Properties of Congruence Lattices of Graph Inverse Semigroups

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Abstract

From any directed graph E one can construct the graph inverse semigroup $G(E)$, whose elements, roughly speaking, correspond to paths in E. Wang and Luo showed that the congruence lattice $L(G(E))$ of $G(E)$ is upper-semimodular for every graph E, but can fail to be lower-semimodular for some E. We provide a simple characterisation of the graphs E for which $L(G(E))$ is lower-semimodular. We also describe those E such that $L(G(E))$ is atomistic, and characterise the minimal generating sets for $L(G(E))$ when E is finite and simple.

Keywords: inverse semigroup, directed graph, congruence lattice, semimodular, atomistic
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1 Introduction

Roughly speaking, a *graph inverse semigroup* $G(E)$ is an inverse semigroup with zero, whose non-zero elements are paths in a (directed) graph E, and where the operation is concatenation of those paths or zero, depending on whether one path ends at the vertex where the next path begins. A precise definition is given in the next section. Graph inverse semigroups were introduced by Ash and Hall [2], who characterised those graph inverse semigroups that are congruence-free, and showed that every partial order is the partial order of the non-zero $\mathscr{J}$-classes of a graph inverse semigroup. These semigroups also generalise the so-called *polycyclic monoids* of Nivat and Perrot [10], and arise in the study of Leavitt path algebras [1] and graph C^*-algebras [11].

There has been a number of more recent papers specifically about graph inverse semigroups; see, for example, [8] and references therein. Of particular relevance here are the papers of Wang [12]; Luo and Wang [6]; and Luo, Wang, and Wei [7]. In [12], Wang gives a description of the congruences of a graph inverse semigroup $G(E)$ in terms of certain sets of vertices of the graph E and integer-valued functions on the cycles in E. This characterisation is used to show that the lattice $L(G(E))$ of any graph inverse semigroup $G(E)$ is Noetherian, i.e., $G(E)$ does not have any infinite strictly ascending chains of congruences. In [6], Luo and Wang show that $L(G(E))$ is always upper-semimodular (Proposition 2.2), but not lower-semimodular in general. These results follow a long tradition of studying lattices naturally associated to various algebraic objects. For one example, among many

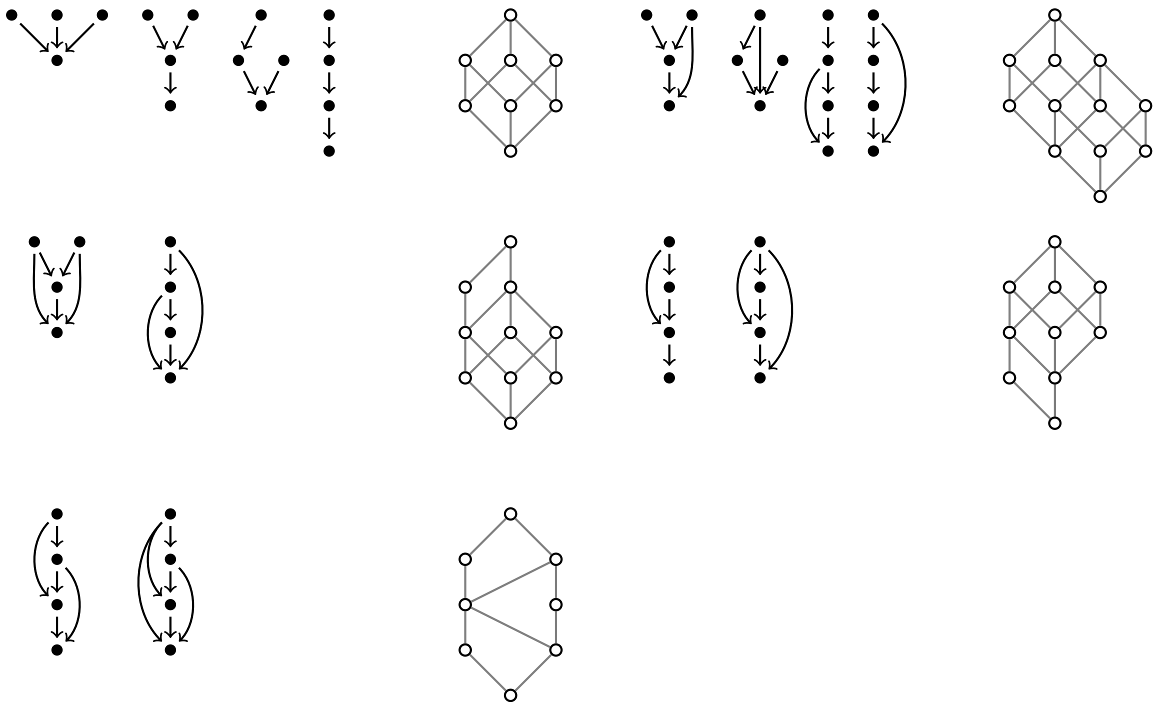

Figure 1: Every connected simple graph E with 4 vertices, such that $L(G(E))$ is lower-semimodular, together with the corresponding lattice $L(G(E))$.

others, it is well-known that the lattice of normal subgroups of a group is modular but not distributive in general. Even if we restrict our attention to the lattices of congruences of semigroups, the literature is rich; see, for example, [3].

The upper-semimodularity result of Luo and Wang [6] naturally raises the following question: is it possible to characterise those graphs E for which $L(G(E))$ is lower-semimodular? We answer this question in Theorem 2.3 for arbitrary graphs E, and provide a somewhat simpler characterisation for finite simple graphs in Corollary 2.4. We also completely describe those graphs E such that $L(G(E))$ is atomistic (Theorem 2.5), and characterise the minimal generating sets for $L(G(E))$ (Theorem 2.7), when E is finite and simple. The results in this paper were initially suggested by experiments performed using the *Semigroups* package [9] for GAP [4]. For example, every connected simple graph E with 4 vertices, up to isomorphism, such that $L(G(E))$ is lower-semimodular, is shown in Fig. 1.

2 Definitions and statements of main results

2.1 Graphs

A *(directed) graph* $E = (E^0, E^1, \mathbf{s}, \mathbf{r})$ is a quadruple consisting of two sets, E^0 and E^1, and two functions $\mathbf{s}, \mathbf{r} : E^1 \longrightarrow E^0$, called *source* and *range*, respectively. The elements of E^0 and E^1 are referred to as *vertices* and *edges*, respectively. A vertex $v \in E^0$ satisfying $\mathbf{s}^{-1}(v) = \varnothing$ is called a *sink*. A sequence $p = e_1 e_2 \cdots e_n$ of (not necessarily distinct) edges $e_i \in E^1$, such that $\mathbf{r}(e_i) = \mathbf{s}(e_{i+1})$ for $1 \leq i \leq n-1$, is a *path* from $\mathbf{s}(e_1)$ to $\mathbf{r}(e_n)$. Here we set $\mathbf{s}(p) = \mathbf{s}(e_1)$ and $\mathbf{r}(p) = \mathbf{r}(e_n)$, and refer to n as the *length* of p. We view the elements of E^0 as paths of length 0, and denote by $\mathrm{Path}(E)$ the set of all paths in E. A path $p = e_1 \cdots e_n$ where $n \geq 1$, $\mathbf{s}(p) = \mathbf{r}(p)$, and $\mathbf{s}(e_i) \neq \mathbf{s}(e_j)$ for all $i \neq j$, is a *cycle*. Two distinct edges $e, f \in E^1$, such that $\mathbf{s}(e) = \mathbf{s}(f)$ and $\mathbf{r}(e) = \mathbf{r}(f)$, are called *parallel*. A graph containing no cycles is called *acyclic*, while an acyclic graph without parallel edges is called *simple*. A graph E is *finite* if both E^0 and E^1 are finite.

Given a graph E and vertices $u, v \in E^0$, we write $u > v$ if $u \neq v$ and there is a path $p \in \mathrm{Path}(E)$ such that $\mathbf{s}(p) = u$ and $\mathbf{r}(p) = v$. It is easy to see that $\geq$, defined in the obvious way from $>$, is a preorder on E^0. Next, let H be a subset of E^0. Then H is *downward directed* if it is non-empty, and for all $u, v \in H$ there exists $w \in H$ such that $u \geq w$ and $v \geq w$. We say that H is *hereditary* if $u \geq v$ implies that $v \in H$, for all $u \in H$ and $v \in E^0$. Supposing that H is non-empty, H is called a *strongly connected component* if $u \geq v$ for all $u, v \in H$, and H is maximal with respect to this property. If H is non-empty and hereditary, then being a strongly connected component amounts to satisfying $u \geq v$ for all $u, v \in H$. Finally, we denote by $E \setminus H$ the subgraph of E induced by H. Specifically, $(E \setminus H)^0 = E^0 \setminus H$,

$$(E \setminus H)^1 = E^1 \setminus \{e \in E^1 \mid \mathbf{s}(e) \in H \text{ or } \mathbf{r}(e) \in H\},$$

and the source and range functions, $\mathbf{s}_{E \setminus H}$ and $\mathbf{r}_{E \setminus H}$, are the restrictions of $\mathbf{s}$ and $\mathbf{r}$, respectively, to $(E \setminus H)^1$.

2.2 Inverse semigroups

Let S be a semigroup, i.e., a set with an associative binary operation. We say that S is an *inverse semigroup*, if for every $x \in S$ there exists a unique $x^{-1} \in S$ satisfying $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

Given a graph E, we define the *graph inverse semigroup* $G(E)$ *of* E to be the semigroup with zero generated by E^0, E^1, and $E^{-1} = \{e^{-1} \mid e \in E^1\}$ that satisfies the following relations, for all $u, v \in E_0$ and $e, f \in E^1$:

(E1) $\mathbf{s}(e)e = e\mathbf{r}(e) = e$,

(E2) $\mathbf{r}(e)e^{-1} = e^{-1}\mathbf{s}(e) = e^{-1}$,

(E3) $e^{-1}f = \begin{cases} \mathbf{r}(e) & \text{if } e = f \\ 0 & \text{if } e \neq f, \end{cases}$

(V) $vu = \begin{cases} v & \text{if } v = u \\ 0 & \text{if } v \neq u. \end{cases}$

For every $v \in E^0$ we define $v^{-1} = v$, and for every $q = e_1 \cdots e_n \in \mathrm{Path}(E)$ we define $q^{-1} = e_n^{-1} \cdots e_1^{-1}$. It follows directly from the above axioms that every non-zero element in $G(E)$ can be written in the form pq^{-1} for some $p, q \in \mathrm{Path}(E)$. It is routine to show that $G(E)$ is an inverse semigroup, with $(pq^{-1})^{-1} = qp^{-1}$ for every non-zero $pq^{-1} \in G(E)$. Moreover, $G(E)$ is finite if and only if E is finite and acyclic.

If S is any semigroup and $\rho \subseteq S \times S$ is an equivalence relation, then ρ is called a *congruence* if $(zx, zy), (xz, yz) \in \rho$, for all $(x, y) \in \rho$ and all $z \in S$. The *diagonal congruence* is $\Delta_S = \{(x, x) \mid x \in S\}$, and the *universal congruence* is $S \times S$. If R is any subset of $S \times S$, then we denote by $R^\sharp$ the least congruence on S containing R; this is called the congruence *generated by* R.

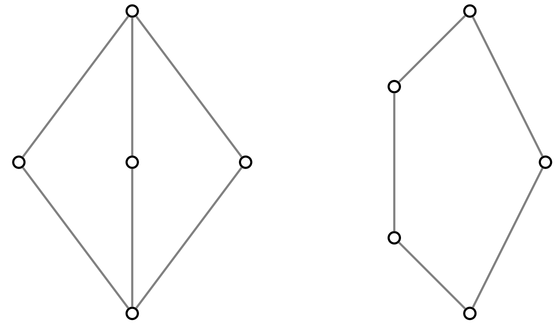

Figure 2: The diamond lattice $\mathfrak{M}_3$ and the pentagon lattice $\mathfrak{N}_5$.

