Matrix Sturm-Liouville problems analytic in a strip
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- $$P_2(z)\frac{d^2Y}{dz^2} + P_1(z)\frac{dY}{dz} + P_0(z)Y = \lambda Y$$
  $$Y(z+2\pi) = Y(z)$$
- Can we say something new about a 200 year old problem?

Eigenvalue problem

- $$\mathcal{L}Y = \lambda Y, \quad \mathcal{L} = P_2(z)D^2 + P_1(z)D + P_0(z), \quad D = \frac{d}{dz} \sim \frac{d}{dx}$$
- Scalar example: $\mathcal{L} = -D^2, \quad \lambda_n = n^2,$
  $Y(z + 2\pi, \lambda) = Y(z, \lambda),$
  $$\psi_n(z) = \exp(inz) = e^{inx}e^{-ny} = \cos(nz) + i\sin(nz).$$
- $P_j(z)$ are $K \times K$, 2π - periodic, complex analytic in
  $$\Omega = \{z = x + i\tau \in \mathbb{C}, -T < \tau < T\}.$$
- $\mathcal{L}$ is often self adjoint on the Hilbert space $\oplus_K \mathbb{L}^2_{per}$, the 2π periodic vector-valued functions square integrable on $[0, 2\pi]$.
- What does analyticity add?

Motivation 1

- Reaction-diffusion models for population or chemical interactions with K-species $u_1(t,x), u_2(t,x), \ldots, u_N(t,x)$ in one space dimension
- Equations with spatially varying diffusion

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial x} k_1(x) \frac{\partial u_1}{\partial x} + F_1(u_1, \ldots, u_K)$$

$$\vdots$$

$$\frac{\partial u_N}{\partial t} = \frac{\partial}{\partial x} k_N(x) \frac{\partial u_N}{\partial x} + F_N(u_1, \ldots, u_K)$$

- Let $U = [u_1, \ldots, u_N]$

Motivation 2 - Pretend $\mathcal{L}$ is a square matrix

- Approximate system with linear equations

$$\frac{\partial U}{\partial t} = \mathcal{L}U, \quad U(0) = U_0$$

- Matrix version (say after discretization) would look like

$$\frac{\partial U}{\partial t} = AU \quad U(0) = U_0$$

- Use expansion in eigenvectors/functions to solve

$$A\psi_j = \lambda_j \psi_j, \quad U(t) = \sum_j c_j(t)\psi_j, \quad U_0 = \sum_j c_j \psi_j$$

- Equation becomes decoupled system

$$\sum_j \frac{dc_j}{dt}\psi_j(x) = \sum_j \lambda_j c_j(t)\psi_j(x).$$

L^2_{per} Hilbert space approach

- Analogy with linear algebra is facilitated in a Hilbert space
- $$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)}\, dx$$
- Good cases self adjoint (integrate by parts)
$$\langle \mathcal{L} f, g \rangle = \langle f, \mathcal{L} g \rangle$$
- Usually (nearly self adjoint) (but not always) we get eigenvalues $\lambda_1, \lambda_2, \cdots \to \infty$ and a basis of eigenfunctions
- The case $\mathcal{L} = -D^2$ has eigenvalues $\lambda_n = n^2$, functions
$$\psi_n(x) = \exp(inx) = \cos(nx) + i \sin(nx).$$

Use weighted Fourier series for better fits

- If $f = \sum_j \alpha_n \exp(inx)$, $\quad g = \sum_n \beta_n \exp(inx)$, then

$$\|f - g\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f - g|^2 = \sum_n |\alpha_n - \beta_n|^2,$$

  or to make derivatives close too,

- $$\|f - g\|_1^2 = \frac{1}{2\pi} \int_0^{2\pi} |f - g|^2 + |f' - g'|^2$$

$$= \sum_n |\alpha_n - \beta_n|^2 + \sum_n n^2 |\alpha_n - \beta_n|^2,$$

  etc.

A Hilbert space of analytic functions

- $\mathbb{H}^2(\Omega)$ is the set of Fourier series $f = \sum_n c_n \exp(inz)$ satisfying the exponential weighted summability condition

$$\|f\|_H^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \cosh(2nT) < \infty,$$

  These Fourier series converge uniformly to an analytic function on any strip with height $T_1 < T$.

