The Hypergeometric Function, a Heun Equation, and a Solvable Schrodinger Equation

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October 2, 2018

Abstract

Using elementary transformations, the time independent Schrödinger equation

$$W'' - [\frac{\gamma^2 + \sigma^2 - 2\gamma\sigma\cosh(x) - 1}{\sinh(x)^2} + \frac{(\sigma - 1)^2}{4}]W = \lambda W$$

is transformed to a special equation of Heun type, which may be solved using the hypergeometric function. With slightly modified parameters, this equation has the previously known hyperbolic Pöschel-Teller potential.

Analytic Adjoints

- In spaces $\mathcal{H}_\beta$ of analytic functions on the unit disk, some differential expressions have (formal) adjoints

  $$L_1 = (a_2 z^2 + a_1 z + a_0)D + \sigma a_2 z,\ L_1^+ = (\overline{a_0} z^2 + \overline{a_1} z + \overline{a_2})D + \sigma \overline{a_0} z$$

  and higher order cases use $(L_1 L_2)^+ = L_2^+ L_1^+$, etc.
- This exercise started with the "adjointable" (Gegenbauer, ultraspherical) expression

  $$L = (x^2 - 1)D^2 + (2\alpha + 1)xD, \quad 2\alpha + 1 > 0,$$

  with ortho-polynomial eigenfunctions, eigenvalues $n(n + 2\alpha)$. Specialize to Chebyshev, Legendre, generalize to Jacobi.

Wishful thinking

- Roots of $z^2 - 1$ lie on the unit circle, making problem "singular".
- Real eigenvalues of L might mean same eigenvalues for L^+, with analytic eigenfunctions.
- The eigenvalue equation

$$(x^2 - 1)D^2 y + (2\alpha + 1)xDy = n(n + 2\alpha)y$$

has 3 regular singular points: $\pm 1, \infty$.

Regular singular points (second order)

- Consider an equation

$$y'' + a_1(z)y' + a_0(z)y = 0,$$

  where $a_j(z)$ is analytic except at isolated (singular) points.
- Singular point z_0 will be regular if local solutions are linear combinations of $(z - z_0)^r(\log(z - z_0))^k p(z)$, with $p(z)$ analytic. Usually Frobenius series work:

$$y(z) = (z - z_0)^r \sum_{k=0}^{\infty} c_k (z - z_0)^k.$$

- $z_0 = \infty$ can be regular by mapping $w = 1/z$, study $w = 0$.

Gauss hypergeometric function

- A popular special function. It appears that Gauss' study of convergence initiated the modern theory of infinite series.
- $$F(a,b;c;Z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{Z^n}{n!},\quad c\neq 0,-1,-2,\ldots,$$

  $(a)_n$ is the Pochhammer (or rising factorial) symbol

  $$(a)_n=\left\{\begin{array}{cc} 1, & n=0 \\ a(a+1)\cdots(a+n-1), & n=1,2,3,\ldots \end{array}\right\}$$

  If a or b is a nonpositive integer, $F(a,b;c;z)$ is a polynomial.

Gauss hypergeometric equation

- The hypergeometric equation

$$Z(1-Z)\frac{d^2u}{dZ^2} + \{c-(a+b+1)Z\}\frac{du}{dZ} - abu = 0, \quad (0.1)$$

  with regular singular points at $0, 1, \infty$, is the prototypical second order equation with three regular singular points.
- Independent set of solutions around $Z = 0$ given by

$$u_1(Z) = F(a,b;c;Z), \quad u_2(Z) = Z^{1-c}F(1+a-c, 1+b-c; 2-c; Z), \quad (0.2)$$

- Elementary transformations convert any second order linear equation with three regular singular points to (0.1) (Riemann-Papperitz).

Back to Gegenbauer/Jacobi

- The change of variables $z = 1 - 2Z$ carries Hyper (0.1) to

$$(1 - z^2)D^2 y + \{(a + b + 1 - 2c) - (a + b + 1)z\}Dy - aby = 0. \tag{0.3}$$

Generalizing Gegenbauer,(0.3) arises in the study of Jacobi polynomials. The general solution is a linear combination of

$$y_1(z) = F(a, b; c; \frac{1 - z}{2}),$$

$$y_2(z) = \left(\frac{1 - z}{2}\right)^{1-c} F(1 + a - c, 1 + b - c; 2 - c; \frac{1 - z}{2}).$$

