Math 340 Professor Carlson
Homework 8

Section 3.5

#1 *Find the general solution of the equation*

$$y'' - 2y' - 3y = 3e^{2t}.$$

Start by writing the equation as

$$(D^2 - 2D - 3)y = (D - 3)(D + 1)y = 3e^{2t}.$$

The nonhomogeneous term on the right is annihilated by $D - 2$, so the solution must be a linear combination

$$y(t) = c_1 e^{3t} + c_2 e^{-t} + c_3 e^{2t}.$$

One solution must have the form $y_p(t) = c_3 e^{2t}$. Putting this into the equation gives

$$4c_3 e^{2t} - 4c_3 e^{2t} - 3c_3 e^{2t} = 3e^{2t},$$

so $c_3 = -1$. The general solution is

$$y(t) = c_1 e^{3t} + c_2 e^{-t} - e^{2t}.$$

#2 *Find the general solution of the equation*

$$y'' - y' - 2y = -2t + 4t^2.$$

Start by writing the equation as

$$(D^2 - D - 2)y = (D - 2)(D + 1)y = -2t + 4t^2.$$

The nonhomogeneous term on the right is annihilated by D^3, so the solution must be a linear combination

$$y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 + c_4 t + c_5 t^2.$$

One solution must have the form $y_p(t) = c_3 + c_4 t + c_5 t^2$. Putting this into the equation gives

$$2c_5 - (2c_5 t + c_4) - 2(c_3 + c_4 t + c_5 t^2) = -2t + 4t^2$$

so $c_5 = -2$, $c_4 = 3$, $c_3 = -7/2$. The general solution is

$$y(t) = c_1 e^{2t} + c_2 e^{-t} - 7/2 + 3t - 2t^2.$$

#3 *Find the general solution of the equation*

$$y'' + y' - 6y = 12e^{3t} + 12e^{-2t}.$$

Start by writing the equation as

$$(D^2 + D - 6)y = (D - 2)(D + 3)y = 12e^{3t} + 12e^{-2t}.$$

The nonhomogeneous term on the right is annihilated by $(D-3)(D+2)$, so the solution must be a linear combination

$$y(t) = c_1e^{2t} + c_2e^{-t} + c_3e^{3t} + c_4e^{-2t}.$$

One solution must have the form $y_p(t) = c_3e^{3t} + c_4e^{-2t}$. Putting this into the equation gives

$$(9c_3e^{3t} + 4c_4e^{-2t}) + (3c_3e^{3t} - 2c_4e^{-2t}) - 6(c_3e^{3t} + c_4e^{-2t}) = 12e^{3t} + 12e^{-2t}$$

so $c_3 = 2$, $c_4 = -3$. The general solution is

$$y(t) = c_1e^{2t} + c_2e^{-t} + 2e^{3t} - 3e^{-2t}.$$

#4 *Find the general solution of the equation*

$$y'' - 2y' - 3y = -3te^{-t}.$$

Start by writing the equation as

$$(D^2 - 2D - 3)y = (D-3)(D+1)y = -3te^{-t}.$$

The nonhomogeneous term on the right is annihilated by $(D+1)^2$, so the solution must be a linear combination

$$y(t) = c_1e^{3t} + c_2e^{-t} + c_3te^{-t} + c_4t^2e^{-t}.$$

One solution must have the form $y_p(t) = c_3te^{-t} + c_4t^2e^{-t}$. Putting this into the equation gives

$$(-2c_3 + c_3t + 2c_4 - 4c_4t + c_4t^2)e^{-t} - 2(c_3 - c_3t + 2c_4t - c_4t^2)e^{-t} - 3(c_3t + c_4t^2)e^{-t} = -3te^{-t},$$

or

$$(-2c_3 + c_3t + 2c_4 - 4c_4t + c_4t^2) - 2(c_3 - c_3t + 2c_4t - c_4t^2) - 3(c_3t + c_4t^2) = -3t,$$

so $c_3 = 3/16$, $c_4 = 3/8$. The general solution is

$$y(t) = c_1e^{3t} + c_2e^{-t} + \frac{3}{16}te^{-t} + \frac{3}{8}t^2e^{-t}.$$

#5 *Find the general solution of the equation*

$$y'' + 2y' = 3 + 4\sin(2t).$$

Start by writing the equation as

$$(D^2 + 2D)y = (D+2)Dy = 3 + 4\sin(2t).$$

The nonhomogeneous term on the right is annihilated by $D(D^2+4)$, so the solution must be a linear combination