2.3 Lattices

A partially ordered set $(L, \leq)$ is a *lattice*, if for all $a, b \in L$ there exists an infimum $a \wedge b$, called the *meet* of a and b, and a supremum $a \vee b$, called the *join* of a and b. When the order is clear from the context we will write L instead of $(L, \leq)$. For instance, if X is any set, then the power set $\mathcal{P}(X)$ of X forms a lattice under containment, where $\wedge$ is intersection and $\vee$ is union. Similarly, the collection of all congruences $L(S)$ on a semigroup S forms a lattice, with respect to containment, where $\rho \vee \sigma = (\rho \cup \sigma)^\sharp$ and $\rho \wedge \sigma = \rho \cap \sigma$, for all $\rho, \sigma \in L(S)$. By convention, the diagonal congruence Δ_S on S is the join of the empty set of congruences. A lattice L is *complete* if every subset $K \subseteq L$ has an infimum $\bigwedge K$ and a supremum $\bigvee K$. Examples of complete lattices include all finite lattices, the power set lattice $\mathcal{P}(X)$ of any set X, and the lattice of congruences $L(S)$ of any semigroup S.

Two lattices L_1 and L_2 are *order-isomorphic* if there exists a bijection $\Psi : L_1 \longrightarrow L_2$, such that $\Psi(a \vee b) = \Psi(a) \vee \Psi(b)$ and $\Psi(a \wedge b) = \Psi(a) \wedge \Psi(b)$, for all $a, b \in L_1$. A subset L' of a lattice L is called a *sublattice* of L if it forms a lattice under the same join and meet operations as L. We say that L is *generated* by a subset X if every element of L can be expressed as a join of finitely many elements of X. In this situation, the elements of X are called *generators* of L. For $a, b \in L$, we say that b *covers* a, and write $a \prec b$, if $a < b$ and there is no element $c \in L$, such that $a < c < b$. If L is a lattice with a least element 0, then $a \in L$ is an *atom* in case $0 \prec a$. A lattice is *atomistic* if it can be generated using only atoms.

Let L be a lattice. Then L is *modular* if $a \leq c$ implies that $(a \vee b) \wedge c = a \vee (b \wedge c)$, for all $a, b, c \in L$. Moreover, L is *upper-semimodular* if $a \wedge b \prec a, b$ implies that $a, b \prec a \vee b$, for all $a, b \in L$. Likewise, L is *lower-semimodular* if $a, b \prec a \vee b$ implies that $a \wedge b \prec a, b$, for all $a, b \in L$. Finally, L is *distributive* if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, for all $a, b, c \in L$.

It is well-known that every distributive lattice is modular. Moreover, every modular lattice is both upper- and lower-semimodular, and the converse also holds for finite lattices. A lattice L is distributive if and only if neither the pentagon $\mathfrak{N}_5$ nor the diamond $\mathfrak{M}_3$, shown in Fig. 2, is a sublattice of L. Similarly, a lattice L is modular if and only if the pentagon $\mathfrak{N}_5$ is not a sublattice of L. See, for example, [5] for further details.

2.4 Congruence lattices on graph inverse semigroups

Let E be a graph. Given a subset H of E^0, we denote by $C(H)$ the set of cycles $c = e_1 \dots e_n \in \text{Path}(E)$ such that $\mathbf{s}(e_i) \in H$ for each i. As defined in [6], a *Wang triple*

(H, W, f) on E consists of a hereditary set $H \subseteq E^0$, a set

$$W \subseteq \{v \in E^0 \setminus H \mid |\mathbf{s}^{-1}_{E\setminus H}(v)| = 1\},$$

and a *cycle* function $f : C(E^0) \longrightarrow \mathbb{Z}^+ \cup \{\infty\}$ (i.e., $f(c) = 1$ for all $c \in C(H)$, $f(c) = \infty$ for all $c \notin C(H \cup W)$, and the restriction of f to $C(W)$ is invariant under cyclic permutations). In [6], the term "congruence triple" is used for this concept.

Given a Wang triple (H, W, f) on a graph E, we define $\varrho(H, W, f)$ to be the corresponding congruence, generated by the following set:

$$(H \times \{0\}) \cup \{(w, ee^{-1}) \mid w \in W,\ \mathbf{s}(e) = w,\ \mathbf{r}(e) \notin H\} \\ \cup \{(c^{f(c)}, \mathbf{s}(c)) \mid c \in C(W),\ f(c) \in \mathbb{Z}^+\}.$$

Also, given two Wang triples (H_1, W_1, f_1) and (H_2, W_2, f_2) on E, write $(H_1, W_1, f_1) \leq (H_2, W_2, f_2)$ if $H_1 \subseteq H_2$, $W_1 \setminus H_2 \subseteq W_2$, and $f_2(c) \mid f_1(c)$ for all $c \in C(E^0)$. (Here $\mid$ denotes "divides", and it is understood that $\infty \mid \infty$, and $n \mid \infty$ for all $n \in \mathbb{Z}^+$.) According to [6, Corollary 1.1] and [12, Lemma 2.18], $\leq$ is a partial order on the set of all Wang triples on a graph. Moreover, Luo and Wang characterise the lattice of congruences on a graph inverse semigroup, according to the associated Wang triples. We will repeatedly require this characterisation, and so we state it in the next proposition.

Proposition 2.1 (Proposition 1.2 in [6]). *Let E be a graph, let $G(E)$ be the graph inverse semigroup of E, and let $L(G(E))$ be the lattice of congruences on $G(E)$. The function $(H, W, f) \mapsto \varrho(H, W, f)$ is an order-isomorphism between the set of all Wang triples on E, ordered by $\leq$, and $L(G(E))$, ordered by containment.*

In light of this proposition, we will abuse notation by identifying Wang triples with the corresponding congruences, writing (H, W, f) instead of $\varrho(H, W, f)$, and $(H_1, W_1, f_1) \subseteq (H_2, W_2, f_2)$ instead of $(H_1, W_1, f_1) \leq (H_2, W_2, f_2)$. If E is acyclic, then the component f in a Wang triple (H, W, f) is redundant, and we will write $(H, W, \varnothing)$ instead.

Next, we record another frequently used result, mentioned in the introduction.

Proposition 2.2 (Theorem 1.3 in [6]). *Let E be a graph, let $G(E)$ be the graph inverse semigroup of E, and let $L(G(E))$ be the lattice of congruences on $G(E)$. Then $L(G(E))$ is upper-semimodular.*

In [6, Example 2] Luo and Wang produce a graph E such that $L(G(E))$ is not lower-semimodular. For another example, if E is the graph in Fig. 3, then it can be shown using the *Semigroups* package [9] for GAP [4], that $L(G(E))$ is isomorphic to the lattice in Fig. 3, which is easily seen to not be lower-semimodular.

2.5 Main results

To state our first main theorem, which characterises those graphs E such that the congruence lattice $L(G(E))$ of the corresponding graph inverse semigroup $G(E)$ is lower-semimodular, we require the following definition. Let E be a graph and $v \in E^0$. We refer to v as a *forked* vertex, if there exist distinct edges $e, f \in \mathbf{s}^{-1}(v)$ such that the following properties hold:

(i) $\mathbf{r}(g) \not\geq \mathbf{r}(e)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e\}$;

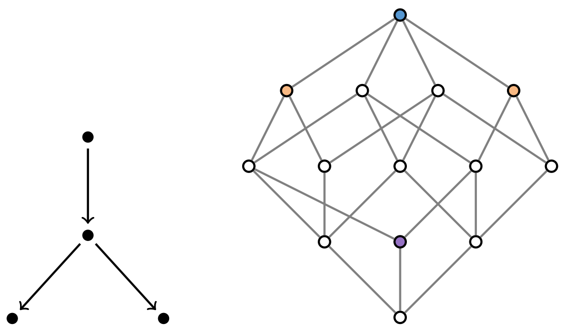

Figure 3: A graph E, together with $L(G(E))$, which is not lower-semimodular. The vertices of $L(G(E))$ shown in orange are covered by their join, shown in blue, but they do not cover their meet, shown in purple.

(ii) $\mathbf{r}(g) \not\geq \mathbf{r}(f)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{f\}$.

Theorem 2.3. *Let E be a graph, let $G(E)$ be the graph inverse semigroup of E, and let $L(G(E))$ be the lattice of congruences on $G(E)$. Then E has no forked vertices if and only if $L(G(E))$ is lower-semimodular.*

Theorem 2.3 has a somewhat simpler form when restricted to finite simple graphs, in which case additional characterisations can be given.

Corollary 2.4. *Let E be a finite acyclic graph, let $G(E)$ be the graph inverse semigroup of E, and let $L(G(E))$ be the lattice of congruences of $G(E)$. Then the following are equivalent:*

(i) *$L(G(E))$ is lower-semimodular;*

(ii) *$L(G(E))$ is modular;*

(iii) *$L(G(E))$ is distributive.*

If E is simple, then these conditions are also equivalent to the following:

(iv) *$\mathbf{r}(e) \geq \mathbf{r}(f)$ or $\mathbf{r}(f) \geq \mathbf{r}(e)$ for all $e, f \in E^1$ such that $\mathbf{s}(e) = \mathbf{s}(f)$.*

In the next of our main theorems, we characterise those graphs E such that $L(G(E))$ is atomistic.

Theorem 2.5. *Let E be a graph, let $G(E)$ be the graph inverse semigroup of E, and let $L(G(E))$ be the lattice of congruences of $G(E)$. Then every congruence in $L(G(E))$ is the join of a (possibly infinite) collection of atoms if and only if for every $v \in E^0$ one of the following holds:*

(i) *$|\mathbf{s}^{-1}(v)| = 0$;*

(ii) *$|\mathbf{s}^{-1}(v)| = 1$, v does not belong to a cycle, and $v > u$ for some $u \in E^0$ such that $|\mathbf{s}^{-1}(u)| \neq 1$;*

(iii) *$|\mathbf{s}^{-1}(v)| \geq 2$, and $\mathbf{r}(e) \geq v$ for all $e \in \mathbf{s}^{-1}(v)$.*

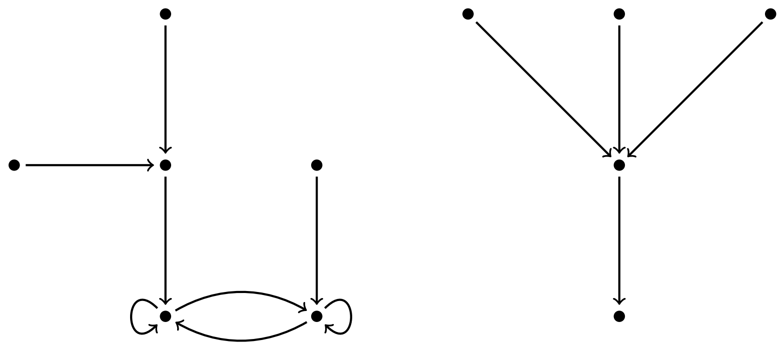

Figure 4: An example of a graph satisfying the conditions of Theorem 2.5.

Moreover, $L(G(E))$ is atomistic if and only if, in addition to the above conditions on all vertices, E^0 has only finitely many strongly connected components and vertices v such that $|\mathbf{s}^{-1}(v)| = 1$.

An example of a graph satisfying the conditions of Theorem 2.5 is given in Fig. 4.

The conditions of Theorem 2.5 simplify significantly when the graph is finite and acyclic.

Corollary 2.6. *Let E be a finite acyclic graph, let $G(E)$ be the graph inverse semigroup of E, and let $L(G(E))$ be the lattice of congruences of $G(E)$. Then the following are equivalent:*

(i) $|s^{-1}(v)| \leq 1$ *for all* $v \in E^0$*;*

(ii) $L(G(E))$ *is isomorphic to the power set lattice* $\mathcal{P}(E^0)$*;*

(iii) $L(G(E))$ *is atomistic.*

For graphs E such that $G(E)$ is infinite, $L(G(E))$ being isomorphic to $\mathcal{P}(E^0)$ is generally not equivalent to $L(G(E))$ being atomistic. For example, if $|E^0| = \aleph_0$, then the number of atoms in $L(G(E))$ is at most $\aleph_0$ (this follows from Proposition 5.1 and Proposition 2.1), and so the cardinality of the lattice generated by atoms is at most $\aleph_0$ also. On the other hand, $|\mathcal{P}(E^0)| = 2^{\aleph_0} > \aleph_0$. Hence $L(G(E))$ is not atomistic if it is isomorphic to $\mathcal{P}(E^0)$.

