Quantum Cayley Graphs for Free Groups

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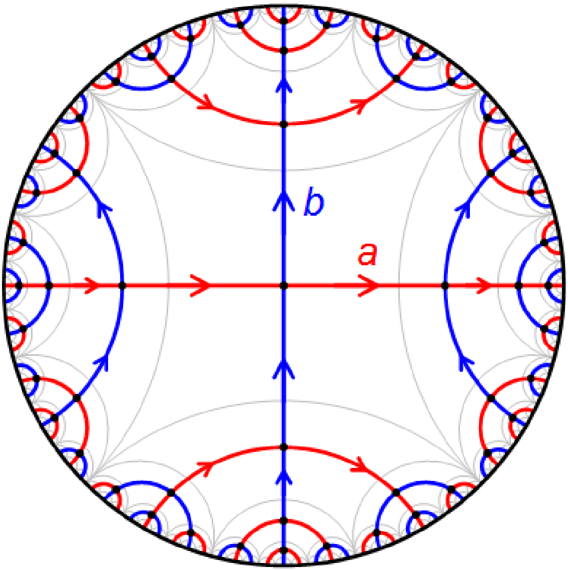
Overview

- Treat spectral theory of invariant differential operators of Schrödinger type $\mathcal{L} = -D^2 + q_m$ (with usual vertex conditions) on the metric Cayley graphs of free groups.
- The graphs are 'metric' or 'quantum' graphs. Edges are identified with intervals $[0, l_e]$. Hilbert space is $\oplus_e L^2[0, l_e]$.
- Work extends the 'Floquet theory' of Hill's equation.

Cayley graphs of $\mathbb{F}_M$

- Free group $\mathbb{F}_M$ on M generators
  $\mathbb{F}_M$: equivalence classes of finite length words generated by distinct $s_1, \ldots, s_M$ and inverse symbols $s_1^{-1}, \ldots, s_M^{-1}$. Remove adjacent symbol pairs $s_m s_m^{-1}$ or $s_m^{-1} s_m$ to get equivalent words.
- Cayley graph $\Gamma_{\mathbb{G},\mathbb{S}}$ for the group $\mathbb{G}$ with (minimal) generating set $\mathbb{S}$ is the directed (undirected) graph with $\mathbb{G}$ elements as vertices and directed edges (v, vs) with $s \in \mathbb{S}$. There are M edge types, type m having length l_m.
- The undirected Cayley graph $\mathcal{T}_M$ of $\mathbb{F}_M$ is a tree whose vertices have degree $2M$.

$\mathcal{T}_2$ from $\mathbb{F}_2$



Hill's equation, $M = 1$, Hilbert space is $L^2(\mathbb{R})$

- Eigenvalue equation $-y'' + q(x)y = \lambda y$, with $q \geq 0$, $q(x+1) = q(x)$, $q(-x) = q(x)$, has 2-dim solution space with standard basis

$$C(x,\lambda) \sim \cos(\sqrt{\lambda}x), \quad S(x,\lambda) \sim \sin(\sqrt{\lambda}x)/\sqrt{\lambda}.$$

- Translation by 1 is linear. Two eigenvalues $\mu(\lambda), \nu(\lambda) = 1/\mu(\lambda)$ with $|\mu(\lambda)| < 1$ for $\lambda \in \mathbb{C} \setminus [0,\infty)$.
- Spectrum of $\mathcal{L}$ is

$$\{|\mu(\sigma)| = 1, \sigma \in \mathbb{R}\} = \{\mu(\eta) \notin \mathbb{R}, \eta \to \sigma \in \mathbb{R}\}$$

$$= \overline{\{\lim_{\epsilon \downarrow 0^+} \mu(\sigma + i\epsilon) - \mu(\sigma - i\epsilon) \neq 0\}} \subset [0,\infty).$$

No obvious small space of solutions on $\mathcal{T}_M$ of $\mathbb{F}_M$

- For $\lambda \in \mathbb{C} \setminus [0, \infty)$, edge e, functional analysis shows existence of a distinguished one-dimensional space $\mathbb{X}_e$ of square integrable eigensolutions y_+ on the half-tree $\mathcal{T}_e^+$ (same for $-$).
- Abelian subgroups $vs_m^k v^{-1}$ act to give a nested family of subtrees $\mathcal{T}_{e(k)}^+$. The induced action on $\mathbb{X}_{e(k)}$ is by multiplication by $\mu_m^k(\lambda) \in \mathbb{C}$.

Theorem

Assume $\lambda \in \mathbb{C} \setminus [0, \infty)$, $e = (v, vs_m)$ and $y_+ \in \mathbb{X}_e^+$, $y_+(v) = 1$. Suppose w is a vertex in $\mathcal{T}_e^+$, and the path from v to w is given by the reduced word $s_m s_{k(1)}^{\pm 1} \dots s_{k(n)}^{\pm 1}$. Then

$$y_+(w) = \mu_m \mu_{k(1)} \dots \mu_{k(n)}. \tag{0.1}$$

By ODE the vertex values of y_+ can be interpolated to the edges.

Coupled equations

- q_m can vary by edge type with eigensolutions $C_m(x,\lambda), S_m(x,\lambda)$.

