Math 340 Professor Carlson
Homework 9

Section 5.2
Solve using power series. If $x_0 = 0$ we may express solutions as

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad y' = \sum_{k=1}^{\infty} k a_k x^{k-1}, \quad y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

#1 $y'' - y = 0, \quad x_0 = 0.$
In series form the equation is

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} a_k x^k = 0.$$

Use the substitution $j = k - 2$ to rewrite the first series as

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = \sum_{j=0}^{\infty} (j+2)(j+1) a_{j+2} x^j = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k.$$

The equation can now be written as

$$\sum_{k=0}^{\infty} [(k+2)(k+1) a_{k+2} - a_k] x^k = 0.$$

(a) The recurrence relation is

$$a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

(b) Solutions satisfying

$$y_1(0) = 1 = a_0, \ y_1'(0) = 0 = a_1, \quad y_2(0) = 0 = a_0, \ y_2'(0) = 1 = a_1,$$

will be independent, with $W[y_1, y_2](x_0) = 1$. Computing the first few terms,

$$y_1 = 1 + \frac{1}{2} x^2 + \frac{1}{2 \cdot 3 \cdot 4} x^4 + \frac{1}{6!} x^6 + \dots,$$

$$y_2 = x + \frac{1}{3 \cdot 2} x^3 + \frac{1}{5!} x^5 + \frac{1}{7!} x^7 + \dots.$$

(d) These are the series for

$$y_1(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}, \quad y_2(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}.$$

#2 $\quad y'' + 3y' = 0, \quad x_0 = 0.$

In series form the equation is

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + 3\sum_{k=1}^{\infty} k a_k x^{k-1} = 0.$$

Use the substitution $j = k-2$ to rewrite the first series as

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k.$$

Use the substitution $j = k-1$ to rewrite the second series as

$$\sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{j=0}^{\infty}(j+1)a_{j+1}x^j = \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k.$$

The equation can now be written as

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2} + 3(k+1)a_{k+1}]x^k = 0.$$

(a) The recurrence relation is

$$a_{k+2} = -\frac{3a_{k+1}}{(k+2)}.$$

(b) Solutions satisfying

$$y_1(0) = 1 = a_0,\ y_1'(0) = 0 = a_1, \quad y_2(0) = 0 = a_0,\ y_2'(0) = 1 = a_1,$$

will be independent, with $W[y_1, y_2](x_0) = 1$. Computing the first few terms,

$$y_1 = 1,$$

$$y_2 = x - \frac{3}{2}x^2 + \frac{9}{2\cdot 3}x^3 - \frac{27}{4!}x^4 + \dots.$$

(d) (By looking for exponential solutions,) these are the series for

$$y_1(x) = 1, \quad y_2(x) = \frac{1}{3}[1 - e^{-3x}].$$

#3 $y'' - xy' - y = 0, \quad x_0 = 0.$
In series form the equation is

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - x\sum_{k=1}^{\infty} k a_k x^{k-1} - \sum_{k=0}^{\infty} a_k x^k = 0.$$

Use the substitution $j = k - 2$ to rewrite the first series as

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}x^j = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k.$$

The equation can now be written as

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0.$$

(a) The second series has no $k = 0$ term, so first we have

$$2a_2 - a_0 = 0,$$

and then the general recurrence relation is

$$a_{k+2} = \frac{a_k}{(k+2)}, \quad k \geq 1.$$

(b) Solutions satisfying

$$y_1(0) = 1 = a_0,\ y_1'(0) = 0 = a_1, \quad y_2(0) = 0 = a_0,\ y_2'(0) = 1 = a_1,$$

will be independent, with $W[y_1, y_2](x_0) = 1$. Computing the first few terms,

$$y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 + \frac{1}{2\cdot 4\cdot 6}x^6 + \dots,$$

$$y_2 = x + \frac{1}{3}x^3 + \frac{1}{3\cdot 5}x^5 + \frac{1}{3\cdot 5\cdot 7}x^7 + \dots.$$

#4 $y'' - xy' - y = 0, \quad x_0 = 1.$

In this case $x_0 = 1$, so we express solutions as

$$y = \sum_{k=0}^{\infty} a_k (x-1)^k, \quad y' = \sum_{k=1}^{\infty} k a_k (x-1)^{k-1}, \quad y'' = \sum_{k=2}^{\infty} k(k-1) a_k (x-1)^{k-2}.$$

We also rewrite the equation as

$$y'' - (x-1)y' - y' - y = 0.$$

In series form the equation is

$$\sum_{k=2}^{\infty} k(k-1)a_k(x-1)^{k-2} - (x-1)\sum_{k=1}^{\infty} k a_k (x-1)^{k-1} - \sum_{k=1}^{\infty} k a_k (x-1)^{k-1} - \sum_{k=0}^{\infty} a_k (x-1)^k = 0.$$

Use the substitution $j = k-2$ to rewrite the first series as

$$\sum_{k=2}^{\infty} k(k-1)a_k(x-1)^{k-2} = \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}(x-1)^j = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}(x-1)^k.$$

Similarly,

$$\sum_{k=1}^{\infty} k a_k (x-1)^{k-1} = \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^k.$$

The equation can now be written as

$$\sum_{k=0}^{\infty}[(k+2)(k+1)a_{k+2}(x-1)^k - \sum_{k=1}^{\infty} k a_k (x-1)^k - \sum_{k=0}^{\infty}(k+1)a_{k+1}(x-1)^k - \sum_{k=0}^{\infty} a_k (x-1)^k = 0.$$

(a) The second series has no $k = 0$ term, but we can add it since it is zero. The general recurrence relation is

$$a_{k+2} = \frac{a_k + a_{k+1}}{k+2}, \quad k \ge 1.$$

(b) Solutions satisfying

$$y_1(0) = 1 = a_0, \ y_1'(0) = 0 = a_1, \quad y_2(0) = 0 = a_0, \ y_2'(0) = 1 = a_1,$$

will be independent, with $W[y_1, y_2](x_0) = 1$. Computing the first few terms,

$$y_1 = 1 + \frac{1}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + \frac{1}{6}(x-1)^4 + \dots,$$

$$y_2 = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{4}(x-1)^4 + \dots.$$

#5 $\quad y'' + C^2 x^2 y = 0, \quad x_0 = 0.$

In series form the equation is

$$\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + C^2 \sum_{k=0}^{\infty} a_k x^{k+2} = 0.$$

Use the substitution $j = k+2$ to rewrite the second series as

$$\sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{j=2}^{\infty} a_{j-2} x^j = \sum_{k=2}^{\infty} a_{k-2} x^k.$$

The equation can now be written as

$$\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + C^2 \sum_{k=2}^{\infty} a_{k-2} x^k = 0.$$

(a) The first series contributes

$$a_2 = 0, \quad a_3 = 0.$$

The general recurrence relation is then

$$a_{k+2} = -C^2 \frac{a_{k-2}}{(k+2)(k+1)}.$$

(b) Solutions satisfying

$$y_1(0) = 1 = a_0,\ y_1'(0) = 0 = a_1, \quad y_2(0) = 0 = a_0,\ y_2'(0) = 1 = a_1,$$

will be independent, with $W[y_1, y_2](x_0) = 1$. Computing the first few terms,

$$y_1 = 1 - C^2 \frac{1}{4 \cdot 3} x^4 + C^4 \frac{1}{8 \cdot 7 \cdot 4 \cdot 3} x^8 + \dots,$$

$$y_2 = x - C^2 \frac{1}{5 \cdot 4} x^5 + C^4 \frac{1}{4 \cdot 5 \cdot 8 \cdot 9} x^9 + \dots.$$