The last of our main theorems establishes a generating set for $L(G(E))$ in terms of the graph E, when it is finite and simple.

Theorem 2.7. *Let E be a finite simple graph, let $G(E)$ be the graph inverse semigroup of E, let $L(G(E))$ be the lattice of congruences on $G(E)$, and let $\mathcal{A} \subseteq L(G(E))$. Then $\mathcal{A}$ generates $L(G(E))$ if and only if $\mathcal{A}$ contains all the congruences of the following types:*

(i) $(\{v\}, \varnothing, \varnothing)$*, such that* $v \in E^0$ *and* $|\mathbf{s}^{-1}(v)| = 0$*;*

(ii) $(H, \{v\}, \varnothing)$*, such that* $v \in E^0$*,* $|\mathbf{s}^{-1}(v)| > 0$*, and H is a minimal (with respect containment) hereditary subset of E^0 satisfying* $|\mathbf{s}^{-1}_{E\setminus H}(v)| = 1$*.*

The statement in Theorem 2.7 does not hold for graphs with parallel edges. For example, if E is the graph given in Fig. 5, then the only congruences on $G(E)$ of types (i) and (ii) in Theorem 2.7 are of the form $(\{v\}, \varnothing, \varnothing)$, where $v \in E^0$ and $|\mathbf{s}^{-1}(v)| = 0$. It follows (using Proposition 3.1) that the congruence $(E^0, \varnothing, \varnothing)$ on $G(E)$ is not a join of congruences of types (i) and (ii).

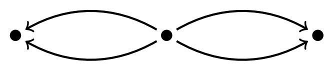

Figure 5: A graph E with parallel edges, for which the conclusion of Theorem 2.7 does not hold.

3 Meets, joins, and covers

In this section, we describe the circumstances under which one congruence covers another in a graph inverse semigroup, in terms of Wang triples. This fact will be used repeatedly in the paper.

We begin by stating a result from [6] that describes the meets and joins of Wang triples, for convenience of reference.

Proposition 3.1 (Lemmas 2.7 and 2.8 in [6]). *Let E be a graph, let (H_1, W_1, f_1) and (H_2, W_2, f_2) be Wang triples on E, and set*

$$V_0 = \{v \in (W_1 \cup W_2) \setminus (H_1 \cup H_2) \mid \mathbf{s}^{-1}_{E\setminus(H_1\cup H_2)}(v) = \varnothing\}$$

and

$$J = \{v \in (W_1 \cup W_2) \setminus (H_1 \cup H_2) \mid \exists e_1 \cdots e_n \in \mathrm{Path}(E)\ \forall i \in \{2, \dots, n\} \\ (\mathbf{s}(e_1) = v,\ \mathbf{r}(e_n) \in V_0,\ \mathbf{s}(e_i) \in W_1 \cup W_2)\}.$$

Then

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_1 \cap H_2, (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup ((W_1 \cap W_2) \setminus V_0), \mathrm{lcm}(f_1, f_2)),$$

where $\mathrm{lcm}(f_1, f_2)(c) = \mathrm{lcm}(f_1(c), f_2(c))$ *for all* $c \in C(E^0)$*, and*

$$(H_1, W_1, f_1) \vee (H_2, W_2, f_2) = (H_1 \cup H_2 \cup J, (W_1 \cup W_2) \setminus (H_1 \cup H_2 \cup J), \gcd(f_1, f_2)),$$

where $\gcd(f_1, f_2)(c) = \gcd(f_1(c), f_2(c))$ *for all* $c \in C(E^0)$*.*

Next, we characterise the situations where one Wang triple covers another.

Proposition 3.2. *Let E be a graph, and let (H_1, W_1, f_1) and (H_2, W_2, f_2) be Wang triples on E, such that $(H_1, W_1, f_1) \subsetneq (H_2, W_2, f_2)$. Then $(H_1, W_1, f_1) \prec (H_2, W_2, f_2)$, i.e., (H_2, W_2, f_2) covers (H_1, W_1, f_1), if and only if one of the following holds:*

(i) *$H_1 = H_2$, $W_1 = W_2$, and $f_2 \prec f_1$. (I.e., there is a cycle $c \in C(W_1)$ such that $f_1(c)/f_2(c)$ is a prime integer, and $f_1(d) = f_2(d)$ for all $d \in C(W_1) \setminus \{c\}$.)*

(ii) *$H_1 = H_2$, $|W_2 \setminus W_1| = 1$, and $f_1 = f_2$.*

(iii) *$H_1 \subsetneq H_2$, $W_1 \setminus H_2 = W_2$,*

$$W_1 \cap H_2 = \{v \in H_2 \setminus H_1 \mid |\mathbf{s}^{-1}_{E\setminus H_1}(v)| = 1\},$$

$f_1(c) = f_2(c)$ for all $c \in C(W_1)$, and for each hereditary set $H_1 \subsetneq H' \subsetneq H_2$ there exists $v \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$.

Moreover, if (iii) holds, then $H_2 \setminus H_1$ is downward directed.

Proof. If (i) holds, then $(H_1, W_1, f_1) \prec (H_2, W_2, f_2)$, by [6, Lemma 2.4]. If (ii) holds, then it follows immediately from the definition of the ordering on Wang triples (or [6, Lemma 2.3]) that $(H_1, W_1, f_1) \prec (H_2, W_2, f_2)$. Let us now suppose that (iii) holds and that

$$(H_1, W_1, f_1) \subseteq (H', W', f') \subseteq (H_2, W_2, f_2)$$

for some Wang triple (H', W', f'). We will show that either $(H', W', f') = (H_1, W_1, f_1)$ or $(H', W', f') = (H_2, W_2, f_2)$. Notice that necessarily $f_1(c) = f'(c) = f_2(c)$ for all $c \in C(W_1)$.

Suppose that $H_1 \subsetneq H' \subsetneq H_2$. Then, by hypothesis, there exists $v \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$. Thus $v \in W_1 \setminus (H' \cup W')$, which contradicts $(H_1, W_1, f_1) \subseteq (H', W', f')$. It follows that either $H' = H_1$ or $H' = H_2$. In the first case, $H' = H_1$,

$$W_1 = W_1 \setminus H_1 = W_1 \setminus H' \subseteq W',$$

which implies that $W_1 \setminus H_2 = W' \setminus H_2 = W_2$. Since, by hypothesis, W_1 contains all $v \in H_2 \setminus H_1$ such that $|\mathbf{s}^{-1}_{E \setminus H_1}(v)| = 1$, we see that $W_1 = W'$, and so $(H', W', f') = (H_1, W_1, f_1)$. In the second case, $H' = H_2$,

$$W' = W' \setminus H' = W' \setminus H_2 \subseteq W_2 = W_1 \setminus H_2 = W_1 \setminus H' \subseteq W',$$

which implies that $W' = W_2$. Since $W_2 \subseteq W_1$, it follows that $f' = f_2$, and so $(H', W', f') = (H_2, W_2, f_2)$, as desired.

For the converse, suppose that $(H_1, W_1, f_1) \prec (H_2, W_2, f_2)$. Let us also suppose, for the moment, that $H_1 = H_2$. If $W_1 = W_2$, then $f_2 \prec f_1$, by [6, Lemma 2.4]. If $W_1 \subsetneq W_2$, then $|W_2 \setminus W_1| = 1$, and $f_1 = f_2$, by [6, Lemma 2.3]. Thus, if $H_1 = H_2$, then exactly one of (i) or (ii) must hold. Let us now assume that $H_1 \subsetneq H_2$. Then $W_1 \setminus H_2 = W_2$,

$$W_1 \cap H_2 = \{v \in H_2 \setminus H_1 \mid |\mathbf{s}^{-1}_{E \setminus H_1}(v)| = 1\},$$

and $f_1(c) = f_2(c)$ for all $c \in C(W_1)$, by [6, Lemma 2.1]. Therefore to conclude the proof of the main claim, it suffices to take a hereditary set $H_1 \subsetneq H' \subsetneq H_2$ and show that there exists $v \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$.

Suppose, on the contrary, that $|\mathbf{s}^{-1}_{E \setminus H'}(v)| = 1$ for all $v \in W_1 \setminus H'$. Let $W' = W_1 \setminus H'$, and let $f' : C(E^0) \longrightarrow \mathbb{Z}^+ \cup \{\infty\}$ be the cycle function such that $f_1(c) = f'(c)$ for all $c \in C(W')$ and $f'(c) = 1$ for all $c \in C(H')$. Then (H', W', f') is a Wang triple such that

$$(H_1, W_1, f_1) \subsetneq (H', W', f') \subsetneq (H_2, W_2, f_2),$$

contradicting our hypothesis. Therefore there must exist $v \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$.

For the final claim, suppose that (iii) holds, and let $u, v \in H_2 \setminus H_1$. Now suppose that for all $w \in H_2 \setminus H_1$ either $u \ngeq w$ or $v \ngeq w$. Let $G_1 = H_1 \cup \{x \in E^0 \mid u \geq x\}$, and for each $i > 1$ let

$$G_i = G_{i-1} \cup \{x \in W_1 \setminus G_{i-1} \mid \mathbf{r}(\mathbf{s}^{-1}(x)) \subseteq G_{i-1}\}.$$

Then, clearly, $H' = \bigcup_{i=1}^{\infty} G_i$ is hereditary, $H_1 \subsetneq H' \subseteq H_2$, and there is no $x \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(x)$. Condition (iii) then implies that $H' = H_2$. Notice also that, by hypothesis, $v \notin G_1$, and that if $v \in W_1$, then it cannot be the case that $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq G_{i-1}$ for some $i > 1$. It follows that $v \notin H'$, in contradiction to $H' = H_2$. Therefore there must exist $w \in H_2 \setminus H_1$ such that $u \geq w$ and $v \geq w$, i.e., $H_2 \setminus H_1$ is downward directed. □

If the graph E is finite and acyclic, then the conditions in Proposition 3.2 can be simplified substantially, as the next corollary shows. Corollary 3.3 follows fairly quickly from Proposition 3.2, but the proof is omitted because we will not use this result directly.

Corollary 3.3. *Let E be a finite acyclic graph, and let $(H_1, W_1, \varnothing)$ and $(H_2, W_2, \varnothing)$ be Wang triples on E, such that $(H_1, W_1, \varnothing) \subseteq (H_2, W_2, \varnothing)$. Then $(H_1, W_1, \varnothing) \prec (H_2, W_2, \varnothing)$ if and only if $|(H_2 \cup W_2) \setminus (H_1 \cup W_1)| = 1$.*

4 Modularity

In this section we will prove Theorem 2.3 and Corollary 2.4. We begin with a sequence of lemmas that will culminate in the proof of Theorem 2.3.