- $\mathbb{H}^2(\Omega)$ has an inner product (and e^{inz} are orthogonal)

$$\langle f, g\rangle_H = \frac{1}{4\pi}\int_0^{2\pi} [f(x+iT)\overline{g(x+iT)} + f(x-iT)\overline{g(x-iT)}]\, dx,$$

$$= \int_{\partial\Omega} f(z)\overline{g(z)}.$$

Hardy space basics

- These spaces are named after G.H. Hardy. They come in several flavors.
- Here are some standard facts.

- Proposition

  If $f \in \mathbb{H}^2(\Omega)$, then the boundary values $f(x \pm iT)$ are well-defined elements of $\mathbb{L}^2_{per}$, and the set $\{(f(x+iT), f(x-iT)), f \in H^2\}$ is a closed subspace of $\mathbb{L}^2_{per} \oplus \mathbb{L}^2_{per}$. $H^2(\Omega)$ is the set of all functions analytic in the strip with period 2π and

$$\sup_{-T<\tau<T} \int_0^{2\pi} |f(x+i\tau)|^2 \, dx < \infty. \tag{0.1}$$

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Our main question

- Do the 2π periodic eigenfunctions of 'nice' analytic $\mathcal{L} = P_2(z)D^2 + P_1(z)D + P_0(z)$ have dense span in $\oplus_K \mathbb{H}^2(\Omega)$?
- 'nice' should include those which are self adjoint in L^2_{per}
- Note that the eigenvalues and eigenfunctions 'agree' in L^2_{per} and $\mathbb{H}^2(\Omega)$.

The answer can be YES

- Perhaps the simplest examples are the operators

$$\mathcal{L}_1 = e^{i\phi(z)} D e^{-i\phi(z)} = \frac{d}{dz} - i\phi'(z). \tag{0.2}$$

  Suppose $\phi(z)$ is entire, 2π periodic, and real-valued for $z \in \mathbb{R}$.
- The operator $\mathcal{L}_1$ is skew adjoint and unitarily equivalent to D on $\mathbb{L}^2_{per}$.
- D is still skew adjoint on $\mathbb{H}^2$, but $\mathcal{L}_1$ is only similar to D with eigenfunctions $\psi_n(z) = e^{i\phi(z)} e^{inz}$. Since $\phi(z)$ is entire, the operator similarity holds for any strip height T.

Scalar case: Change of (real) variables can help

- Start with first order operators

$$\mathcal{L}_1 = p_1(z)D + p_0(z).$$

- If $p_1(x)$ is positive, the change of variables,

$$w = C_1 \int_0^z \frac{1}{p_1(s)}\, ds, \quad C_1 = 2\pi / \int_0^{2\pi} \frac{1}{p_1(s)}\, ds. \tag{0.3}$$

  transforms the periodic operator $p_1(z)D$ to $C_1 d/dw$ and carries $[0, 2\pi]$ to $[0, 2\pi]$.

Complex variables version

- With some limits, this idea has a complex variables extension.

- Proposition

  Suppose $p_1(z)$ is analytic and 2π periodic in Ω. If the real part of $1/p_1(z)$ is positive on Ω, then $w(z)$ given in (0.3) is injective on Ω, and the range includes a strip $\Omega_r = \{-r < \Im(w) < r\}$.

  - In the second order scalar case

    $$\mathcal{L}_2 = p_2(z)D^2 + p_1(z)D + p_0(z).$$

    a version (plus a conjugation trick) reduces $\mathcal{L}_2$ to Liouville (Schrödinger) normal form $-D^2 + q(z)$
  - $-D^2 + q(z)$ is close to self adjoint, so eigenfunctions have dense span in a small strip

For $L_1 = p(z)D$ the answer can be NO!

- The eigenvalue equation $p(z)Dy = \lambda y$ has solutions

$$y(z,\lambda) = C\exp(\lambda \int_0^z \frac{1}{p(s)}\ ds). \tag{0.4}$$

- If $\int_0^{2\pi} 1/p(s)\ ds = 2\pi$, then 2π periodic nontrivial solutions have $\lambda_n = in$ for integer n.
- Trouble if

$$w(z) = \int_0^z 1/p(s)\ ds$$

is not one-to-one in Ω_0 (mod 2π).

