Math 340 Professor Carlson
Homework 3

section 2.1 # 19: Consider

$$\frac{dy}{dt} + \frac{1}{4}y = 3 + 2\cos(2t), \quad y(0) = 0.$$

a) Find the solution and describe the large t behavior.
b) Determine the value of t when the solution first hits $y = 12$.
Use the integrating factor $\mu(t) = e^{t/4}$ to get

$$\frac{d}{dt}[e^{t/4}y] = 3e^{t/4} + 2e^{t/4}\cos(2t).$$

Integration gives

$$e^{t/4}y = 12e^{t/4} + 2\int^t e^{s/4}\cos(2s)\ ds + C.$$

There is a trick for evaluating the integral. Let $I = \int^t e^{s/4}\cos(2s)\ ds$. Integrate by parts twice to get

$$I = e^{t/4}\sin(2t)/2 - \int^t e^{s/4}\sin(2s)/8\, ds = e^{t/4}\sin(2t)/2 + e^{t/4}\cos(2t)/16 - \int^t e^{s/4}\cos(2s)/64\, ds.$$

The last term is $I/64$, so

$$I = \frac{64}{65}[e^{t/4}\sin(2t)/2 + e^{t/4}\cos(2t)/16].$$

The general solution of the differential equation is

$$y = 12 + \frac{128}{65}[\sin(2t)/2 + \cos(2t)/16] + Ce^{-t/4}.$$

The initial condition $y(0) = 0$ gives

$$C = -12 - 8/65,$$

so

$$y(t) = 12 + \frac{128}{65}[\sin(2t)/2 + \cos(2t)/16] + [-12 - 8/65]e^{-t/4}.$$

The last term decays to 0 as $t \to \infty$, so the solution oscillates around $y = 12$.

section 2.1 # 20: Find the value of y_0 so that the solution of

$$\frac{dy}{dt} - y = 1 + 3\sin(t), \quad y(0) = y_0,$$

remains finite as $t \to \infty$.

Use the integrating factor $\mu(t) = e^{-t}$ to get

$$\frac{d}{dt}[e^{-t}y] = e^{-t} + 3e^{-t}\sin(t),$$

and

$$y(t) = -1 + 3e^t \int^t e^{-s}\sin(s)\ ds + Ce^t.$$

The integral can again be found by integrating by parts twice, or using a table to get

$$\int^t e^{-s}\sin(s)\ ds = \frac{e^{-t}}{2}[-\sin(t) - \cos(t)],$$

and

$$y(t) = -1 + \frac{3}{2}[-\sin(t) - \cos(t)] + Ce^t.$$

For the solution to remain finite as $t \to \infty$ we need $C = 0$, so

$$y_0 = y(0) = -1 - \frac{3}{2} = -5/2.$$

section 2.2 # 1

$$\frac{dy}{dx} = \frac{x^2}{y}, \quad y\ dy = x^2\ dx, \quad y^2/2 = x^3/3 + C.$$

section 2.2 # 2

$$\frac{dy}{dx} + y^2\sin(x) = 0, \quad -y^{-2}\ dy = \sin(x)\ dx, \quad y^{-1} = -\cos(x) + C.$$

section 2.2 # 5

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}, \quad (y + e^y)dy = (x - e^{-x})\ dx, \quad y^2/2 + e^y = x^2/2 + e^{-x} + C.$$

section 2.2 # 6

$$\frac{dy}{dx} = \frac{x^2}{1 + y^2}, \quad (1 + y^2)\ dy = x^2\ dx, \quad y + y^3/3 = x^3/3 + C.$$

section 2.2 # 19
Solve the initial value problem

$$\frac{dy}{dx} = 2y^2 + xy^2, \quad y(0) = 1,$$

and determine where the solution attains its minimum value.

Rewrite the equation as

$$y^{-2}\ dy = (2 + x)\ dx, \quad -y^{-1} = 2x + x^2/2 + C.$$

Since $y(0) = 1$ we have

$$-1 = C.$$

The solution with $y(0) = 1$ is

$$y(x) = \frac{-2}{(x_2)^2 - 6}.$$

At the minimum we should have $dy/dx = 0$, or from the original equation

$$0 = y^2(2 + x).$$

Since $y(x) \neq 0$, $x = -2$. So the minimum is at $x = -2$.

section 2.2 # 23
Solve

$$\frac{dy}{dx} = \frac{ay + b}{cy + d}.$$

Rewrite this equation as

$$\frac{cy + d}{ay + b}\, dy = dx, \qquad \frac{c}{a} \frac{y + b/a + d/c - b/a}{y + b/a}\, dy = dx,$$

giving

$$\frac{c}{a} y + (d/c - b/a) \log(|y + b/a|) = x + K.$$