Quantum Cayley Graphs for Free Groups 2

Robert Carlson
Department of Mathematics
University of Colorado at Colorado Springs
rcarlson@uccs.edu

September 15, 2015

Abstract

The spectral theory of self-adjoint operators provides an abstract framework for solving some of the main differential equations of mathematical physics: the heat equation, wave equation, and Schrödinger equation. When the operators are invariant under a group action, a much more detailed analysis is often possible. This work on invariant differential operators on the metric Cayley graphs of free groups throws differential equations, graphs, group actions, functional analysis, algebraic topology, linear algebra, and a pinch of algebraic geometry into the blender. What emerges is a surprisingly satisfying concoction.

Groups and operators

- Group reps - decompose vector (Hilbert) space into 'smaller' invariant subspaces where group action is (more) transparent.
- Can be tough for infinite groups (noncompact Lie, infinite discrete).
- Group invariant self-adjoint operators to the rescue (sometimes), giving orthogonal projection onto invariant 'eigenspaces'.

Projections from resolvents

- Generally, $R(\lambda)f = (A - \lambda I)^{-1} f$
- Variation of parameters form:

$$u(x, \lambda) = c_1 y_1 + c_2 y_2 + \int_0^x \frac{y_1(t) y_2(x) - y_1(x) y_2(t)}{W(y_1, y_2)} f(t)\, dt.$$

- Hilbert space form: $-\psi_n'' + q(x)\psi_n = \lambda_n \psi_n$,

$$u(x, \lambda) = \int_\alpha^\beta \sum_n \frac{\psi_n(x)\overline{\psi_n(t)}}{\lambda_n - \lambda} f(t)\, dt.$$

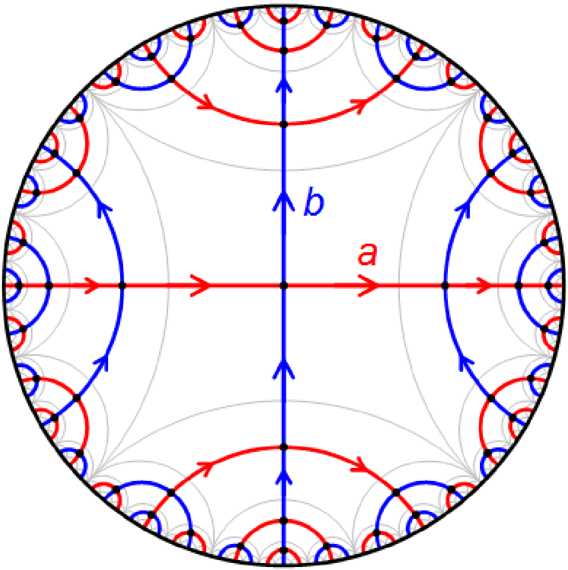
- Spatial domain bigger, denser λ_n, but resolvent ok for $\lambda \notin \mathbb{R}$

$$\frac{1}{2}[P_{[a,b]} + P_{(a,b)}]f = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_a^b [R(\sigma + i\epsilon) - R(\sigma - i\epsilon)] f \, d\sigma.$$

Cayley graphs $\mathbb{F}_M$

- Free group $\mathbb{F}_M$ on M generators
  $\mathbb{F}_M$: equivalence classes of finite length words generated by distinct $s_1, \ldots, s_M$ and inverse symbols $s_1^{-1}, \ldots, s_M^{-1}$.
- Cayley graph $\mathcal{T}_M$ for the group $\mathbb{F}_M$ with (minimal) generating set $\mathbb{S}$ is the directed/undirected graph with $\mathbb{F}_M$ elements as vertices and directed edges (v, vs) with $s \in \mathbb{S}$. $\mathbb{F}_M$ acts on $\mathcal{V}, \mathcal{E}$ by left multiplication.
- The undirected Cayley graph $\mathcal{T}_M$ of $\mathbb{F}_M$ is a tree whose vertices have degree $2M$.

$\mathcal{T}_2$ from $\mathbb{F}_2$



Quantum Cayley graphs $\mathbb{F}_M$

- Edges are intervals with all (v, vs_m) same length l_e.
- $L^2(\mathcal{T}_M)$ is $\oplus_e L^2[0, l_e]$ with the inner product

$$\langle f, g \rangle = \int_{\mathcal{T}_M} f\overline{g} = \sum_e \int_0^{l_e} f_e(x)\overline{g_e(x)}\, dx.$$

- Self-adjoint $\Delta + q = -\frac{d^2}{dx^2} + q$ acts component-wise. Continuity plus derivative conditions for domain.
- Functions $q \geq 0$ are even on each edge, same on (v, vs_m).

