Math 340 Professor Carlson

Homework 4

section 2.2 # 25: Consider

$$\frac{dy}{dt} = \frac{y - 4x}{x - y}.$$

a) Divide numerator and denominator on the right by x to get

$$\frac{dy}{dt} = \frac{y/x - 4}{1 - y/x}.$$

Since the right hand side is a function of y/x, it is homogeneous as described on the previous page.

b) Let $y = xv(x)$. By the product rule,

$$\frac{dy}{dx} = v(x) + x\frac{dv}{dx}.$$

c) The equation from part a) becomes

$$v(x) + x\frac{dv}{dx} = \frac{v - 4}{1 - v},$$

or

$$x\frac{dv}{dx} = \frac{v - 4}{1 - v} - \frac{v - v^2}{1 - v} = \frac{v^2 - 4}{1 - v}.$$

d) This equation is separable, with

$$\frac{1 - v}{v^2 - 4}\ dv = \frac{1}{x}\ dx.$$

Write the left hand side using partial fractions:

$$\frac{1 - v}{v^2 - 4} = \frac{A}{v - 2} + \frac{B}{v + 2}.$$

$$A + B = -1, \quad 2A - 2B = 1.$$

The equation becomes

$$-\frac{1/4}{v - 2} - \frac{3/4}{v + 2}\ dv = \frac{1}{x}\ dx.$$

Integration gives

$$\log(|v - 2|)^{-1/4} + \log(|v + 2|)^{-3/4} = \log(|x|) + C.$$

e) Returning to the variables x, ywe get

$$\log(|y/x - 2|)^{-1/4} + \log(|y/x + 2|)^{-3/4} = \log(|x|) + C.$$

section 2.3 # 1:

A tank contains $200l$ *of dye solution with a concentration of* $1g/l$. *The tank is rinsed with fresh water flowing at* $2l/min$. *When will the concentration reach 1 percent of its original value.*

Let Q be the amount of dye. The equation is

$$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out}) = 0 - (2)(Q/200),$$

or

$$\frac{dQ}{dt} = -.01Q.$$

The form of the solution is

$$Q(t) = Q_0 e^{-.01t}.$$

The concentration reach 1 percent of its original value when

$$e^{-.01t} = .01,$$

or

$$-.01t = log(.01) = -\log(100), \quad t = 460 \text{ minutes}.$$

Notice that the initial concentration is irrelevant.

section 2.3 # 2:

A tank contains 120 liters of pure water. A mixture containing a concentration of $\gamma g/l$ *of salt enters at* $2l/min$, *and fluid leaves at the same rate. Find the amount of salt in the tank, and the limiting amount.*

The modeling technique is the same as 1, with the equation

$$\frac{dQ}{dt} = 2\gamma - 2(Q/120).$$

Rewrite as

$$\frac{dQ}{dt} + Q/60 = 2\gamma.$$

$$\frac{d}{dt}(e^{t/60}Q) = 2\gamma e^{t/60}.$$

$$e^{t/60}Q = C + \int^t 2\gamma e^{s/60}\, ds = C + 120\gamma e^{t/60}.$$

$$Q(t) = Ce^{-t/60} + 120\gamma.$$

We have $Q(0) = 0$, so

$$0 = C + 120\gamma, \quad C = -120\gamma.$$

Thus

$$Q(t) = 120\gamma[1 - e^{t/60}].$$

The limiting amount is 120γ.

section 2.3 # 9:

a) Let $Q' = -rQ$ with $r > 0$. The general solution is

$$Q(t) = Q_0 e^{-rt}.$$

If the half life is 5730 years, then

$$Q_0/2 = Q_0 \exp(-5730r),$$

so $-5730r = \log(1/2)$, or

$$r = 1.2097 \times 10^{-4}.$$

b) The amount of Carbon-14 at time t (years) is

$$Q(t) = Q_0 \exp(-1.2097 \times 10^{-4} t).$$

c) If $Q(t) = Q_0/5$, then

$$Q_0/5 = Q_0 \exp(-1.2097 \times 10^{-4} t),$$

or $t = 13,304$ years.

section 2.3 # 12:
Newton's law of cooling states that

$$\frac{dT_1}{dt} = k(T_1 - T_0),$$

where T_0 is the environmental temperature. Fresh coffee is $200F$. One minute later it is $190F$. The room temperature is $70F$. When does the coffee reach $150F$?

We have

$$\frac{dT_1}{dt} - kT_1 = -kT_0 = -70k,$$

$$\frac{d}{dt}(e^{-kt}T_1) = -70ke^{-kt}.$$

Integration gives

$$e^{-kt}T_1 = C + 70e^{-kt},$$

$$T_1 = Ce^{kt} + 70.$$

We then have

$$T_1(0) = 200 = C + 70, \quad C = 130.$$

$$T_1(1) = 190 = 130e^k + 70, \quad k = -.08.$$

Finally,

$$150 = 130e^{-.08t} + 70, \quad t = 6 \text{ minutes}.$$

Here are partial solutions (the nongraphical parts) to p.67# 1-4 .
section 2.5 # 1:

$$\frac{dy}{dt} = ay + by^2, \quad a, b > 0.$$

Equilibrium points:
y=0 : unstable
y = -a/b: asymptotically stable

section 2.5 # 2:

$$\frac{dy}{dt} = y(y-1)(y-2)$$

Equilibrium points:
y =0:unstable
y=1:asymptotically stable
y=2: unstable

section 2.5 # 3:

$$\frac{dy}{dt} = e^y - 1$$

Equilibrium points:
y=0: unstable

section 2.5 # 4:

$$\frac{dy}{dt} = e^{-y} - 1$$

Equilibrium points:
y=0: asymptotically stable