Lemma 4.1. *Let E be a graph containing a forked vertex. Then $L(G(E))$ is not lower-semimodular.*

Proof. By hypothesis, there exist $v \in E^0$ and distinct $e, f \in \mathbf{s}^{-1}(v)$, such that $\mathbf{r}(g) \not\geq \mathbf{r}(e)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e\}$, and $\mathbf{r}(g) \not\geq \mathbf{r}(f)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{f\}$. Let $u = \mathbf{r}(e)$, $w = \mathbf{r}(f)$, $X = \{x \in E^0 \mid v \geq x\}$, $H_u = \{x \in X \mid x \not\geq u\}$, and $H_w = \{x \in X \mid x \not\geq w\}$. Then, clearly, X, H_u, and H_w are hereditary. Next, let

$$W_u = \{y \in X \setminus H_u \mid |\mathbf{s}^{-1}_{E\setminus H_u}(y)| = 1\} \text{ and } W_w = \{y \in X \setminus H_w \mid |\mathbf{s}^{-1}_{E\setminus H_w}(y)| = 1\}.$$

Also, for each set $H \subseteq E^0$ let us denote by $f_H : C(E^0) \longrightarrow \mathbb{Z}^+ \cup \{\infty\}$ the function such that $f_H(c) = 1$ for all $c \in C(H)$ and $f_H(c) = \infty$ for all $c \in C(E^0 \setminus H)$. Then (H_u, W_u, f_u) and (H_w, W_w, f_w) are Wang triples, where $f_u = f_{H_u \cup W_u}$ and $f_w = f_{H_w \cup W_w}$. Also, by construction, $v \in W_u$ and $v \in W_w$. By Proposition 3.1,

$$(H_u, W_u, f_u) \vee (H_w, W_w, f_w) = (X, \varnothing, f_X),$$

since $|\mathbf{s}^{-1}_{E\setminus(H_u \cup H_w)}(v)| = 0$ and $v \in (W_u \cup W_w) \setminus (H_u \cup H_w)$. Using Proposition 3.1 again, since $v \in V_0$,

$$(H_u, W_u, f_u) \wedge (H_w, W_w, f_w) = (H_u \cap H_w, W, f_{uw}),$$

for some set W such that $W \setminus H_u \subseteq W_u \setminus \{v\}$, and $f_{uw} = \operatorname{lcm}(f_u, f_v)$. Then

$$(H_u \cap H_w, W, f_{uw}) \subsetneq (H_u, W_u \setminus \{v\}, f_{H_u \cup (W_u \setminus \{v\})}) \subsetneq (H_u, W_u, f_u),$$

since $w \in H_u \setminus H_w$ implies that $H_u \cap H_w \subsetneq H_u$. Therefore, to conclude that $L(G(E))$ is not lower-semimodular it suffices to show that $(H_u, W_u, f_u) \prec (X, \varnothing, f_X)$ and $(H_w, W_w, f_w) \prec (X, \varnothing, f_X)$. Given the symmetry of the situation, we shall only show that $(H_w, W_w, f_w) \prec (X, \varnothing, f_X)$. By Proposition 3.2, it is enough to prove that for any hereditary set $H_w \subsetneq H' \subsetneq X$ there exists $y \in W_w \setminus H'$ such that $\mathbf{r}(g) \in H'$ for all $g \in \mathbf{s}^{-1}(y)$.

Suppose that $H_w \subsetneq H' \subsetneq X$ for some hereditary set H', and let $x \in H' \setminus H_w$. Then $x \geq w$, and so $w \in H'$. Hence, by construction, $\mathbf{r}(g) \in H'$ for all $g \in \mathbf{s}^{-1}(v)$. Moreover $v \in W_w \setminus H'$, since $H' \neq X$, giving the desired conclusion. □

Lemma 4.2. *Let E be a graph with no forked vertices. Then for any pair (H_1, W_1, f_1) and (H_2, W_2, f_2) of Wang triples on E, we have $V_0 \cap W_1 \cap W_2 = \varnothing$, where V_0 is the set defined in Proposition 3.1, and*

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_1 \cap H_2, (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2), \operatorname{lcm}(f_1, f_2)).$$

Proof. Suppose that there exists $v \in V_0 \cap W_1 \cap W_2$. Since $v \in V_0$, we have $\mathbf{s}^{-1}_{E \setminus (H_1 \cup H_2)}(v) = \varnothing$. Since $v \in W_1 \cap W_2$, there must exist (distinct) $e, f \in \mathbf{s}^{-1}(v)$ such that $\mathbf{r}(e) \in H_1 \setminus H_2$, $\mathbf{r}(f) \in H_2 \setminus H_1$, and $\mathbf{r}(g) \in H_1 \cap H_2$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e, f\}$. Since H_1 and H_2 are hereditary, it follows that $\mathbf{r}(g) \not\geq \mathbf{r}(e)$ and $\mathbf{r}(g) \not\geq \mathbf{r}(f)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e, f\}$, $\mathbf{r}(e) \not\geq \mathbf{r}(f)$, and $\mathbf{r}(f) \not\geq \mathbf{r}(e)$. That is, $v \in E^0$ is forked.

Thus if E has no forked vertices, then $V_0 \cap W_1 \cap W_2 = \varnothing$. The claim about $(H_1, W_1, f_1) \wedge (H_2, W_2, f_2)$ now follows from Proposition 3.1. □

Lemma 4.3. *Let E be a graph, and suppose that (H_1, W_1, f_1) and (H_2, W_2, f_2) are Wang triples on E, such that the set J defined in Proposition 3.1 is empty, and*

$$(H_1, W_1, f_1) \prec (H_1, W_1, f_1) \vee (H_2, W_2, f_2).$$

Then

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) \prec (H_2, W_2, f_2).$$

Proof. By Proposition 3.1,

$$(H_1, W_1, f_1) \vee (H_2, W_2, f_2) = (H, (W_1 \cup W_2) \setminus H, \gcd(f_1, f_2)),$$

where $H = H_1 \cup H_2 \cup J = H_1 \cup H_2$. Since

$$(H_1, W_1, f_1) \prec (H_1, W_1, f_1) \vee (H_2, W_2, f_2),$$

by Proposition 3.2, there are three possible cases, which we examine individually.

Case 1: $H_1 = H$, $W_1 = (W_1 \cup W_2) \setminus H$, and $\gcd(f_1, f_2) \prec f_1$. Then $H_2 \subseteq H_1$, $W_2 \setminus H_1 \subseteq W_1$, and $f_2 \prec \mathrm{lcm}(f_1, f_2)$. Given that $V_0 \subseteq J = \varnothing$ and $W_1 \cap H_1 = \varnothing$, it follows that

$$(W_1 \cap H_2) \cup (W_2 \cap H_1) \cup ((W_1 \cap W_2) \setminus V_0) = \varnothing \cup (W_2 \cap H_1) \cup (W_1 \cap W_2) = W_2.$$

Therefore, by Proposition 3.1 and Proposition 3.2,

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_2, W_2, \mathrm{lcm}(f_1, f_2)) \prec (H_2, W_2, f_2).$$

Case 2: $H_1 = H$, $|((W_1 \cup W_2) \setminus H) \setminus W_1| = 1$, and $f_1 = \gcd(f_1, f_2)$. Then $H_2 \subseteq H_1$, $f_2 = \mathrm{lcm}(f_1, f_2)$, and

$$\begin{aligned} &|W_2 \setminus ((W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2))| = |W_2 \setminus (H_1 \cup W_1)| \\ &= |(W_1 \cup W_2) \setminus (H_1 \cup H_2 \cup W_1)| = |((W_1 \cup W_2) \setminus H) \setminus W_1| = 1. \end{aligned}$$

Therefore, again using the fact that $V_0 = \varnothing$, by Proposition 3.1 and Proposition 3.2,

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_2, (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2), f_2) \prec (H_2, W_2, f_2).$$

Case 3: $H_1 \subsetneq H$, $W_1 \setminus H = (W_1 \cup W_2) \setminus H$,

$$W_1 \cap H = \{v \in H \setminus H_1 \mid |\mathbf{s}^{-1}_{E \setminus H_1}(v)| = 1\},$$

$f_1(c) = \gcd(f_1, f_2)(c)$ for all $c \in C(W_1)$, and for each hereditary set $H_1 \subsetneq H' \subsetneq H$ there exists $v \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$. Then $W_1 \setminus H = (W_1 \cup W_2) \setminus H$

implies that $W_2 \subseteq H_1 \cup W_1$. Moreover, $f_2(c) = \operatorname{lcm}(f_1, f_2)(c)$ for all $c \in C(W_1)$, from which it follows that $f_2(c) = \operatorname{lcm}(f_1, f_2)(c)$ for all $c \in C(W_1 \cup W_2)$, since $f_1(c) = 1$ for all $c \in C(H_1)$. Notice also that given a hereditary set $H_1 \subsetneq H' \subsetneq H$ and $v \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$, it must be the case that $v \in W_1 \cap H_2$, since otherwise $v \in V_0 \subseteq J$. Therefore, by Proposition 3.1,

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_1 \cap H_2, (W_1 \cap H_2) \cup W_2, \operatorname{lcm}(f_1, f_2)).$$

Now, since $H_1 \subsetneq H_1 \cup H_2$, we have $H_1 \cap H_2 \subsetneq H_2$. Also $((W_1 \cap H_2) \cup W_2) \setminus H_2 = W_2$, and

$$\begin{aligned}((W_1 \cap H_2) \cup W_2) \cap H_2 = W_1 \cap H_2 = W_1 \cap H &= \{v \in H_2 \setminus H_1 \mid |\mathbf{s}^{-1}_{E \setminus H_1}(v)| = 1\} \\ &= \{v \in H_2 \setminus (H_1 \cap H_2) \mid |\mathbf{s}^{-1}_{E \setminus (H_1 \cap H_2)}(v)| = 1\},\end{aligned}$$

since $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H_2$ for any $v \in H_2$. Thus, by Proposition 3.2, to conclude that

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) \prec (H_2, W_2, f_2)$$

it suffices to check that for each hereditary set $H_1 \cap H_2 \subsetneq H' \subsetneq H_2$ there exists $v \in ((W_1 \cap H_2) \cup W_2) \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$. Given such a hereditary set H', the set $H_1 \cup H'$ is also hereditary, and $H_1 \subsetneq H_1 \cup H' \subsetneq H$. Hence, as noted above, there exists $v \in (W_1 \cap H_2) \setminus (H_1 \cup H')$ such that $\mathbf{r}(e) \in H_1 \cup H'$ for all $e \in \mathbf{s}^{-1}(v)$. That is, $v \in (W_1 \cap H_2) \setminus H'$. Since $v \in H_2$, we see that

$$\mathbf{r}(e) \in (H_1 \cup H') \cap H_2 = (H_1 \cap H_2) \cup (H' \cap H_2) = (H_1 \cap H_2) \cup H' = H'$$

for all $e \in \mathbf{s}^{-1}(v)$, as desired. □

Lemma 4.4. *Let E be a graph, and suppose that (H_1, W_1, f_1) and (H_2, W_2, f_2) are Wang triples on E, such that $H_2 \subseteq H_1$, $J \neq \varnothing$, and*

$$(H_1, W_1, f_1) \prec (H_1, W_1, f_1) \vee (H_2, W_2, f_2).$$

Then $V_0 = W_2 \setminus (H_1 \cup W_1)$ and $|V_0| = 1$. (See Proposition 3.1 for the definitions of J and V_0.)

Proof. By Proposition 3.1,

$$(H_1, W_1, f_1) \vee (H_2, W_2, f_2) = (H, (W_1 \cup W_2) \setminus H, \gcd(f_1, f_2)),$$

where $H = H_1 \cup H_2 \cup J$. Since

$$(H_1, W_1, f_1) \prec (H_1, W_1, f_1) \vee (H_2, W_2, f_2),$$

by Proposition 3.2, there are three possible cases. However, the hypothesis that $J \neq \varnothing$ implies that $H_1 \neq H$, which rules out two of those cases. Thus $H_1 \subsetneq H$, $W_1 \setminus H = (W_1 \cup W_2) \setminus H$, and $f_1(c) = \gcd(f_1, f_2)(c)$ for all $c \in C(W_1)$, among other conditions. Since $H_2 \subseteq H_1$ and $W_1 \cap H_1 = \varnothing$, we have

$$W_2 \setminus (H_1 \cup J) = W_2 \setminus H \subseteq W_1 \setminus H = W_1 \setminus J,$$

which implies that $W_2 \setminus (H_1 \cup W_1) \subseteq J$. We begin by showing that $V_0 = W_2 \setminus (H_1 \cup W_1)$.

Note that since $H_2 \subseteq H_1$, for all $v \in W_1$ we have $1 = |\mathbf{s}^{-1}_{E\setminus H_1}(v)| = |\mathbf{s}^{-1}_{E\setminus(H_1\cup H_2)}(v)|$, and so $v \notin V_0$. Thus $V_0 \cap W_1 = \varnothing$. Hence $V_0 \subseteq W_2 \setminus (H_1 \cup H_2) = W_2 \setminus H_1$, and therefore $V_0 \subseteq W_2 \setminus (H_1 \cup W_1)$.