$$y(t) = c_1e^{-2t} + c_2 + c_3t + c_4\sin(2t) + c_5\cos(2t).$$

One solution must have the form $y_p(t) = c_3 t + c_4 \sin(2t) + c_5 \cos(2t)$. Putting this into the equation gives

$$-4c_4 \sin(2t) - 4c_5 \cos(2t) + 2c_3 + 4c_4 \cos(2t) - 4c_5 \sin(2t) = 3 + 4\sin(2t).$$

so $c_3 = 3/2$, $c_4 = -1/2$, $c_5 = -1/2$. The general solution is

$$y(t) = c_1 e^{-2t} + c_2 + \frac{3}{2}t - \frac{1}{2}\sin(2t) - \frac{1}{2}\cos(2t).$$

Section 3.6

#1 *Solve by using variation of parameters and the method of undetermined coefficients.*

$$(NH) \quad y'' - 5y' + 6y = 2e^t.$$

The associated homogeneous equation is

$$(H) \quad y'' - 5y' + 6y = 0.$$

The characteristic equation is

$$r^2 - 5r + 6 = (r-3)(r-2) = 0, \quad r = 3, 2.$$

Take

$$y_1(t) = e^{2t}, \quad y_2(t) = e^{3t}.$$

The Wronskian for these solutions is

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = 3e^{5t} - 2e^{5t} = e^{5t}.$$

The variation of parameters formula give a particular solution

$$y_p(t) = -y_1(t) \int^t \frac{y_2(s)g(s)}{W(y_1, y_2)(s)}\, ds + y_2(t) \int^t \frac{y_1(s)g(s)}{W(y_1, y_2)(s)}\, ds.$$

In this case we get

$$\begin{aligned} y_p(t) &= -e^{2t} \int^t \frac{2e^{3s}e^s}{e^{5s}}\, ds + e^{3t} \int^t \frac{2e^{2s}e^s}{e^{5s}}\, ds \\ &= -2e^{2t} \int^t e^{-s}\, ds + 2e^{3t} \int^t e^{-2s}\, ds = 2e^t - e^t = e^t. \end{aligned}$$

The general solution is

$$y(t) = e^t + c_1 e^{2t} + c_2 e^{3t}.$$

#2 *Solve by using variation of parameters and the method of undetermined coefficients.*

$$(NH) \quad y'' - y' - 2y = 2e^{-t}.$$

The associated homogeneous equation is

$$(H) \quad y'' - y' - 2y = 0.$$

The characteristic equation is

$$r^2 - r - 2 = (r-2)(r+1) = 0, \quad r = 2, -1.$$

Take

$$y_1(t) = e^{2t}, \quad y_2(t) = e^{-t}.$$

The Wronskian for these solutions is

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = -e^t - 2e^t = -3e^t.$$

The variation of parameters formula give a particular solution

$$y_p(t) = -y_1(t) \int^t \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\ ds + y_2(t) \int^t \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\ ds.$$

In this case we get

$$y_p(t) = -e^{2t} \int^t \frac{2e^{-s}e^{-s}}{-3e^s}\ ds + e^{-t} \int^t \frac{2e^{2s}e^{-s}}{-3e^s}\ ds$$

$$= \frac{2}{3}e^{2t} \int^t e^{-3s}\ ds - \frac{2}{3}e^{-t} \int^t 1\ ds = -\frac{2}{9}e^{-t} - \frac{2}{3}te^{-t}.$$

Since e^{-t} is a solution of (H), another particular solution is

$$y_p(t) = -\frac{2}{3}te^{-t}.$$

The method of undetermined coefficients leads to the third order homogeneous equation

$$(D+1)(D-2)(D+1)y = 0.$$

The root $r = -1$ is repeated, so

$$y(t) = c_1 e^{2t} + c_2 e^{-t} + c_3 te^{-t}.$$

Putting this form into (NH) leads to

$$[-2c_3 e^{-t} + c_3 te^{-t}] - [c_3 e^{-t} - c_3 te^{-t}] - 2c_3 te^{-t} = 2e^{-t},$$

or

$$c_3 = -\frac{2}{3}.$$

#3 *Solve by using variation of parameters and the method of undetermined coefficients.*

$$(NH) \quad 4y'' - 4y' + y = 16e^{t/2}.$$

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