Outline 1: Find one-dim vector spaces: FA + ODE

- (i) Treat $R_e(\lambda)f$, where $\mathrm{support}(f) \subset e$.
- (ii) With this restriction

$$R_e(\lambda)f = \int_e R_e(x,t,\lambda)f(t)\,dt,$$

$$R_e(x,t,\lambda) = \begin{matrix} y_-(x,\lambda)y_+(t,\lambda)/W_k(\lambda), & 0 \le x \le t \le l_e, \\ y_-(t,\lambda)y_+(x,\lambda)/W_k(\lambda), & 0 \le t \le x \le l_e. \end{matrix}$$

  and $y_\pm(x,\lambda)$ extend past e to whole tree.
- (iii) The y_+ on the half-tree $\mathcal{T}_e^+$ (same for $-$) are square integrable satisfying de and vertex conditions - this is a distinguished one-dimensional space $\mathbb{X}_e$ of functions

Outline 2: multipliers from $\mathbb{F}_M$ action on $\mathcal{T}_M$

- The abelian subgroups $vs_m^k v^{-1}$ act to give a nested family of subtrees $\mathcal{T}^+_{e(k)}$. The induced action on $\mathbb{X}_{e(k)}$ is by complex multiplication by $\mu_m(\lambda)$.

Theorem

Assume $\lambda \in \mathbb{C} \setminus [0,\infty)$, $e = (v, vs_m)$ and $y_+ \in \mathbb{X}_e^+$ with $y_+(v) = 1$. Suppose w is a vertex in $\mathcal{T}_e^+$, and the path from v to w is given by the reduced word $s_m s_{k(1)}^{\pm 1} \dots s_{k(n)}^{\pm 1}$. Then

$$y_+(w) = \mu_m \mu_{k(1)} \dots \mu_{k(n)}. \tag{0.1}$$

By ODE the vertex values of y_+ can be interpolated to the edges.

Outline 3: μ_m satisfy equations

- Solutions $y_\pm$ satisfy $-y'' + q_m y = \lambda y$ on each edge. Use standard basis

$$C_m(x,\lambda) \sim \cos(\sqrt{\lambda}x), \quad S_m(x,\lambda) \sim \sin(\sqrt{\lambda}x)/\sqrt{\lambda}.$$

Vertex conditions + multipliers + basis lead to

Theorem

For $\lambda \in \mathbb{C} \setminus [0,\infty)$ and $m = 1,\dots,M$, the multipliers $\mu_m(\lambda)$ satisfy the system of equations

$$\frac{\mu_m^2(\lambda) - 1}{S_m(l_m,\lambda)\mu_m(\lambda)} - 2\sum_{k=1}^{M} \frac{\mu_k(\lambda) - C_k(l_k,\lambda)}{S_k(l_k,\lambda)} = 0. \tag{0.2}$$

Outline 4: Equations (0.2) not pathological

- The μ_m equations (0.2) have solutions $\xi_m(\lambda)$ including $\mu_m(\lambda)$.
- Write as $P_m(\xi_1,\ldots,\xi_M)=0$, compute gradients, do linear algebra, use Implicit Function Theorem.

Corollary

Suppose $\lambda \in \mathbb{C}$, $S_m(\lambda, l_m) \neq 0$, and

$$\xi_m(\lambda)^2+1\neq 0, \quad \sum_{m=1}^{M}\frac{\xi_m^2}{\xi_m^2+1}\neq 1/2, \quad m=1,\ldots,M.$$

Solutions $\xi(\lambda)$ are locally holomorphic $\mathbb{C}^M$- valued functions of λ.

Outline 5: Elimination gives $1 - var$ polynomials

- Rewrite (0.2) as

$$F_m(\xi_m, \lambda) = \frac{\xi_m^2(\lambda) - 1}{S_m(l_m)\xi_m(\lambda)} = g = 2\sum_{k=1}^{M} \frac{\xi_k(\lambda) - C_k(l_k)}{S_k(l_k)}.$$

- Subtraction gives $F_m = F_{m-1}$ for $M - 1$ equations. Move up the equations, reducing to degree 1 in ξ_k, using the quadratic formula to eliminate ξ_k, then 'squaring' to eliminate roots.

- **Theorem**
  There is a discrete set $Z_0 \subset \mathbb{R}$ and a positive integer N such that for all $\lambda \in \mathbb{C} \setminus Z_0$ the equations satisfied by the multipliers $\mu_m(\lambda)$ have at most N solutions $\xi_1(\lambda), \dots, \xi_M(\lambda)$. For $\lambda \in \mathbb{C} \setminus Z_0$, the functions $\mu_m(\lambda)$ are solutions of $1 - var$ polynomial equations $p_m(\xi_m) = 0$ of positive degree, with coefficients entire in λ.

Outline 6

- For $\sigma \in [0, \infty)$ there are limiting values $\mu^{\pm}(\sigma)$.
- Let $Z_m \subset \mathbb{C}$ be the discrete set where the leading coefficient of $p_m(\xi_m)$ vanishes. For $\lambda \in \mathbb{C} \setminus Z_m$ the roots of $p_m(\lambda)$, in particular $\mu_m(\lambda)$, are holomorphic if root is simple. Otherwise the roots extend continuously The limiting values $\mu^{\pm}(\sigma)$ need not agree.