Now, suppose that $v \in (W_2 \setminus (H_1 \cup W_1)) \setminus V_0$. Then, in particular $v \in J \setminus V_0$, and so $|\mathbf{s}^{-1}_{E\setminus H_1}(v)| = |\mathbf{s}^{-1}_{E\setminus H_2}(v)| = 1$, but $v \notin W_1$. Therefore,

$$\begin{aligned}(H_1, W_1, f_1) \subsetneq (H_1, W_1 \cup \{v\}, f_1) \subsetneq (H_1 \cup J, (W_1 \cup W_2) \setminus H, \gcd(f_1, f_2)) \\ = (H_1, W_1, f_1) \vee (H_2, W_2, f_2),\end{aligned}$$

contrary to hypothesis. Thus $W_2 \setminus (H_1 \cup W_1) \subseteq V_0$, and so $V_0 = W_2 \setminus (H_1 \cup W_1)$.

It remains to show that $|V_0| = 1$. Since $J \neq \varnothing$, there exists $v \in V_0 = W_2 \setminus (H_1 \cup W_1)$. Let

$$\begin{aligned}J_v = \{u \in J \mid \exists e_1 \cdots e_n \in \mathrm{Path}(E)\ \forall i \in \{2, \dots, n\} \\ (\mathbf{s}(e_1) = u,\ \mathbf{r}(e_n) = v,\ \mathbf{s}(e_i) \in W_1 \cup W_2)\}.\end{aligned}$$

Then it is easy to see that $H_1 \cup J_v$ is a hereditary set, and that $|\mathbf{s}^{-1}_{E\setminus(H_1\cup J_v)}(w)| = 1$ for all $w \in W_1 \setminus J_v$. Thus $(H_1 \cup J_v, W_1 \setminus J_v, f)$ is a well-defined Wang triple, where $f(c) = f_1(c)$ for all $c \in W_1 \setminus J_v$, $f(c) = 1$ for all $c \in C(H_1 \cup J_v)$, and $f(c) = \infty$ for all $c \notin C(H_1 \cup J_v \cup W_1)$. Now, since $v \in J_v$, and hence $J_v \neq \varnothing$, we have

$$(H_1, W_1, f_1) \subsetneq (H_1 \cup J_v, W_1 \setminus J_v, f) \subseteq (H, (W_1 \cup W_2) \setminus H, \gcd(f_1, f_2)),$$

which implies that $H_1 \cup J_v = H = H_1 \cup J$, and so $J = J_v$ (since $J \cap H_1 = \varnothing$). It follows from the definition of J that v is the unique element of V_0, and hence $|V_0| = 1$. □

Lemma 4.5. *Let E be a graph, and suppose that (H_1, W_1, f_1) and (H_2, W_2, f_2) are Wang triples on E, such that $H_2 \not\subseteq H_1$, $J \neq \varnothing$ (see Proposition 3.1), and*

$$(H_1, W_1, f_1) \prec (H_1, W_1, f_1) \vee (H_2, W_2, f_2).$$

Then $J \subseteq W_1$.

Proof. By Proposition 3.1,

$$(H_1, W_1, f_1) \vee (H_2, W_2, f_2) = (H_1 \cup H_2 \cup J, (W_1 \cup W_2) \setminus (H_1 \cup H_2 \cup J), \gcd(f_1, f_2)).$$

Now suppose that $v \in J \setminus W_1$, let

$$\begin{aligned}J_v = \{u \in J \mid \exists e_1 \cdots e_n \in \mathrm{Path}(E)\ \forall i \in \{2, \dots, n\} \\ (\mathbf{s}(e_1) = u,\ \mathbf{r}(e_n) = v,\ \mathbf{s}(e_i) \in W_1 \cup W_2)\},\end{aligned}$$

and let $H = H_1 \cup H_2 \cup (J \setminus J_v)$. We note that for all $w \in J \setminus J_v$, either $\mathbf{r}(e) \in H_1 \cup H_2$ for all $e \in \mathbf{s}^{-1}(w)$, or there is a unique $e \in \mathbf{s}^{-1}(w)$ such that $\mathbf{r}(e) \notin H_1 \cup H_2$, in which case $\mathbf{r}(e) \in J \setminus J_v$. It follows that H is a hereditary set.

Next, suppose that $w \in W_1 \setminus H$. We claim that $|\mathbf{s}^{-1}_{E\setminus H}(w)| = 1$. If $w \notin J_v$, then $w \in W_1 \setminus (H_1 \cup H_2 \cup J)$, and so $|\mathbf{s}^{-1}_{E\setminus(H_1\cup H_2\cup J)}(w)| = 1$, by [6, Lemma 2.8] (Proposition 3.1). Since $|\mathbf{s}^{-1}_{E\setminus H}(w)| \leq 1$ for all $w \in W_1 \cup W_2$, in this case it follows that $|\mathbf{s}^{-1}_{E\setminus H}(w)| = 1$.

Therefore we may suppose that $w \in J_v$. Then $w \neq v$, since $w \in W_1$ and $v \in J \setminus W_1$, from which it follows that $w \notin V_0$ (since $V_0 \cap J_v \subseteq \{v\}$). Hence there is a unique $e \in \mathbf{s}^{-1}(w)$ such that $\mathbf{r}(e) \in J_v$, and so once again $|\mathbf{s}_{E\setminus H}^{-1}(w)| = 1$.

Since H is hereditary and $|\mathbf{s}_{E\setminus H}^{-1}(w)| = 1$ for all $w \in W_1 \setminus H$, we conclude that $(H, W_1 \setminus H, f)$ is a well-defined Wang triple, where $f(c) = f_1(c)$ for all $c \in W_1 \setminus H$, $f(c) = 1$ for all $c \in C(H)$, and $f(c) = \infty$ for all $c \notin C(H \cup W_1)$. Since $H_2 \nsubseteq H_1$ and $J_v \neq \varnothing$, it follows that

$$(H_1, W_1, f_1) \subsetneq (H, W_1 \setminus H, f) \subsetneq (H_1 \cup H_2 \cup J, (W_1 \cup W_2) \setminus (H_1 \cup H_2 \cup J), \gcd(f_1, f_2)),$$

contradicting the hypothesis that

$$(H_1, W_1, f_1) \prec (H_1, W_1, f_1) \vee (H_2, W_2, f_2).$$

Therefore $J \setminus W_1 = \varnothing$. □

Proof of Theorem 2.3. ($\Longleftarrow$): It follows immediately from Lemma 4.1 that if $L(G(E))$ is lower-semimodular, then E has no forked vertices.

($\Longrightarrow$): Suppose that E has no forked vertices. It suffices to take Wang triples (H_1, W_1, f_1) and (H_2, W_2, f_2) such that

$$(H_1, W_1, f_1) \prec (H_1, W_1, f_1) \vee (H_2, W_2, f_2),$$

and show that

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) \prec (H_2, W_2, f_2).$$

By Lemma 4.3, we may assume that $J \neq \varnothing$. By Proposition 3.1,

$$(H_1, W_1, f_1) \vee (H_2, W_2, f_2) = (H, (W_1 \cup W_2) \setminus H, \gcd(f_1, f_2)),$$

where $H = H_1 \cup H_2 \cup J$. Since $J \neq \varnothing$, it cannot be the case that $H_1 = H$. Hence, by Proposition 3.2, the following conditions hold:

(a) $H_1 \subsetneq H$,

(b) $W_1 \setminus H = (W_1 \cup W_2) \setminus H$,

(c) $W_1 \cap H = \{v \in H \setminus H_1 \mid |\mathbf{s}_{E\setminus H_1}^{-1}(v)| = 1\}$,

(d) $f_1(c) = \gcd(f_1, f_2)(c)$ for all $c \in C(W_1)$,

(e) for each hereditary set $H_1 \subsetneq H' \subsetneq H$ there exists $v \in W_1 \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$.

Since E has no forked vertices, by Lemma 4.2,

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_1 \cap H_2, W, \operatorname{lcm}(f_1, f_2)),$$

where

$$W = (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2).$$

Moreover, by (d), $\operatorname{lcm}(f_1, f_2)(c) = f_2(c)$ for all $c \in C(W_1)$. Since $f_1(c) = 1$ for all $c \in H_1$, it follows that $\operatorname{lcm}(f_1, f_2)(c) = f_2(c)$ for all $c \in C(W)$.

Let us next consider the case where $H_2 \subseteq H_1$, i.e., where $H_1 \cap H_2 = H_2$. Then, by Lemma 4.4, $V_0 = W_2 \setminus (H_1 \cup W_1)$ and $|V_0| = 1$. Thus, using the fact that $W_2 \cap H_2 = \varnothing$, gives

$$W_2 \setminus W = W_2 \setminus (W_2 \cap (H_1 \cup W_1)) = W_2 \setminus (H_1 \cup W_1) = V_0,$$

and so $|W_2 \setminus W| = 1$. Since no vertex in V_0 can belong to a cycle, and $H_2 \subseteq H_1$, it also follows that $\operatorname{lcm}(f_1, f_2)(c) = f_2(c)$ for all

$$c \in C(H_2 \cup (W_2 \setminus W) \cup W) = C(H_2 \cup W_2),$$

and so $\operatorname{lcm}(f_1, f_2) = f_2$. Therefore, by Proposition 3.2,

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_2, W, f_2) \prec (H_2, W_2, f_2),$$

as desired.

Hence we may assume that $H_2 \nsubseteq H_1$, i.e., that $H_1 \cap H_2 \subsetneq H_2$. Then $J \subseteq W_1$, by Lemma 4.5. Since, by (b), $W_2 \setminus H \subseteq W_1 \setminus H$, it follows that

$$W_2 \subseteq H_1 \cup W_1 \cup J = H_1 \cup W_1.$$

Therefore

$$W \setminus H_2 = (W_2 \cap (H_1 \cup W_1)) \setminus H_2 = W_2 \cap (H_1 \cup W_1) = W_2.$$

Next, using (c) and $J \subseteq W_1$, gives

$$(W_1 \cap H_2) \cup J = W_1 \cap (H_2 \cup J) = W_1 \cap H = \{v \in (H_2 \setminus H_1) \cup J \mid |\mathbf{s}^{-1}_{E \setminus H_1}(v)| = 1\},$$

from which it follows that

$$W \cap H_2 = W_1 \cap H_2 = (W_1 \cap H) \setminus J = \{v \in H_2 \setminus (H_1 \cap H_2) \mid |\mathbf{s}^{-1}_{E \setminus (H_1 \cap H_2)}(v)| = 1\},$$

since $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H_2$ for any $v \in H_2$. Therefore, recalling that

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) = (H_1 \cap H_2, W, \operatorname{lcm}(f_1, f_2))$$

and that $\operatorname{lcm}(f_1, f_2)(c) = f_2(c)$ for all $c \in C(W)$, by Proposition 3.2, to conclude that

$$(H_1, W_1, f_1) \wedge (H_2, W_2, f_2) \prec (H_2, W_2, f_2),$$

it suffices to show that for each hereditary set $H_1 \cap H_2 \subsetneq H' \subsetneq H_2$ there exists $v \in W \setminus H'$ such that $\mathbf{r}(e) \in H'$ for all $e \in \mathbf{s}^{-1}(v)$.

Let $H_1 \cap H_2 \subsetneq H' \subsetneq H_2$ be a hereditary set. Then $H_1 \cup H'$ is a hereditary set, and

$$H_1 \subsetneq H_1 \cup H' \subsetneq H_1 \cup H_2 \subseteq H.$$

Hence, by (e), there exists $v \in W_1 \setminus (H_1 \cup H') = W_1 \setminus H'$ such that $\mathbf{r}(e) \in H_1 \cup H'$ for all $e \in \mathbf{s}^{-1}(v)$. If $v \in H_2$ for some such v, then $v \in (W_1 \cap H_2) \setminus H' \subseteq W \setminus H'$, and $\mathbf{r}(e) \in (H_1 \cup H') \cap H_2 = H'$ for all $e \in \mathbf{s}^{-1}(v)$, as required. Thus to conclude the proof, it is enough to show that it cannot be the case that $v \notin H_2$ for all $v \in W_1 \setminus H'$ satisfying $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H_1 \cup H'$.