- Theorem
  For $\lambda \in \mathbb{C} \setminus [0,\infty)$ and $m = 1, \ldots, M$, the multipliers $\mu_m(\lambda)$ satisfy the system of equations

$$\frac{\mu_m^2(\lambda) - 1}{S_m(l_m,\lambda)\mu_m(\lambda)} - 2\sum_{k=1}^{M} \frac{\mu_k(\lambda) - C_k(l_k,\lambda)}{S_k(l_k,\lambda)} = 0.$$

- Theorem
  (Elimination) There is a discrete set $Z_0 \subset \mathbb{R}$ such that for all $\lambda \in \mathbb{C} \setminus Z_0$, the functions $\mu_m(\lambda)$ are solutions of $1 - var$ polynomial equations $p_m(\xi_m) = 0$ of positive degree, with coefficients entire in λ.

Boundary values

- For $\sigma \in [0, \infty)$ there are limiting values $\mu^{\pm}(\sigma) = \lim_{\epsilon \downarrow 0^+} \mu(\sigma \pm i\epsilon)$.
- Let $Z_m \subset \mathbb{C}$ be the discrete set where the leading coefficient of $p_m(\xi_m)$ vanishes. For $\lambda \in \mathbb{C} \setminus Z_m$ the roots of $p_m(\lambda)$, in particular $\mu_m(\lambda)$, are holomorphic if root is simple. Otherwise the roots extend continuously. The limiting values $\mu^{\pm}(\sigma)$ need not agree.

The spectrum

- Theorem

  Assume $\sigma \in [0,\infty) \setminus Z_0$ *and for* $m = 1,\dots,M$ *suppose* $\mu_m^{\pm}(\sigma) \neq \pm 1$. *Then* σ *is in the resolvent set of* $\Delta + q$ *if and only if* $\mu_m^{+}(\sigma)$ *is real valued in open neighborhood of* σ *for* $m = 1,\dots,M$.

- Theorem

  Suppose $M \geq 2$ *and the lengths* l_m *are rational. Then the resolvent set of* Δ *includes an unbounded sequence of pairwise disjoint open intervals in* $[0,\infty)$.

Computations 1

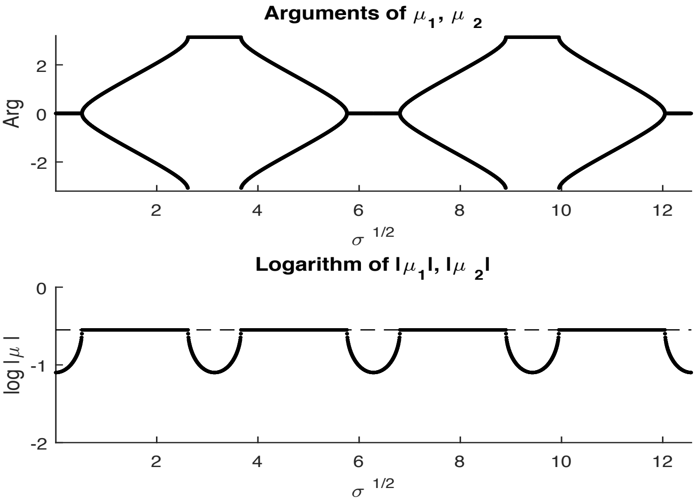


Figure: Case $l_1 = 1$, $l_2 = 1$

Computations 2

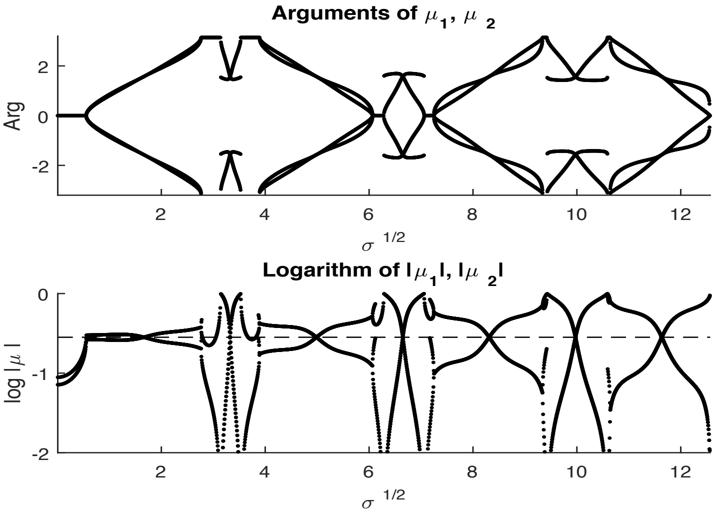


Figure: Case $l_1 = 1$, $l_2 = .89$

Computations 3

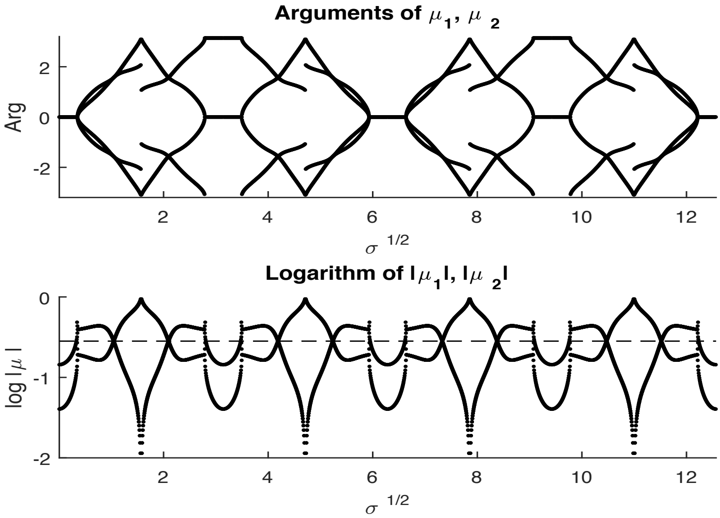


Figure: Case $l_1 = 1$, $l_2 = 2$

Bibliography

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