) - t(e^*e) = t(ee^*) - t(ee^*) = 0,$$

and hence t cannot be faithful.

Example 35. Let $K = \mathbb{R}$ be the field of the real numbers, endowed with the identity involution, which is easily seen to be positive definite. (Here a nonzero element is positive, in the sense of Definition 22, precisely when it is positive in the usual sense.) Also let E be the graph consisting of two vertices v, w and infinitely many edges e_1, e_2, e_3, \dots , each having source v and range w , as pictured below.



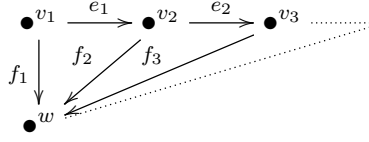
Then E is a no-exit graph with no infinite paths, but it is not row-finite. We shall show that there are no faithful \mathbb{R} -linear traces $t : L_{\mathbb{R}}(E) \rightarrow \mathbb{R}$, and hence that here condition (6) of Theorem 33 is satisfied, but not condition (1).

Suppose that, on the contrary, $t : L_{\mathbb{R}}(E) \rightarrow \mathbb{R}$ is a faithful \mathbb{R} -linear trace. Then, by Proposition 29(4), we have $t(v), t(w) > 0$, and by Proposition 29(3), we see that for all $n \geq 1$,

$$t(v) - nt(w) = t(v) - \sum_{i=1}^n t(\mathbf{r}(e_i)) \geq 0.$$

Thus, $t(v) \geq nt(w) > 0$ for all $n \geq 1$, which is absurd.

Example 36. Again, let $K = \mathbb{R}$ be the field of the real numbers, endowed with the identity involution. Now let E be the following graph.



Then E is a row-finite no-exit graph, but with an infinite path that does not end in a sink or a cycle, namely $e_1 e_2 e_3 \dots$. Again, we shall show that there are no faithful \mathbb{R} -linear traces $t : L_{\mathbb{R}}(E) \rightarrow \mathbb{R}$, and hence that condition (6) of Theorem 33 is satisfied, but not condition (1).

Suppose that, on the contrary, $t : L_{\mathbb{R}}(E) \rightarrow \mathbb{R}$ is a faithful \mathbb{R} -linear trace. For all $i \geq 1$ we have $v_i = e_i e_i^* + f_i f_i^*$ and hence also

$$t(v_i) = t(e_i^* e_i) + t(f_i^* f_i) = t(v_{i+1}) + t(w).$$

Iterating this relation gives $t(v_1) = t(v_{i+1}) + it(w)$ for all $i \geq 1$. This produces the desired contradiction upon noting that $t(v_{i+1}), t(w) > 0$, by Proposition 29(4).

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