Convert to eigenvalue problem

- Introduce new parameters $\sigma = a + b + 1$, $\lambda = -ab$. Then (0.3) becomes

$$(z^2 - 1)D^2 y + ((2c - \sigma) + \sigma z)Dy = \lambda y, \tag{0.4}$$

which is an eigenvalue equation $L_J y = \lambda y$ for the differential expression

$$L_J = \Big((z^2 - 1)D + \sigma z + (2c - \sigma)\Big)D. \tag{0.5}$$

- $\mathcal{H}_\beta$ adjoint is

$$L_J^+ = \Big(z^2 D + \sigma z\Big)\Big((1 - z^2)D - \sigma z + (2c - \sigma)\Big). \tag{0.6}$$

$$L_J^+ = \left(z^2 D + \sigma z\right)\left((1-z^2)D - \sigma z + (2c-\sigma)\right).$$

- $$L_J^+ y - \lambda y = 0 \tag{0.7}$$

  has 4 (regular) singular points: $0, \pm 1, \infty$ - a Heun eqn.
- Some Heun eqns ((0.7)) reduce to hypergeometric equation.
- With $w = 1/z$, and $Y(w) = y(1/z)$, (0.7) becomes

$$w^2(w^2-1)\frac{d^2Y}{dw^2} + [(2-\sigma)w^3 + 2\sigma w - (2c-\sigma)w^2]\frac{dY}{dw} \tag{0.8}$$

$$-\sigma[\sigma + 1 - (2c-\sigma)w]Y = \lambda w^2 Y.$$

- Write $Y = w^\sigma \phi(w)$, drop factor $w^{2+\sigma}$ to get

$$(w^2-1)\phi'' + [(2+\sigma)w - (2c-\sigma)]\phi' = (\lambda - \sigma)\phi. \tag{0.9}$$

  Same form as the Jacobi (0.4) with altered parameters.

Back to L_J^+

- We're hoping for analytic eigenfunctions $L_J^+ y = \lambda y$. When $\lambda = (n+1)(n+\sigma)$ the previous subs give elementary nonanalytic solutions $z^{-\sigma} p(1/z)$:

$$J_1(\sigma, \lambda, c; z) = z^{-\sigma} F(-n, n+\sigma+1; \sigma - c, \frac{1-1/z}{2}). \quad (0.10)$$

- Try reduction of order, $(y_2 = J_1 v)$. This second independent solution has the form

$$y_2(z) = J_1(z) \int^z J_1^{-2}(s) s^{-\sigma} (s-1)^{c-\sigma-1} (s+1)^{-c-1} \, ds. \quad (0.11)$$

L_J^+ has analytic eigenfunctions

- Theorem

  The equation $L_J^+ y = \lambda y$ has nonzero solutions analytic in $\mathbb{D}$ if and only if $\lambda = 0$ or $\lambda_n = (n+1)(n+\sigma)$ for $n = 0, 1, 2 \ldots$.
  If $\lambda = 0$ an analytic solution is $y = (z-1)^{c-\sigma}(z+1)^{-c}$.
  If $\lambda = (n+1)(n+\sigma)$ and $2n \neq \sigma - 1$, a nonzero solution analytic in $\mathbb{D}$ is

$$y_2(z) = J_1(z) \int_0^z J_1^{-2}(s) s^{-\sigma} (s-1)^{c-\sigma-1}(s+1)^{-c-1}\, ds.$$

Proof

- Proof:
  Use Frobenius method on $L_J^+ y - \lambda y = 0$,
  $$y = \sum_{k=0}^{\infty} c_k z^{k+r}, \quad c_0 \neq 0.$$
  Vanishing of the lowest order coefficients gives the indicial equation
  $$r(r-1) + \sigma r - \lambda = 0. \tag{0.12}$$
  Indicial equation has nonnegative integer root r when $\lambda = n(n-1) + \sigma n$.

More Frobenius

- If $\lambda = 0$ a solution analytic in the disk is $y = (z-1)^{c-\sigma}(z+1)^{-c}$; y annihilated by right factor of L_J^+.
- The values of $n(n-1)+\sigma n$ for $n = 1, 2, 3, \ldots$ are the same as $(n+1)(n+\sigma)$ for $n = 0, 1, 2, \ldots$. Nonanalytic solution of $L_J^+ y - \lambda y = 0$ given by (0.10). Reduction of order solution given by (0.11). The integrand in (0.11) has the form

$$J_1^{-2}(s)s^{-\sigma}(s-1)^{c-\sigma-1}(s+1)^{-c-1} = s^{2n}s^{\sigma}f(s),$$

where $f(s)$ is analytic near $s = 0$. Insert a power series centered at $s = 0$ and integrate termwise. If $2n \neq \sigma - 1$, the terms have the form $c_m z^{2n+1+m+\sigma}$, so that $y_2(z) = z^{n+1}g(z)$, with $g(z)$ analytic near $z = 0$.

