Complex Analytic Functions and Differential Operators

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Some motivation

- Suppose L is a differential expression (formal operator)

$$L = \sum_{k=0}^{N} p_k(z) D^k, \quad D = \frac{d}{dz} \tag{0.1}$$

  with $p_k(z) = \sum_{j=0}^{\infty} b_j z^j$ analytic on the unit disk $\mathbb{D} \subset \mathbb{C}$.
- Think of L as a (big) matrix with eigenvalues and eigenvectors. Use these ($N = 2$) to understand

$$\frac{du}{dt} = Lu, \quad (heat\ equation)$$

$$\frac{du}{dt} = iLu, \quad (Schrodinger\ equation)$$

$$\frac{d^2u}{dt^2} = Lu, \quad (wave\ equation)$$

Some hair splitting

- This plan works best in a Hilbert space $\mathcal{H}$ with inner product $\langle f, g\rangle$, and when $\mathcal{L}$ is (nearly) self adjoint.
- Distinguish an adjoint pair L, L^+

  $$\langle Lf, g\rangle = \langle f, L^+ g\rangle, \quad f, g \in \mathcal{D}_{min}$$

  and (formal) symmetry $L = L^+$ from the adjoint $\mathcal{L}^*$ (graph)

  $$\mathcal{L}^* = \{(g, h) \mid \langle \mathcal{L}f, g\rangle = \langle f, h\rangle, \ \textit{all } f \textit{ in domain of } \mathcal{L}\}$$

  with self adjoint $\mathcal{L} = \mathcal{L}^*$.
- Typically, $\mathcal{L}$ is only densely defined, and a symmetric $\mathcal{L}$ may have no self-adjoint extensions, exactly one, or many.

Previous work

- Often for real variables, $\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}\, dx$. What if L acts on a Hilbert space of analytic functions on $\mathbb{D}$? When is L symmetric/self adjoint ?
- A. Villone (E.A. Coddington) initiated just such a project in his dissertation (1966), and in subsequent papers (1970-1981). Hilbert space was the Bergman space $\mathcal{A}^2$ of analytic functions square integrable with respect to area measure on $\mathbb{D}$.

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dxdy$$

- This project was extended in the dissertation of W. Stork (J. Weidmann), pubs (1975).

Extend to weighted Hardy spaces

- On $\mathcal{D}_{min}$ = polynomials $f = \sum_{n=0}^{K} a_n z^n$, use a sequence $\beta = \{\beta_n > 0, n = 0, 1, 2, \ldots\}$ to define an inner product

$$\langle f_1, f_2 \rangle_\beta = \sum_{n=0}^{\infty} a_n \overline{c_n} \beta_n^2, \quad \|f_1\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2.$$

- The Hilbert space $\mathcal{H}_\beta$ is $g = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|g\|_\beta^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty.$$

  $\mathcal{H}_\beta$ has an orthonormal basis $e_n = z^n/\beta_n$, $n = 0, 1, 2, \ldots$.
- Examples include the classical Hardy space $\mathcal{H}^2$, $\beta_n = 1$, and the Bergman space $\mathcal{A}^2$, $\beta_n = \sqrt{\pi/(n+1)}$.

Linking β to $\mathbb{D}$

- Assume (M_z is bounded)

$$C_\beta = \sup_n \beta_{n+1}/\beta_n < \infty, \tag{0.2}$$

and

$$\lim_{n\to\infty} \beta_n^{1/n} = r, \quad r > 0. \tag{0.3}$$

Proposition

If (0.3) holds, then every $f \in \mathcal{H}_\beta$ is analytic in the open disk $\mathbb{D}_r$. (RKHS) For $j = 0, 1, 2, \ldots$ and $z_1 \in \mathbb{D}_r$, the linear functional $f^{(j)}(z_1)$ given by derivative evaluation is uniformly bounded in the $\mathcal{H}_\beta$ norm on compact subsets of $\mathbb{D}_r$. For any $R > r$, every function analytic in $\mathbb{D}_R$ belongs to $\mathcal{H}_\beta$, and some functions in $\mathcal{H}_\beta$ do not extend analytically to $\mathbb{D}_R$.

Formal adjoints are rare

- Villone has the next result in $\mathcal{A}^2$ when $L = L^+$.

Theorem

Suppose $p_k(z)$ and $q_k(z)$ are in $\mathcal{H}_\beta$ for $k = 0, \ldots, N$, with β satisfying (0.2) *and* (0.3)*. If $L = \sum_{k=0}^{N} p_k(z) D^k$ has a formal adjoint $L^+ = \sum_{k=0}^{N} q_k(z) D^k$, then L has polynomial coefficients, with* $\deg(p_k(z)) \leq N + k$.

- Idea of proof: Use $e_n = z^n/\beta_n$ to form matrices $A_{mn} = \langle Le_m, e_n \rangle$ for L and L^+. Since $D^k e_m = Ce_{m-k}$ while multiplication by $p_k(z)$ raises degree, $A_{mn} = 0$ for $n < m - N$. Matrix for L^+ should be (i) same form, and (ii) the conjugate transpose of A. So the nonzero entries of A must be located near the diagonal.