Assume that $v \notin H_2$ for all $v \in W_1 \setminus H'$ satisfying $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H_1 \cup H'$. Then, in particular, $v \in V_0$ for all such vertices. Let

$$V' = \{u \in V_0 \mid \mathbf{r}(\mathbf{s}^{-1}(u)) \subseteq H_1 \cup H'\},$$

and let

$$\begin{aligned} J' = \{u \in J \mid \exists e_1 \cdots e_n \in \mathrm{Path}(E)\ \forall i \in \{2, \dots, n\} \\ (\mathbf{s}(e_1) = u,\ \mathbf{r}(e_n) \in V',\ \mathbf{s}(e_i) \in W_1 \cup W_2)\}. \end{aligned}$$

Then, using the fact that $J \subseteq W_1$, we see that $K = H_1 \cup H' \cup J'$ is hereditary. Also, $H_1 \subsetneq K \subsetneq H = H_1 \cup H_2 \cup J$, since $J \cap (H_1 \cup H_2) = \varnothing$.

We claim that $|\mathbf{s}^{-1}_{E\setminus K}(u)| = 1$ for all $u \in W_1 \setminus K$. Let $u \in W_1 \setminus K$, and let $e \in \mathbf{s}^{-1}(u)$ be the unique edge such that $\mathbf{r}(e) \notin H_1$. Since $W_1 \setminus K \subseteq W_1 \setminus H'$, if $\mathbf{r}(e) \in H'$ $(\subseteq H_1 \cup H')$, then, by assumption, $u \notin H_2$, and so $u \in V'$, contradicting $u \notin J'$. Thus $\mathbf{r}(e) \notin H'$. Moreover, $\mathbf{r}(e) \notin J'$, since $u \notin J'$, and therefore $\mathbf{r}(e) \notin K$. Since $|\mathbf{s}^{-1}_{E\setminus H_1}(u)| = 1$, it follows that $|\mathbf{s}^{-1}_{E\setminus K}(u)| = 1$.

Therefore, defining $f(c) = f_1(c)$ for all $c \in W_1 \setminus K$, $f(c) = 1$ for all $c \in C(K)$, and $f(c) = \infty$ for all $c \notin C(K \cup W_1)$, we have

$$(H_1, W_1, f_1) \subsetneq (K, W_1 \setminus K, f) \subsetneq (H, (W_1 \cup W_2) \setminus H, \gcd(f_1, f_2)),$$

contrary to hypothesis. (The existence of K with the above properties also contradicts (e).) Hence it cannot be the case that $v \notin H_2$ for all $v \in W_1 \setminus H'$ satisfying $\mathbf{r}(\mathbf{s}^{-1}(v)) \subseteq H_1 \cup H'$, as required. □

We need one more lemma to prove Corollary 2.4.

Lemma 4.6. *Let E be a finite acyclic graph. If $(H_1, W_1, \varnothing)$ and $(H_2, W_2, \varnothing)$ are Wang triples on E such that $H_1 \cup W_1 = H_2 \cup W_2$, then $H_1 = H_2$ and $W_1 = W_2$.*

Proof. Seeking a contradiction, suppose that there exists $v \in H_1 \cap W_2$. Since $v \in W_2$, there exists $e_0 \in \mathbf{s}^{-1}(v)$ such that $\mathbf{r}(e_0) \notin H_2$. Since $v \in H_1$ and H_1 is hereditary, $\mathbf{r}(e_0) \in H_1$ also. Since $H_1 \cup W_1 = H_2 \cup W_2$ and $\mathbf{r}(e_0) \notin H_2$, it follows that $\mathbf{r}(e_0) \in W_2$. Repeating this construction gives $e_1 \in \mathbf{s}^{-1}(\mathbf{r}(e_0))$ such that $\mathbf{r}(e_1) \in H_1 \cap W_2$, and so on. Since E is finite and acyclic, this process must yield a path $e_0 e_1 \cdots e_n$ where $\mathbf{r}(e_i) \in H_1 \cap W_2$ for all i, and where $\mathbf{r}(e_n)$ is a sink. But then $|\mathbf{s}^{-1}_{E\setminus H_2}(\mathbf{r}(e_n))| = 0$, contradicting $\mathbf{r}(e_n) \in W_2$. Therefore $H_1 \cap W_2 = \varnothing$, and so $H_1 \subseteq H_2$. By symmetry, $H_2 \cap W_1 = \varnothing$, and so $H_2 \subseteq H_1$, which implies that $H_1 = H_2$. Since $H_1 \cap W_1 = \varnothing = H_2 \cap W_2$, it follows that $W_1 = W_2$. □

Proof of Corollary 2.4. (i) $\Longrightarrow$ (ii): Suppose that $L(G(E))$ is lower-semimodular. By Proposition 2.2, $L(G(E))$ is also upper-semimodular. Since $L(G(E))$ is finite, it follows that $L(G(E))$ is modular (see [5, IV.2, Corollary 3]).

(ii) $\Longrightarrow$ (iii): Suppose that $L(G(E))$ is modular. Then the pentagon lattice $\mathfrak{N}_5$ (see Fig. 2) is not a sublattice of $L(G(E))$, as discussed above. Therefore to show that $L(G(E))$ is distributive it suffices to prove that the diamond lattice $\mathfrak{M}_3$ (also shown in Fig. 2) is not a sublattice of $L(G(E))$.

Seeking a contradiction, suppose that there exist distinct Wang triples $(H_1, W_1, \varnothing)$, $(H_2, W_2, \varnothing)$, and $(H_3, W_3, \varnothing)$ on E, such that the joins and meets of any two are equal. In this case, none of these three is contained in any of the others. We denote $(H_1, W_1, \varnothing) \vee (H_2, W_2, \varnothing)$ by $(H^\vee, W^\vee, \varnothing)$, and $(H_1, W_1, \varnothing) \wedge (H_2, W_2, \varnothing)$ by $(H^\wedge, W^\wedge, \varnothing)$.

By Proposition 3.1,

$$H^\vee \cup W^\vee = H_i \cup W_i \cup H_j \cup W_j = H_i \cup W_i \cup [(H_j \cup W_j) \setminus (H_i \cup W_i)] \tag{4.1}$$

for all distinct $i, j \in \{1, 2, 3\}$. Therefore

$$(H_j \cup W_j) \setminus (H_i \cup W_i) = (H_k \cup W_k) \setminus (H_i \cup W_i), \tag{4.2}$$

whenever $\{i, j, k\} = \{1, 2, 3\}$. Using Proposition 3.1 again,

$$(H^\wedge, W^\wedge, \varnothing) = (H_1 \cap H_2, (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup ((W_1 \cap W_2) \setminus V_0), \varnothing).$$

Since $L(G(E))$ is modular, and therefore lower-semimodular, by Theorem 2.3, E has no forked vertices. Hence, by Lemma 4.2, $W_1 \cap W_2 \cap V_0 = \varnothing$, and so

$$H^\wedge \cup W^\wedge = (H_1 \cap H_2) \cup (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2) = (H_1 \cup W_1) \cap (H_2 \cup W_2).$$

By symmetry,

$$H^\wedge \cup W^\wedge = (H_i \cup W_i) \cap (H_j \cup W_j) \tag{4.3}$$

for all distinct $i, j \in \{1, 2, 3\}$.

If $\{i, j, k\} = \{1, 2, 3\}$, then, by Eq. (4.1),

$$\begin{aligned} H^\vee \cup W^\vee &= [(H_i \cup W_i) \setminus (H_j \cup W_j)] \cup [(H_i \cup W_i) \cap (H_j \cup W_j)] \\ &\quad \cup [(H_j \cup W_j) \setminus (H_i \cup W_i)] \\ &= [(H_i \cup W_i) \setminus (H_k \cup W_k)] \cup [(H_i \cup W_i) \cap (H_k \cup W_k)] \\ &\quad \cup [(H_k \cup W_k) \setminus (H_i \cup W_i)]. \end{aligned}$$

From Eq. (4.2) and Eq. (4.3) it follows that

$$(H_i \cup W_i) \setminus (H_j \cup W_j) = (H_i \cup W_i) \setminus (H_k \cup W_k),$$

and so $H_j \cup W_j = H_k \cup W_k$. Thus, by Lemma 4.6, $(H_j, W_j, \varnothing) = (H_k, W_k, \varnothing)$, producing a contradiction. Hence the diamond $\mathfrak{M}_3$ is not a sublattice of $L(G(E))$, as required.

(iii) $\Longrightarrow$ (i): Suppose that $L(G(E))$ is distributive. Since, as mentioned above, every distributive lattice is modular, $L(G(E))$ is modular, and hence also lower-semimodular.

Now, let us assume that E is simple, and prove that (i) $\Longleftrightarrow$ (iv).

(i) $\Longleftrightarrow$ (iv): It suffices, by Theorem 2.3, to show that there exists a forked vertex in E if and only if there are $e, f \in E^1$ such that $\mathbf{s}(e) = \mathbf{s}(f)$, $\mathbf{r}(e) \ngeq \mathbf{r}(f)$, and $\mathbf{r}(f) \ngeq \mathbf{r}(e)$.

For the forward direction, suppose that v is a forked vertex in E. Then there exist distinct $e, f \in \mathbf{s}^{-1}(v)$ such that $\mathbf{r}(g) \ngeq \mathbf{r}(e)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e\}$ and $\mathbf{r}(g) \ngeq \mathbf{r}(f)$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{f\}$. In particular, $\mathbf{s}(e) = \mathbf{s}(f)$, $\mathbf{r}(e) \ngeq \mathbf{r}(f)$, and $\mathbf{r}(f) \ngeq \mathbf{r}(e)$.

For the converse, suppose that there exist $v \in E^0$ and $e, f \in \mathbf{s}^{-1}(v)$, such that $\mathbf{r}(e) \ngeq \mathbf{r}(f)$ and $\mathbf{r}(f) \ngeq \mathbf{r}(e)$. For convenience, we will refer to this situation as *v splitting at e and f*. Since E is finite and acyclic, we may assume that v is $\leq$-minimal, among vertices that spilt (i.e., no $u \in E^0$ satisfying $v > u$ splits). Next, using the fact that E is finite and has no parallel edges, among the $g \in \mathbf{s}^{-1}(v)$ satisfying $\mathbf{r}(g) \geq \mathbf{r}(e)$ we can find one, denoted e', which is $\leq$-maximal. That is, $\mathbf{r}(g) \not\geq \mathbf{r}(e')$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{e'\}$. Likewise, among the $g \in \mathbf{s}^{-1}(v)$ satisfying $\mathbf{r}(g) \geq \mathbf{r}(f)$ we can find one, denoted f', such that $\mathbf{r}(g) \not\geq \mathbf{r}(f')$ for all $g \in \mathbf{s}^{-1}(v) \setminus \{f'\}$. To conclude that v is forked it suffices to show that $e' \neq f'$.

Suppose that $e' = f'$. Then $\mathbf{r}(e') \geq \mathbf{r}(e)$ and $\mathbf{r}(f') \geq \mathbf{r}(f)$ imply that $e' \neq e$ and $e' \neq f$. Since E has no parallel edges, there must exist $g, h \in E^1$ such that $\mathbf{s}(g) = \mathbf{r}(e') = \mathbf{s}(h)$, $\mathbf{r}(g) = \mathbf{r}(e)$, and $\mathbf{r}(h) = \mathbf{r}(f)$. But then $u = \mathbf{r}(e') = \mathbf{r}(f')$ splits at g and h, and satisfies $v > u$, contrary to the choice of v. Thus $e' \neq f'$, as desired. □

5 Atoms and atomistic congruence lattices

In this section, we describe the atoms in the congruence lattice of a graph inverse semigroup, and prove Theorem 2.5 and Corollary 2.6.