More NO

- Use $w(z+2\pi) = w(z) + 2\pi$ and find $z_1 \neq z_2$ with $w(z_1) = w(z_2)$.
- Begin with a nonconstant, entire and 2π periodic function $q(z)$, with $q(x) > 0$ for $x \in \mathbb{R}$. Take

$$p(z) = C_1 \exp(q(z)), \quad C_1 = \frac{1}{2\pi} \int_0^{2\pi} \exp(-q(x))\, dx,$$

so that $p(z)$ is 2π periodic, $w(z+2\pi) = w(z) + 2\pi$, and $p(z)$ and $w(z)$ are both entire with essential singularities at ∞.

NO for T large

- Proposition

  If $w(z)$ is entire with an essential singularity at ∞, then for a dense set of points $w_0 \in \mathbb{C}$, $w_0^{-1} = \{z \in \mathbb{C} \mid w(z) = w_0\}$ is infinite.

- Proposition

  There will be disjoint nonempty open sets $U_1, U_2 \in \Omega_0$ with the property that if $z_1 \in U_1$, then there is a $z_2 \in U_2$ such that

  $$\exp(inw(z_1)) = \exp(inw(z_2)), \quad n = 0, \pm 1, \pm 2, \dots.$$

- Proposition

  There is an infinite dimensional subspace $\mathcal{N} \subset \mathbb{H}^2$, and for all eigenfunctions $y_n(z)$

  $$\langle y_n(z), g(z) \rangle_H = 0, \quad g \in \mathcal{N}$$

Leap ahead to Main result

- **Proposition**

  Suppose the leading coefficient of the regular operator

$$\mathcal{L} = -DP_2(z)D + P_1(z)D + P_0(z) \tag{0.5}$$

  satisfies the positive real part condition

$$\Re(V^* P_2(z)V) \geq C_0 \|V\|_E^2, \quad V \in \mathbb{C}^K, \quad C_0 > 0, \quad z \in \overline{\Omega}. \tag{0.6}$$

  Then $\mathcal{L}$ generates a holomorphic semigroup $S(t)$ in $\oplus_K \mathbb{H}^2$.

- **Theorem**

  Suppose the regular operator $\mathcal{L}$ of (0.5) is self adjoint and positive as an operator on $\oplus_K \mathbb{L}^2_{per}$ and satisfies (0.6) on Ω. Then the periodic eigenfunctions of $\mathcal{L}$ have dense span in $\oplus_K \mathbb{H}^2$.

Sketch proof 1

- Since $\mathcal{L}$ has compact resolvent there is an orthonormal basis of eigenfunctions ψ_n with eigenvalues λ_n. As a semigroup on $\oplus_K \mathbb{L}^2_{per}$, $S(t)$ can be represented using an eigenfunction expansion,

$$S_L(t)f = \exp(-t\mathcal{L})f(x) = \sum_n c_n \exp(-t\lambda_n)\psi_n \qquad (0.7)$$

  where

$$c_n = \langle f, \psi_n \rangle_L = \frac{1}{2\pi}\int_0^{2\pi} \psi_n(s)^* f(s)\, ds.$$

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Sketch proof 2

- Estimates: $|\psi_n(z)| \leq C_1 \exp(C_2\lambda_n^{1/2})$ for $z \in \Omega$, $\lambda_n^{1/2} \geq \alpha n$ for some $\alpha > 0$, so there is a pointwise estimate

$$\|e^{-t\lambda_n}\psi_n(z)\|_E \leq C_1 \exp(C_2\lambda_n^{1/2} - t\lambda_n).$$

- For $t > 0$ the series in (0.7) converges uniformly to a function analytic in Ω and continuous on $\overline{\Omega}$.
- Analytic continuation of $S_L(t) =$ analytic semigroup $S(t)$.
- Finally, suppose $h \in \oplus_K \mathbb{H}^2$ is othogonal to all eigenfunctions of $\mathcal{L}$. Then

$$\langle S(t)f, h\rangle_H = \langle \sum_n c_n \exp(-t\lambda_n)\psi_n, h\rangle = 0, \quad t > 0.$$

But $S(t)f$ converges in $\oplus_K \mathbb{H}^2$ to f for all f in $\oplus_K \mathbb{H}^2$, implying that $h = 0$. Consequently, the periodic eigenfunctions of $\mathcal{L}$ have dense span in $\oplus_K \mathbb{H}^2$.