Interesting β are rare

- Theorem
  Let $\{\beta_n\}$ be a positive sequence. Suppose $L_1 = [a_2 z^2 + a_0]D + b_1 z$ has a formal adjoint $L_1^+ = [c_2 z^2 + c_1 z + c_0]D + [d_1 z + d_0]$ on $\mathcal{H}_\beta$. If $a_2 \neq 0$, then $b_1 = \sigma a_2$ for some $\sigma > 0$. The weights satisfy

$$\beta_{m+1}^2 = \beta_m^2 \frac{\beta_1^2}{\beta_0^2} \frac{(m+1)\sigma}{m+\sigma}, \tag{0.4}$$

  and $\mathcal{H}_\beta$ is a Hilbert space of functions analytic on $\mathbb{D}_r$ with $r^2 = \beta_1^2 \sigma / \beta_0^2$.

  For convenience assume restricted weight sequences β with

$$\sigma > 0, \quad \beta_0 = 1, \quad \beta_1^2 = 1/\sigma, \quad \beta_{m+1}^2 = \beta_m^2 \frac{m+1}{m+\sigma}, \quad m \geq 1. \tag{0.5}$$

  $\mathbb{D}$ is the natural domain for $\mathcal{H}_\beta$.

Existence of adjoint pairs

- Theorem
  Let β be a weight sequence satisfying (0.5). *On $\mathcal{H}_\beta$ every*
  $$L_1 = [a_2 z^2 + a_1 z + a_0]D + \sigma a_2 z + b_0$$
  has a formal adjoint
  $$L_1^+ = [\overline{a_0} z^2 + \overline{a_1} z + \overline{a_2}]D + \sigma \overline{a_0} z + \overline{b_0}.$$

  - Let $\mathbb{A}_\beta$ be expressions with a formal $\mathcal{H}_\beta$ adjoint L^+.

  Theorem
  As an algebra, $\mathbb{A}_\beta$ is generated by its expressions with order at most one. If $L = \sum_{k=0}^{N} p_k(z) D^k \in \mathbb{A}_\beta$, with $p_N(z) = \sum_{j=0}^{2N} c_j z^j$, then the leading coefficient of L^+ is $q_N(z) = \sum_{j=0}^{2N} \overline{c_j} z^{2N-j}$. If $z_1 \neq 0$, then $p_N(z_1) = 0$ if and only if $q_N(1/\overline{z_1}) = 0$.

Classical examples in $\mathbb{A}_\beta$

- Suppose $N = 2$. Then $\mathbb{A}_\beta$ includes the Jacobi (Gauss) operators

$$(1 - z^2)D^2y + \{(\beta - \alpha) - (\alpha + \beta + 2)z\}Dy, \tag{0.6}$$

  and other operators with polynomial eigenfunctions.
- It is easy to show that $L \in \mathbb{A}_\beta$ of order 2 can have any nonzero polynomial of order ≤ 4 with distinct roots as leading coefficient. This produces many equations $Ly = \lambda y$ with $L \in \mathbb{A}_\beta$ with three (Riemann-Papperitz), four (Heun), and five regular singular points (∞ being one).
- Starting from the classical (Riemann-Papperitz, Heun) operators, the adjoint map (typically?) increases the number of regular singular points.

When does $L = L^+$?

- Villone and Stork consider characterizing formally symmetric expressions L on $\mathcal{A}^2$.
- Villone succeeds with a fairly complex recursive technique. Applied when $N = 1$!!
- Stork shows that for $c \neq 0$, $n = 1, 2, 3, \ldots$, and $0 \leq r \leq n$ the examples

$$l_{n,r} = (cz^{n+r} + \overline{c}z^{n-r})D^n + \sum_{k=1}^{r} c \binom{r}{k} \frac{(n+1)!}{(n+1-k)!} z^{n+r-k} D^{n-k},$$

are formally symmetric in $\mathcal{A}^2$ without characterizing formal symmetry. Of course you can take real linear combinations of the $l_{n,r}$.

Adjoint formulas give a characterization in $\mathcal{H}_\beta$

- Let $B_{0,0} = 1$ and

$$B_{n,r} = (zD)^{n-r} D^r, \quad n = 1, 2, \ldots, \quad 0 \leq r \leq n.$$

Taking the adjoint factors in reverse order gives

$$B^+_{n,r} = (z^2 D + \sigma z)^r (zD)^{n-r}.$$

Since $L + L^+$ is formally symmetric,

Lemma

For $n = 0, 1, 2, \ldots, 0 \leq r \leq n$, and $c_{n,r} \in \mathbb{C}$, the expressions $c_{n,r} B_{n,r} + \overline{c_{n,r}} B^+_{n,r}$ are formally symmetric, with highest order term $(c_{n,r} z^{n-r} + \overline{c_{n,r}} z^{n+r}) D^n$ when written in the standard form (0.1).

Main result 1

- Theorem
  An differential expression L is formally symmetric in $\mathcal{H}_\beta$ if and only if it can be written in the form

$$L = \sum_{n=0}^{N} \sum_{r=0}^{n} [c_{n,r} B_{n,r} + \overline{c_{n,r}} B_{n,r}^{+}]. \tag{0.7}$$

  - Idea of proof: Show that these examples allow you to match any possible leading coefficient. Subtract and use induction.