For convenience, given appropriate $H, W \subseteq E^0$, by $(H, W, 1_H)$ or $(H, W, 1)$ we denote the Wang triple on E with the trivial cycle function, relative to H. That is, $1_H : C(E^0) \longrightarrow \mathbb{Z}^+ \cup \{\infty\}$ is defined by $1_H(c) = 1$ for all $c \in C(H)$, and $1_H(c) = \infty$ for all $c \in C(E^0 \setminus H)$.

Proposition 5.1. *Let E be a graph, and let (H, W, f) be a Wang triple on E. Then (H, W, f) is an atom in the lattice of Wang triples on E if and only if one of the following holds:*

(i) *$H = \varnothing$, $|W| = 1$, and $f = 1_\varnothing$;*

(ii) *H is a strongly connected component of E, $W = \varnothing$, $|\mathbf{s}^{-1}(v)| \geq 2$ for each $v \in H$ that is not a sink, and $f = 1_H$.*

Proof. It is easy to see that the least element in the lattice of Wang triples on E is $(\varnothing, \varnothing, 1_\varnothing)$.

If (H, W, f) satisfies (i), then clearly $(\varnothing, \varnothing, 1_\varnothing) \prec (H, W, f)$, by Proposition 3.2, and so (H, W, f) is an atom. Now suppose that (H, W, f) satisfies (ii). Then $\{v \in H \mid |\mathbf{s}^{-1}(v)| = 1\} = \varnothing$, and there are no hereditary sets H' satisfying $\varnothing \subsetneq H' \subsetneq H$. Thus $(\varnothing, \varnothing, 1_\varnothing) \prec (H, W, f)$, by Proposition 3.2, once again.

Conversely, suppose that (H, W, f) is an atom, i.e., $(\varnothing, \varnothing, 1_\varnothing) \prec (H, W, f)$. By Proposition 3.2, there are three possible cases, which we examine individually.

Case 1: $H = \varnothing$, $W = \varnothing$, and $f \prec 1_\varnothing$. The last clause cannot be satisfied by any f, and so this case is not actually possible.

Case 2: $H = \varnothing$, $|W \setminus \varnothing| = 1$, and $f = 1_\varnothing$. This is condition (i) above.

Case 3: $H \neq \varnothing$, $W = \varnothing$, $\{v \in H \mid |\mathbf{s}^{-1}(v)| = 1\} = \varnothing$, and there are no non-empty hereditary sets $H' \subsetneq H$. The last clause amounts to saying that $\{u \in E^0 \mid v \geq u\} = H$ for all $v \in H$, which implies that H is strongly connected. Finally, given that $W = \varnothing$, it must be the case that $f(c) = 1_H$. Thus condition (ii) is satisfied. □

Corollary 5.2. *Let E be a graph, let (H, W, f) be a Wang triple on E, and suppose that (H, W, f) is the join of a (possibly infinite) collection of atoms in the lattice of Wang triples on E. Then one of the following conditions holds for each $v \in H$:*

(i) *$|\mathbf{s}^{-1}(v)| = 0$;*

(ii) *$|\mathbf{s}^{-1}(v)| = 1$, v does not belong to a cycle, and $v > u$ for some $u \in E^0$ such that $|\mathbf{s}^{-1}(u)| \neq 1$;*

(iii) $|\mathbf{s}^{-1}(v)| \geq 2$, *and* $\mathbf{r}(e) \geq v$ *for all* $e \in \mathbf{s}^{-1}(v)$.

Moreover, if (H, W, f) *is the join of finitely many atoms, then* H *has finitely many strongly connected components and finitely many* $v \in H$ *such that* $|\mathbf{s}^{-1}(v)| = 1$.

Proof. Write

$$(H, W, f) = \bigvee_{i\in I}(\varnothing, W_i, 1) \vee \bigvee_{i\in K}(H_i, \varnothing, 1), \tag{5.1}$$

where each $(\varnothing, W_i, 1)$ is of the form described in (i) of Proposition 5.1, and each $(H_i, \varnothing, 1)$ is of the form described in (ii) of Proposition 5.1.

We claim that $\bigvee_{i\in I}(\varnothing, W_i, 1) = (\varnothing, \bigcup_{i\in I} W_i, 1)$. (Note that, since this is a potentially infinite join, Proposition 3.1 does not apply.) Clearly $(\varnothing, W_i, 1) \subseteq (\varnothing, \bigcup_{i\in I} W_i, 1)$ for each $i \in I$. Now suppose that each $(\varnothing, W_i, 1) \subseteq (H', W', f')$ for some Wang triple (H', W', f'). Then $\bigcup_{i\in I} W_i \subseteq H' \cup W'$, and so $(\varnothing, \bigcup_{i\in I} W_i, 1) \subseteq (H', W', f')$. It follows that $\bigvee_{i\in I}(\varnothing, W_i, 1) = (\varnothing, \bigcup_{i\in I} W_i, 1)$. An even simpler argument shows that $\bigvee_{i\in K}(H_i, \varnothing, 1) = (\bigcup_{i\in K} H_i, \varnothing, 1)$.

By the definition of Wang triples and Proposition 5.1, each W_i consists of vertices v satisfying $|\mathbf{s}^{-1}(v)| = 1$, while each H_i consists of vertices v satisfying $|\mathbf{s}^{-1}(v)| \neq 1$, and so $(\bigcup_{i\in I} W_i) \cap (\bigcup_{i\in K} H_i) = \varnothing$. Thus, by Proposition 3.1,

$$(H, W, f) = \Big(\varnothing, \bigcup_{i\in I} W_i, 1\Big) \vee \Big(\bigcup_{i\in K} H_i, \varnothing, 1\Big) = \Big(J \cup \bigcup_{i\in K} H_i, \Big(\bigcup_{i\in I} W_i\Big) \setminus J, 1\Big), \tag{5.2}$$

where

$$V_0 = \Big\{u \in \bigcup_{i\in I} W_i \;\Big|\; \mathbf{s}^{-1}_{E\setminus(\bigcup_{i\in K} H_i)}(u) = \varnothing\Big\}$$

and

$$J = \Big\{u \in \bigcup_{i\in I} W_i \;\Big|\; \exists e_1 \cdots e_n \in \mathrm{Path}(E)\ \forall i \in \{2, \dots, n\} \Big(\mathbf{s}(e_1) = u,\ \mathbf{r}(e_n) \in V_0,\ \mathbf{s}(e_i) \in \bigcup_{i\in I} W_i\Big)\Big\}.$$

Now, let $v \in H$, and suppose that $|\mathbf{s}^{-1}(v)| \neq 0$. If $|\mathbf{s}^{-1}(v)| \geq 2$, then $v \in \bigcup_{i\in K} H_i$, since $J \subseteq \bigcup_{i\in I} W_i$ and $H = J \cup \bigcup_{i\in K} H_i$. Thus $v \in H_i$ for some $i \in K$, where H_i is hereditary and strongly connected, by Proposition 5.1. It follows that $\mathbf{r}(e) \geq v$ for all $e \in \mathbf{s}^{-1}(v)$, and hence (iii) is satisfied.

Next, suppose that $|\mathbf{s}^{-1}(v)| = 1$. Then, by similar reasoning, $v \in J$, and so, in particular, v does not belong to a cycle. Moreover $v > u$ for some $u \in H_i$ and $i \in K$. By Proposition 5.1, either $|\mathbf{s}^{-1}(u)| = 0$ or $|\mathbf{s}^{-1}(u)| \geq 2$, and so (ii) holds.

For the final statement, suppose that (H, W, f) satisfies Eq. (5.1), and therefore also Eq. (5.2), with I and K finite. By construction, the strongly connected components in H are precisely the H_i, and the $v \in H$ satisfying $|\mathbf{s}^{-1}(v)| = 1$ are elements of the singleton sets W_i. It follows that H can contain only finitely many strongly connected components and $v \in H$ such that $|\mathbf{s}^{-1}(v)| = 1$. □

We can now give the proof of Theorem 2.5.

Proof of Theorem 2.5. ($\Longrightarrow$): If every congruence on $G(E)$ is a join of atoms, then conditions (i), (ii), and (iii) hold for every $v \in E^0$, by Corollary 5.2, applied to $(E^0, \varnothing, 1)$. Moreover, if every congruence on $G(E)$ is the join of finitely many atoms (i.e., $L(G(E))$ is atomistic), then, again applying Corollary 5.2 to $(E^0, \varnothing, 1)$, shows that E^0 has only finitely many strongly connected components and vertices v such that $|s^{-1}(v)| = 1$.

($\Longleftarrow$): Suppose that each $v \in E^0$ satisfies one of the conditions (i), (ii), and (iii) in the statement of the theorem, and that (H, W, f) is a Wang triple on E. Then W consists entirely of vertices $v \in E^0$ such that $|\mathbf{s}^{-1}(v)| = 1$ and v does not belong to a cycle. Thus $C(W) = \varnothing$, and so necessarily $f = 1_H = 1$. Moreover, $(\varnothing, W, 1)$ and $(\varnothing, \{v\}, 1)$ are well-defined Wang triples, for all $v \in W$. It is easy to see (as in the proof of Corollary 5.2) that $(\varnothing, W, 1) = \bigvee_{v\in W}(\varnothing, \{v\}, 1)$, which is a join of atoms, by Proposition 5.1.

Next, again by hypothesis, we can write

$$H = U \cup \bigcup_{i\in I} H_i \cup \bigcup_{i\in K}\{w_i\}, \tag{5.3}$$

where each w_i is a sink, each H_i is hereditary and strongly connected, with $|\mathbf{s}^{-1}(v)| \geq 2$ for all $v \in H$, and U consists of vertices satisfying condition (ii). Then $\bigcup_{i\in I} H_i$ and $\bigcup_{i\in K}\{w_i\}$ are hereditary. Also, once again, it is easy to see that $(\varnothing, U, 1) = \bigvee_{v\in U}(\varnothing, \{v\}, 1)$, $(\bigcup_{i\in I} H_i, \varnothing, 1) = \bigvee_{i\in I}(H_i, \varnothing, 1)$, and $(\bigcup_{i\in K}\{w_i\}, \varnothing, 1) = \bigvee_{i\in K}(\{w_i\}, \varnothing, 1)$, where all the Wang triples involved are well-defined. Moreover, by condition (ii), for each $v \in U$, either $v > u$ for some $i \in I$ and $u \in H_i$, or $v > w_i$ for some $i \in K$. It follows, using Proposition 3.1, that

$$(H, \varnothing, 1) = (\varnothing, U, 1) \vee \Big(\bigcup_{i\in I} H_i, \varnothing, 1\Big) \vee \Big(\bigcup_{i\in K}\{w_i\}, \varnothing, 1\Big),$$

and so

$$(H, \varnothing, 1) = \bigvee_{v\in U}(\varnothing, \{v\}, 1) \vee \bigvee_{i\in I}(H_i, \varnothing, 1) \vee \bigvee_{i\in K}(\{w_i\}, \varnothing, 1), \tag{5.4}$$

which is a join of atoms, by Proposition 5.1. Noting that, by Proposition 3.1,

$$(H, W, f) = (H, W, 1) = (H, \varnothing, 1) \vee (\varnothing, W, 1) = (H, \varnothing, 1) \vee \bigvee_{v\in W}(\varnothing, \{v\}, 1), \tag{5.5}$$

we conclude that (H, W, f) is a join of atoms.

Finally, if E^0 has only finitely many strongly connected components and vertices v such that $|s^{-1}(v)| = 1$, then W, as well as U, I, and K in Eq. (5.3), must be finite. Hence, (H, W, f) is a finite join of atoms, by Eq. (5.4) and Eq. (5.5). □

Proof of Corollary 2.6. (i) $\Longrightarrow$ (ii): Suppose that $|\mathbf{s}^{-1}(v)| \leq 1$ for all $v \in E^0$. We define a mapping $\Psi : L(G(E)) \longrightarrow \mathcal{P}(E^0)$ as follows:

$$\Psi((H, W, \varnothing)) = H \cup W,$$

for every Wang triple $(H, W, \varnothing)$ on E. We will show that Ψ is a lattice isomorphism.