Theorem

Suppose $(\alpha, \beta) \cap Z_0 = \emptyset$. For $m = 1, \ldots, M$ also assume that $(\alpha, \beta) \cap Z_m = \emptyset$ and that $\mu_m^{\pm}(\sigma) \neq \pm 1$ for all $\sigma \in (\alpha, \beta)$. If $[a, b] \subset (\alpha, \beta)$, e is an edge of type m, and $f \in L^2(e)$, then

$$P_{[a,b]}f = \frac{1}{2\pi i} \int_a^b [R_e^+(\sigma) - R_e^-(\sigma)]f \; d\sigma. \tag{0.3}$$

If $\sigma_1 \in (\alpha, \beta)$, then σ_1 is not an eigenvalue of $\Delta + q$.

Outline 7

- The spectrum can be recognized.

Corollary

Assume $\sigma \in [0, \infty) \setminus Z_0$ and for $m = 1, \dots, M$ suppose $\mu_m^{\pm}(\sigma) \neq \pm 1$. Then σ is in the resolvent set of $\Delta + q$ if and only if $\mu_m^{+}(\sigma)$ is real valued in open neighborhood of σ for $m = 1, \dots, M$.

Corollary

Suppose $M \geq 2$ and the lengths l_m are rational. Then the resolvent set of Δ includes an unbounded sequence of pairwise disjoint open intervals in $[0, \infty)$.

Computations 1

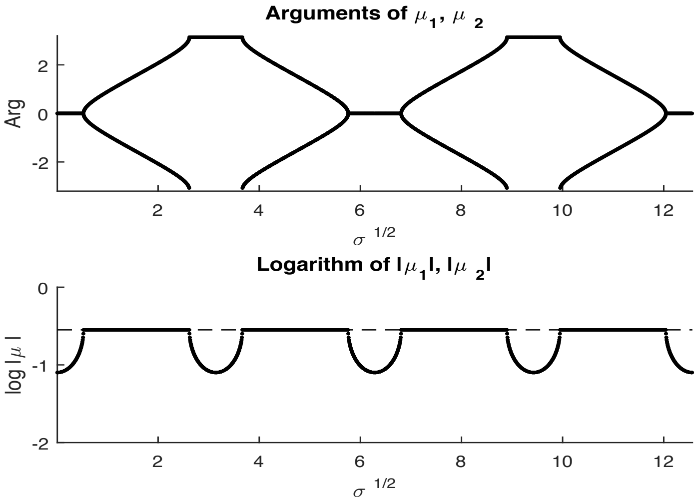


Figure: Case $l_1 = 1,\ l_2 = 1$

Computations 2

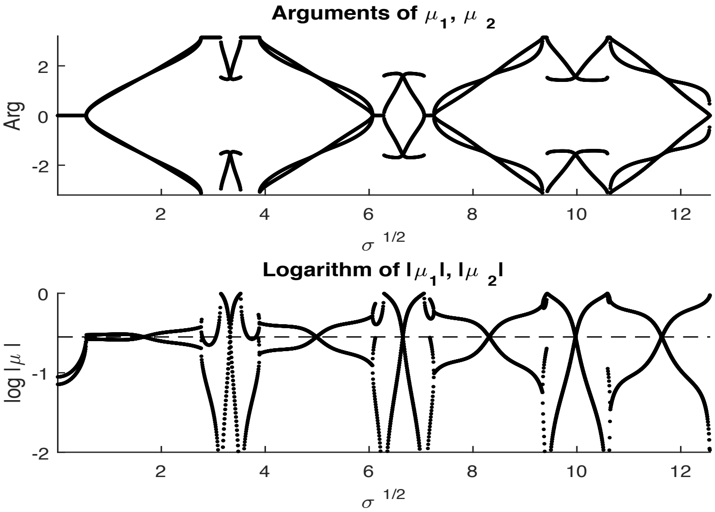


Figure: Case $l_1 = 1$, $l_2 = .89$

Computations 3

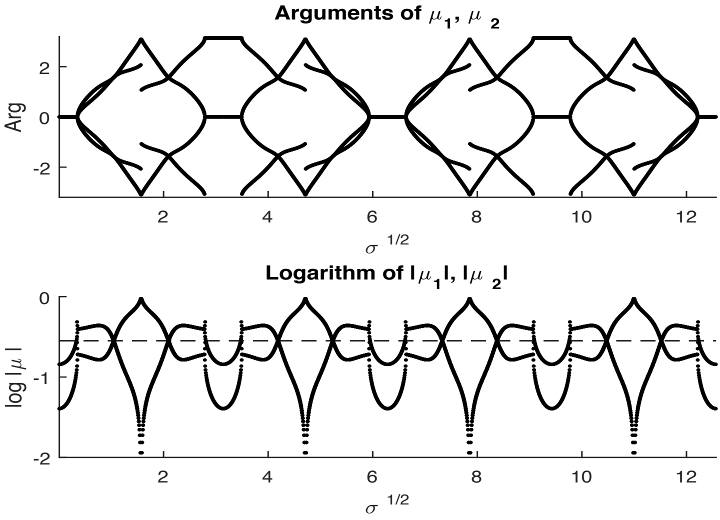


Figure: Case $l_1 = 1$, $l_2 = 2$