Solvable Schrödinger equations

- Let $\gamma = 2c - \sigma$ in (0.7).
  To put $L_J^+ y = \lambda y$ into the Schrödinger form, start with

$$t = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right), \quad z = \frac{1}{\cosh(t)}, \quad 0 < z \leq 1. \quad (0.13)$$

- The change of variables $Y(t) = y(z)$ converts (0.7) to

$$Y'' + q_1(t)Y' - q_0(t)Y = \lambda Y$$

Solvable Schrödinger equations 2

- Use $\xi(t) = 1/\cosh(t)$,

$$q_1(t) = \frac{(1-\sigma)\xi - \gamma\xi^2 + 2\sigma\xi^3}{\xi\sqrt{1-\xi^2}} = \frac{1}{\sinh(t)}[2\sigma\xi - \gamma + (1-\sigma)/\xi]$$

and

$$q_0(t) = \sigma(1+\sigma)\xi^2 - \gamma\sigma\xi.$$

If $r'/r = -q_1/2$, or

$$r(t) = \exp(-\int^t q_1(s)/2 \, ds), \quad W(t) = r^{-1}(t)Y(t).$$

Then

$$W'' - [(q_1/2)' + q_1^2/4 + q_0]W = \lambda W. \tag{0.14}$$

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Solvable Schrödinger equations 3

- Use the abbreviations $C = \cosh(t)$, $S = \sinh(t)$. Some lengthy algebraic manipulations lead to

$$V(t) = (q_1/2)' + q_1^2/4 + q_0 =$$

$$\frac{\gamma^2 + \sigma^2 - 2\gamma\sigma C - 1}{4S^2} + \frac{(\sigma-1)^2}{4}.$$

Thus the solutions of (Pöschl-Teller potential)

$$W'' - [\frac{\gamma^2 + \sigma^2 - 2\gamma\sigma C - 1}{4S^2} + \frac{(\sigma-1)^2}{4}]W = \lambda W \quad (0.15)$$

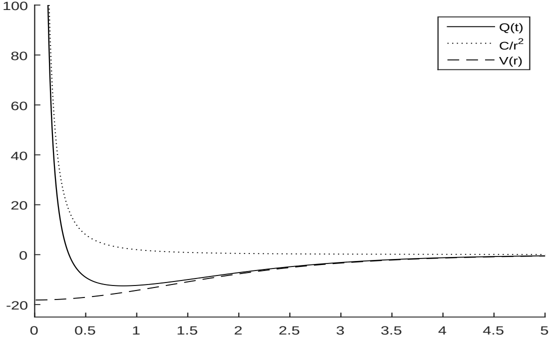
may be expressed in terms of the hypergeometric function. The singularity at 0 is correct for the radial equation for a spherically symmetric operator of Schrödinger form.

Solvable Schrödinger equations 4

- An alternative form is also interesting. The coefficient appearing in (0.15) is analytic in t. Suppose $\tau = it$. Then the equation (0.15) becomes

$$-\frac{d^2U'}{d\tau^2} + [\frac{\gamma^2 + \sigma^2 - 2\gamma\sigma\cos(\tau) - 1}{4\sin^2(\tau)} - \frac{(\sigma-1)^2}{4}]U = \lambda U \tag{0.16}$$

- Here are some plots of $V(t) - (\sigma-1)^2/4$ (solid), $(\sigma-\gamma)^2/4x^2$ (dotted) and the difference (dashed). The first has $\sigma = 7, \gamma = 2$, while the second has $\sigma = 7, \gamma = 12$.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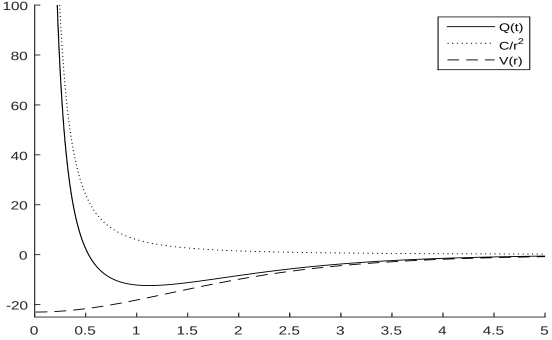
Q(t)
C/r2
V(r)



Q(t)
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