By Lemma 4.6, Ψ is injective. To show that is it surjective, let $V \in \mathcal{P}(E^0)$. By (i), we can write $V = U \cup W$, where U consists of sinks and W consists of vertices v satisfying

$|\mathbf{s}^{-1}(v)| = 1$ (with either set possibly empty). Then $(\{v\}, \varnothing, \varnothing)$ is a Wang triple for each $v \in U$, and $(\varnothing, \{v\}, \varnothing)$ is a Wang triple for each $v \in W$. By Proposition 3.1,

$$\bigvee_{v\in U} (\{v\}, \varnothing, \varnothing) \vee \bigvee_{v\in W} (\varnothing, \{v\}, \varnothing) = (H', W', \varnothing)$$

for some sets H' and W' such that $V = H' \cup W'$, and so $\Psi((H', W', \varnothing)) = V$, from which it follows that Ψ is surjective. It remains to show that Ψ preserves meets and joins.

Let $(H_1, W_1, \varnothing)$ and $(H_2, W_2, \varnothing)$ be Wang triples on E, and let J be the set from Proposition 3.1. Then

$$\begin{aligned}\Psi((H_1, W_1, \varnothing) \vee (H_2, W_2, \varnothing)) &= \Psi((H_1 \cup H_2 \cup J, (W_1 \cup W_2) \setminus (H_1 \cup H_2 \cup J), \varnothing)) \\ &= (H_1 \cup H_2 \cup J) \cup ((W_1 \cup W_2) \setminus (H_1 \cup H_2 \cup J)) \\ &= (H_1 \cup W_1) \cup (H_2 \cup W_2) \\ &= \Psi((H_1, W_1, \varnothing)) \cup \Psi((H_2, W_2, \varnothing)).\end{aligned}$$

Next, since $|\mathbf{s}^{-1}(v)| \leq 1$ for all $v \in E^0$, the graph E cannot have forked vertices. Thus, by Lemma 4.2,

$$(H_1, W_1, \varnothing) \wedge (H_2, W_2, \varnothing) = (H_1 \cap H_2, (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2), \varnothing),$$

and so

$$\begin{aligned}\Psi((H_1, W_1, \varnothing) \wedge (H_2, W_2, \varnothing)) &= \Psi((H_1 \cap H_2, (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2), \varnothing)) \\ &= (H_1 \cap H_2) \cup (W_1 \cap H_2) \cup (W_2 \cap H_1) \cup (W_1 \cap W_2) \\ &= (H_1 \cup W_1) \cap (H_2 \cup W_2) \\ &= \Psi((H_1, W_1, \varnothing)) \cap \Psi((H_2, W_2, \varnothing)).\end{aligned}$$

(ii) $\Longrightarrow$ (iii): The power set lattice $\mathcal{P}(E^0)$ is, by definition, atomistic.

(iii) $\Longrightarrow$ (i): Suppose that $L(G(E))$ is atomistic. Then one of the conditions (i), (ii), or (iii) in Theorem 2.5 holds for every $v \in E^0$. Since E is assumed to be acyclic, no $v \in E^0$ satisfies condition (iii) in that theorem. Hence $|\mathbf{s}^{-1}(v)| \leq 1$ for every $v \in E^0$. □

6 Generating congruences

The purpose of this section is to prove Theorem 2.7.

Proof of Theorem 2.7. ($\Longleftarrow$): Suppose that $\mathcal{A}$ contains all the congruences of types (i) and (ii) in the statement of the theorem, and let $(H, W, \varnothing)$ be any Wang triple on E. Since E is finite and acyclic, we can write $H = \bigcup_{i=0}^{n} H_i$, for some $n \geq 0$, where for each i, the set H_i consists of the vertices $h \in H$ such that there exists $p \in \text{Path}(E)$ of length i, with $\mathbf{s}(p) = h$ and $\mathbf{r}(p)$ a sink, and such that i is maximal for this property of h. For each $i \geq 1$ and $h \in H_i$ choose $e_h \in \mathbf{s}^{-1}(h)$ such that $\mathbf{r}(e_h) \in H_{i-1}$, and let

$$G_h = \{v \in E^0 \mid \mathbf{r}(f) \geq v \text{ for some } f \in \mathbf{s}^{-1}(h) \setminus \{e_h\}\}.$$

By the construction of H_i and the hypothesis that E is simple, here $\mathbf{r}(f) \ngeq \mathbf{r}(e_h)$ for all $f \in \mathbf{s}^{-1}(h) \setminus \{e_h\}$, and so, in particular, $\mathbf{r}(e_h) \notin G_h$. Notice also that $(\{h\}, \varnothing, \varnothing)$ is a well-defined Wang triple for each $h \in H_0$, that $(G_h, \{h\}, \varnothing)$ is a well-defined Wang triple for each $h \in H_i$ with $i \geq 1$, and that both belong to $\mathcal{A}$. We will next show, by induction on n, that

$$(H, \varnothing, \varnothing) = \bigvee_{h \in H_0} (\{h\}, \varnothing, \varnothing) \vee \bigvee_{i=1}^{n} \bigvee_{h \in H_i} (G_h, \{h\}, \varnothing).$$

If $n = 0$, and so $H = H_0$ consists of sinks, then it follows immediately from Proposition 3.1 that $(H, \varnothing, \varnothing) = \bigvee_{h \in H_0} (\{h\}, \varnothing, \varnothing)$. Supposing that $n \geq 1$, let us assume inductively that

$$\Big(\bigcup_{i=0}^{n-1} H_i, \varnothing, \varnothing \Big) = \bigvee_{h \in H_0} (\{h\}, \varnothing, \varnothing) \vee \bigvee_{i=1}^{n-1} \bigvee_{h \in H_i} (G_h, \{h\}, \varnothing).$$

(It is easy to see that $\bigcup_{i=0}^{n-1} H_i$ is hereditary.) By construction, $\mathbf{s}^{-1}_{E \setminus (G_h \cup H_{n-1})}(h) = \varnothing$ for each $h \in H_n$, and so, by Proposition 3.1,

$$(G_h, \{h\}, \varnothing) \vee \Big(\bigcup_{i=0}^{n-1} H_i, \varnothing, \varnothing \Big) = \Big(G_h \cup \{h\} \cup \bigcup_{i=0}^{n-1} H_i, \varnothing, \varnothing \Big).$$

Since $H = \bigcup_{i=0}^{n} H_i$, iterating this computation gives

$$\Big(\bigcup_{i=0}^{n-1} H_i, \varnothing, \varnothing \Big) \vee \bigvee_{h \in H_n} (G_h, \{h\}, \varnothing) = (H, \varnothing, \varnothing),$$

which proves the claim. In particular, if $W = \varnothing$, then $(H, W, \varnothing)$ is a join of congruences from $\mathcal{A}$.

Now suppose that $W \neq \varnothing$, and for each $w \in W$ define K_w to be a minimal hereditary subset of H, such that $|\mathbf{s}^{-1}_{E \setminus K_w}(w)| = 1$. Then clearly $(K_w, \{w\}, \varnothing)$ is a Wang triple belonging to $\mathcal{A}$. Since $|\mathbf{s}^{-1}_{E \setminus H}(w)| = 1$, applying Proposition 3.1 once more, gives

$$(H, \varnothing, \varnothing) \vee (K_w, \{w\}, \varnothing) = (H, \{w\}, \varnothing)$$

for each $w \in W$. It follows that

$$\begin{aligned} (H, W, \varnothing) &= (H, \varnothing, \varnothing) \vee \bigvee_{w \in W} (K_w, \{w\}, \varnothing) \\ &= \bigvee_{h \in H_0} (\{h\}, \varnothing, \varnothing) \vee \bigvee_{i=1}^{n} \bigvee_{h \in H_i} (G_h, \{h\}, \varnothing) \vee \bigvee_{w \in W} (K_w, \{w\}, \varnothing). \end{aligned}$$

Thus $\mathcal{A}$ generates $L(G(E))$.

$(\Longrightarrow)$: It suffices to show that every congruence ρ of type (i) or (ii) in the statement of the theorem is indecomposable, in the sense that if $\rho = \sigma \vee \tau$, then $\sigma = \rho$ or $\tau = \rho$. Any congruence of type (i) is an atom, by Proposition 5.1. As such, any congruence of this sort is not the join of two or more distinct congruences, and therefore the claim trivially holds for every congruence of type (i).

Now let $(H, \{v\}, \varnothing)$ be a congruence of type (ii). Clearly $(H, \varnothing, \varnothing) \subset (H, \{v\}, \varnothing)$. So to show that $(H, \{v\}, \varnothing)$ is indecomposable it suffices to prove that if τ is any other congruence such that $\tau \subseteq (H, \{v\}, \varnothing)$, then $\tau \subseteq (H, \varnothing, \varnothing)$ or $\tau = (H, \{v\}, \varnothing)$. If $\tau = (H', W', \varnothing)$ for some H' and W', then $H' \subseteq H$ and $W' \setminus H \subseteq \{v\}$. Hence either $W' \setminus H = \varnothing$ or $W' \setminus H = \{v\}$. In the first case, $W' \subseteq H$, and so $\tau = (H', W', \varnothing) \subseteq (H, \varnothing, \varnothing)$. In the second case, $W' \setminus H = \{v\}$, the condition that $|\mathbf{s}_{E \setminus H'}^{-1}(v)| = 1$ and the minimality of H imply that $H' = H$. Then

$$W' = W' \setminus H' = W' \setminus H = \{v\},$$

giving $\tau = (H, \{v\}, \varnothing)$, as required. □

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References

[1] G. Abrams, P. Ara, and M. S. Molina. *Leavitt Path Algebras*. Springer London, 2017. DOI: 10.1007/978-1-4471-7344-1. URL: https://doi.org/10.1007/978-1-4471-7344-1.

[2] C. J. Ash and T. E. Hall. "Inverse semigroups on graphs". In: *Semigroup Forum* 11.1 (Dec. 1975), pp. 140–145. URL: https://doi.org/10.1007/bf02195262.

[3] G. G. David. "The lattice of congruences on an inverse semigroup." In: *Pacific Journal of Mathematics* 57.1 (1975), pp. 141–152.

[4] *GAP – Groups, Algorithms, and Programming, Version 4.11.1*. The GAP Group. 2021. URL: https://www.gap-system.org.

[5] G. Grätzer. *General Lattice Theory*. Birkhäuser Basel, 1978. URL: https://doi.org/10.1007/978-3-0348-7633-9.

[6] Y. Luo and Z. Wang. "Semimodularity in congruence lattices of graph inverse semigroups". In: *Communications in Algebra* 49.6 (Feb. 2021), pp. 2623–2632. DOI: 10.1080/00927872.2021.1879826. URL: https://doi.org/10.1080/00927872.2021.1879826.

[7] Y. Luo, Z. Wang, and J. Wei. "Distributivity in congruence lattices of graph inverse semigroups". In: *Communications in Algebra* 51.12 (June 2023), pp. 5046–5053. ISSN: 1532-4125. DOI: 10.1080/00927872.2023.2224450. URL: http://dx.doi.org/10.1080/00927872.2023.2224450.

[8] Z. Mesyan and J. D. Mitchell. "The structure of a graph inverse semigroup". In: *Semigroup Forum* 93.1 (Mar. 2016), pp. 111–130. URL: https://doi.org/10.1007/s00233-016-9793-x.

[9] J. D. Mitchell et al. *Semigroups - GAP package, Version 4.0.1*. Apr. 2022. DOI: 10.5281/zenodo.592893. URL: http://dx.doi.org/10.5281/zenodo.592893.

[10] M. Nivat and J.-F. Perrot. "Une généralisation du monoıde bicyclique". In: *Comptes Rendus de l'Académie des Sciences de Paris* 271 (1970), pp. 824–827.

[11] A. L. Paterson. "Graph inverse semigroups, groupoids and their C*-algebras". In: *Journal of Operator Theory* (2002), pp. 645–662.

[12] Z.-P. Wang. "Congruences on graph inverse semigroups". In: *Journal of Algebra* 534 (Sept. 2019), pp. 51–64. URL: https://doi.org/10.1016/j.jalgebra.2019.06.020.

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