Myopic Models of Population Dynamics on Infinite Networks

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Outline

- Reaction-diffusion population models

$$\frac{dp}{dt} + \Delta p = J(t,p)$$

  on massive (infinite) networks $\mathcal{G}$
- Background on spatially discrete reaction-diffusion theory
- Myopic idea - 'shrink' network by coarse modeling 'at ∞'
  1. $\mathbb{B}$ compactification of $\mathcal{G}$
  2. Diffusion vanishes 'at ∞'
  3. Myopic eigenfunction expansions

General remarks on biological motivation

- Discrete network reaction-diffusion population models:

$$\frac{dp}{dt} + \Delta p = J(t,p).$$

- Biological transport networks (circulatory, rivers, social) are extremely complex ($\simeq 10^9$ arteries) ($\simeq 10^{10}$ people)
- Infinite graph models may help describe
  1. approximation by truncated networks ($\mathcal{G} = \bigcup_n \mathcal{G}_n$)
  2. $\star$ spatial asymptotics of functions as '$x \to \infty$'
  3. $\star$ approximation by models simplifying as '$x \to \infty$'
- Challenge is to simplify infinite graph models.

Networks

- A network (graph) $\mathcal{G}$ has a countable vertex set $\mathcal{V}$ and undirected edge set $\mathcal{E}$. Vertices have $1 \le deg(v) < \infty$ incident edges. $\mathcal{G}$ has vertex weights $\mu(v) > 0$, and edge weights (length) $R(u,v) = R(v,u) > 0$ when $[u,v] \in \mathcal{E}$. Edge conductance is $C(u,v) = 1/R(u,v)$ if $R(u,v) > 0$, or 0.
- Geodesic metric on $\mathcal{G}$ is

$$d_R(u,v) = \inf_{\gamma} \sum_k R(v_k, v_{k+1}), \quad \gamma = (v_1, \dots, v_N).$$

  The metric completion is $\overline{\mathcal{G}}_R$ or $\overline{\mathcal{G}}$.
- For $1 \le p < \infty$ the l^p norms of $f : \mathcal{V} \to \mathbb{R}$ are

$$\|f\|_p = \Big(\sum_{v \in \mathcal{V}} |f(v)|^p \mu(v) \Big)^{1/p}, \quad \|f\|_\infty = \sup_{v \in \mathcal{V}} |f(v)|.$$

Operators Δ on $\mathcal{G}$

- Define two subalgebras of l^∞:
  1. $f \in \mathbb{A}$ if the set of edges $[u, v]$ with $f(u) \neq f(v)$ is finite.
  2. $\mathbb{B}$ is the closure of $\mathbb{A}$ in l^∞.
- Formal Laplace operators are defined by

$$\Delta f(v) = \frac{1}{\mu(v)} \sum_{u \sim v} C(u, v)(f(v) - f(u)).$$

with associated positive symmetric bilinear form
$B(f, g) = \langle \Delta f, g \rangle = \langle f, \Delta g \rangle$, where

$$B(f, g) = \frac{1}{2} \sum_{v \in \mathcal{V}} \sum_{u \sim v} C(u, v)(f(v) - f(u))(g(v) - g(u)).$$

Reaction-diffusion equations

- Model populations (or chemical species) on a network with

$$\frac{dp}{dt} + \Delta p = J(p),$$

  with Δ generating the diffusion, while $J(p)$ or $J(t,p)$ describes the growth and interaction of populations at a site.
- Population models should allow $p \geq \epsilon_1 > 0$ 'at ∞' and $R(u,v) \geq \epsilon_2 > 0$, so l^2 not available, l^∞ is 'too rich'.
- Myopic model - work in $\mathbb{B}$, the l^∞ closure of eventually flat functions $\mathbb{A}$. High fidelity local model, coarse remote model.

Start with bounded Δ

- Δ bounded on l^p, $1 \le p \le \infty$ is

$$\sup_{v \in \mathcal{V}} \frac{1}{\mu(v)} \sum_{u \sim v} C(u,v) < \infty$$

- $$\mathcal{S}(t) = \exp(-t\Delta) = \sum_{n=0}^{\infty} (-t\Delta)^n / n!, \quad t \ge 0.$$

  is positivity preserving contraction semigroup on l^p, preserves the l^1 norm of nonnegative functions, rapid kernel decay.

- Theorem

  $\mathbb{B}$ *is an invariant subspace for* Δ.

The $\mathbb{B}$ compactification of $\mathcal{G}$

- Since $\mathbb{B}$ is a uniformly closed subalgebra of l^∞ with identity, Gelfand promises a compactification $\overline{\mathcal{G}}$ of $\mathcal{G}$, the maximal ideal space of $\mathbb{B}$, on which $\mathbb{B}$ acts as a subalgebra of $C(\overline{\mathcal{G}})$.
- For a given edge weight function $R : \mathcal{E} \to (0, \infty)$, define the volume of a graph to be the sum of its edge lengths,

$$vol_R(\mathcal{G}) = \sum_{[u,v] \in \mathcal{E}} R(u, v).$$

- Introduce a new edge weight function ρ satisfying $vol_\rho(\mathcal{G}) < \infty$. Which ρ is chosen doesn't matter.