Passing from symmetric to self-adjoint in $\mathcal{A}^2$.

- Stork showed that $L = L^+$ and $p_N(z) \neq 0$ for $|z| = 1$ implies $\mathcal{L}$ with polynomial domain is essentially self-adjoint. Also characterizes the domain. (When $z = \exp(i\theta)$ you get a "regular" periodic problem, so the condition is not surprising.)
- Villone later found conditions for a variety of equal (nonzero) deficiency indices, meaning self-adjoint extensions exist. ("Boundary conditions" ?)

Some definitions

- If the differential expression L has coefficients $p_k(z) \in \mathcal{H}_\beta$, the minimal operator $\mathcal{L}_{min}$ may be extended to a maximal operator $\mathcal{L}_{max}$ with the domain,

$$\mathcal{D}_{max} = \{f \in \mathcal{H}_\beta \mid Lf \in \mathcal{H}_\beta\}.$$

- Say that L is $\mathbb{D}-$*regular* if the coefficients $p_k(z)$ are analytic on the closed unit disk $\overline{\mathbb{D}}$, and $p_N(z) \neq 0$ when $|z| = 1$; in particular the set $\{z_k \mid |z_k| < 1, p_N(z_k) = 0\}$ is finite.
- For $k = 0, 1, 2, \ldots$, define additional (Sobolev) Hilbert spaces $\mathcal{H}_\beta^k$ of functions $f \in \mathcal{H}_\beta$ whose k-th derivative is also in $\mathcal{H}_\beta$. For $f = \sum_{n=0}^\infty c_n z^n$, and $g = \sum_{n=0}^\infty b_n z^n$,

$$\langle f, g \rangle_k = \sum_{n=0}^{\infty} (1 + n^{2k}) c_n \overline{b_n} \beta_n^2.$$

Main result 2: Regular case

- After some analysis (zeros in $p_N(z)$ are "singularities"),

- Theorem
  Suppose $L = \sum_{k=0}^{N} p_k(z)D^k$ is $\mathbb{D}-$ regular and formally symmetric on $\mathcal{H}_\beta$, with order $N \geq 1$. Then $\mathcal{L}_{max}$ is the closure of $\mathcal{L}_{min}$, and $\mathcal{L}_{max}$ is self adjoint. The resolvent $R(\lambda) : \mathcal{H}_\beta \to \mathcal{H}_\beta$ is compact, implying the spectrum is a sequence $\{\lambda_n\}$ of eigenvalues, $|\lambda_n| \to \infty$, eigenfunctions orthogonal and complete.
  - A closed operator T is Fredholm if T has a finite dimensional null space and a closed range of finite codimension. The index is $ind(T) = dim(null\ \ T) - codim(range\ \ T)$.

- Theorem
  Suppose $\sigma \geq 1$, and $L = \sum_{k=0}^{N} p_k(z)D^k$ is $\mathbb{D}-$ regular of order $N \geq 1$. If $p_N(z)$ has K roots in $\mathbb{D}$, counted with multiplicity, then $\mathcal{L}$ with domain $\mathcal{H}_\beta^N$ is Fredholm with index $N - K$.

Main result 3: eigenvalue location

- More analysis starting with a periodic problem on $\mathbb{R}$ with good leading coefficient.

- Theorem
  Suppose $L = \sum_{k=0}^{N} p_k(z) D^k$ *is a* $\mathbb{D} - regular\ formally\ symmetric$ *expression of order* $N \geq 2$ *on* $\mathcal{H}_\beta$*, with self-adjoint maximal operator* $\mathcal{L}$*. The eigenvalues of* $\mathcal{L}$ *can be enumerated as a sequence* $\{\lambda_n, n = 0, 1, 2, \dots\}$ *with the following description: for* $0 < \epsilon < 1$ *there are real nonzero constants* C_1 *and* $\tau > 0$*, and a* $C_2 \in \mathbb{C}$ *such that*

$$\lambda_n / C_1 = (n/\tau + C_2)^N + O(n^{N-2+\epsilon}).$$

  In particular, with at most finitely many exceptions, the eigenvalues are either all positive or all negative, and have multiplicity 1.

Comments on monodromy

- Suppose $p_N(z_1) = 0, p_N(z) \neq 0$. Solutions of $Ly = \lambda y$ have analytic continuations around z_1, but may not return to the same value. The local analysis near z_1 is usually easy (regular singular point, Frobenius series, indicial equation).
- Suppose $p(z_k) = 0$ for $k = 1, \ldots, K$ and $z_k \in \mathbb{D}$. Then having an eigenfunction y in $\mathcal{H}_\beta$ means y has trivial monodromy around each z_k. Instead of a simple local problem, we're solving a tricky global one.
- Why should this be linked to self-adjointness/leading coefficient symmetry?
- Hilbert's 21st problem (1900): To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group. Significant work 1908 – 1990. Generalizations in algebraic geometry (Deligne).