More compactification

Theorem

$\overline{\mathcal{G}}_\rho$ compact and totally disconnected.

The compactification $\overline{\mathcal{G}}_\rho$ coming from edge weights with $vol_\rho(\mathcal{G}) < \infty$ varies wildly with the initial network $\mathcal{G}$. If $\mathcal{G}$ is the integer lattice $\mathbb{Z}^d$, $\overline{\mathcal{G}}_\rho$ will be its one point compactification. If $\mathcal{G}$ is an infinite binary tree, $\overline{\mathcal{G}}_\rho$ will include uncountably many points.

Lemma

Functions $f \in \mathbb{B}$ have a unique continuous extension to $\overline{\mathcal{G}}_\rho$, the metric completion of $\mathcal{G}$ with the metric d_ρ.

Theorem

The continuous extension map of $f \in \mathbb{B}$ to $f \in C(\overline{\mathcal{G}}_\rho)$ is a surjective isometry of Banach algebras.

Back to equations

- $$\frac{dp}{dt} + \Delta p = J(t, p),$$
- To localize the nonlinearity, assume (Nemytskii operator) a function $J_v : [0, t_1] \times \mathbb{R}^d \to \mathbb{R}^d$ such that
  $$J(t, p(t))(v) = J_v(t, p(t)(v)), \quad v \in \mathcal{V}.$$
  An example is a variable logistic model,
  $$J_v(t, u) = u(1 - u/K_v(t)).$$
- Such a function $J(t, p)$ is eventually constant if the set of edges $[u, v] \in \mathcal{E}$ such that $J_u \neq J_v$ is finite, independent of t.

Equations at ∞

Let $\partial\overline{\mathcal{G}} = \overline{\mathcal{G}} \setminus \mathcal{V}$. For these J, and $x \in \partial\overline{\mathcal{G}}_\rho$, the diffusion disappears, reducing the problem to an ordinary differential equation.

Theorem

*Assume $J : [0,\infty) \times \mathbb{B}^d \to \mathbb{B}^d$ is continuous for $t \geq 0$ and satisfies a Lipschitz condition. In addition, suppose J eventually constant. For $p_0 \in \mathbb{B}^d$, assume $p(t)$ is a solution of (**??**) for $0 \leq t \leq t_1$. If $x \in \partial\overline{\mathcal{G}}_\rho$, and $q(t)$ solves the initial value problem*

$$\frac{dq}{dt} = J_x(t,q), \quad q(0,x) = p_0(x),$$

then $p(t,x) = q(t,x)$ for $0 \leq t \leq t_1$.

Accelerated diffusion I

- Start with bounded operator edge and vertex weights R, ν, and a second set of finite volume edge weights ρ, and vertex weights μ with $\mu(\mathcal{G}) < \infty$.
- Using the combinatorial distance, pick a vertex r and define edge weights R_n with

$$R_n(u,v) = \begin{Bmatrix} R(u,v), & \max(d_{cmb}(r,u), d_{cmb}(r,v)) \le n \\ \rho(u,v), & \text{otherwise} \end{Bmatrix}$$

  and similarly for vertex weights.
- Pick a closed set $\Omega \subset \partial\overline{\mathcal{G}}_\rho$. Let $\mathcal{S}(t)$ be the $\mathbb{B}$ semigroup generated by Δ, and let $\mathcal{S}_n(t)$ be the $\mathbb{B}$ semigroup generated by $\Delta_n = \Delta_{\Omega,n}$, with coefficients determined by R_n and μ_n.

Accelerated diffusion II

Proposition

Suppose $\mathcal{G}$ is connected, $\overline{\mathcal{G}}_R$ is compact, and $\mu(\mathcal{G})$ is finite. Let S_1 be a symmetric extension of S_K in l^2 with quadratic form

$$\langle S_1 f, f \rangle = B(f, f).$$

Then the Friedrich's extension Δ_1 of S_1 has compact resolvent.

Theorem

Fix $t_1 > 0$. For any $f \in \mathbb{B}_\Omega$,

$$\lim_{n\to\infty} \|\mathcal{S}(t)f - \mathcal{S}_n(t)f\|_\infty = 0$$

uniformly for $0 \le t \le t_1$.

Accelerated diffusion III

Corollary

The conclusion of Theorem remains valid if $S(t)$ and $S_n(t)$ are the solution operators taking initial data in $\mathbb{B}^d$ to the solutions of

$$\frac{dp}{dt} + \Delta p = J(t,p), \quad p(0) = p_0,$$

$$\frac{dP}{dt} + \Delta_n P = J(t,P), \quad P(0) = p_0,$$

where, as before, $J : [0,\infty) \times \mathbb{B}^d \to \mathbb{B}^d$ is continuous for $t \geq 0$, and Lipschitz continuous uniformly in t on bounded intervals.

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