Math 340 Test 1 Professor Carlson

SHOW ALL YOUR WORK

1. (15 points) Determine the order of the differential equation, and whether the equation is linear or nonlinear:

$$a)\ \sin(t)\frac{d^2y}{dt^2}+\cos(t)y = e^t, \quad b)\ t^2\frac{d^3y}{dt^3}+\exp(t)y = \sin(t)+y^3, \qquad c)\ (\frac{d^2y}{dt^2})^2+ty = 0.$$

(a) second order, linear
(b) third order, nonlinear
(c) second order, nonlinear

2. (17 points) Find the (possibly implicit) solution of the initial value problem

$$\frac{dy}{dx} = \frac{2x-1}{3y^2+2}, \quad y(0) = 1.$$

This equation is separable. Rewrite it as

$$(3y^2+2)\ dy = (2x-1)\ dx.$$

Integrate to get

$$y^3 + 2y = x^2 - x + C.$$

The initial condition gives $C = 3$, and the solution we want is

$$y^3 + 2y = x^2 - x + 3.$$

3. (17 points) Find the general solution of

$$y' + 2ty = e^{-t^2}.$$

Find the particular solution satisfying

$$y(0) = 3.$$

Use the integrating factor

$$\mu(t) = \exp(\int^t 2s\ ds) = e^{t^2}.$$

This gives

$$(e^{t^2}y)' = 1, \quad e^{t^2}y = t + C, \quad y = te^{-t^2} + Ce^{-t^2}.$$

The condition $y(0) = 3$ gives $C = 3$, so

$$y(t) = te^{-t^2} + 3e^{-t^2}.$$

4. (17 points) Consider the differential equation

$$\frac{dy}{dt} = f(y), \quad f(y) = y(4 - y^2).$$

Sketch the graph of $f(y)$ vs. y. Find the equilibrium solutions and determine whether the equilibrium solutions are unstable or asymptotically stable. Sketch the graph of solutions $y(t)$ vs. t.

The equilibrium solutions are $y = 0$, which is unstable, and $y = \pm 2$, both of which are asymptotically stable.

5. (17 points) Use Euler's method to find a approximate solution of the initial value problem

$$\frac{dy}{dt} - y = 1, \quad y(0) = 1$$

on the interval $[0, 8]$. Use the step size $h = 2$, so the sample times are $t_0 = 0$, $t_1 = 2$, $t_2 = 4$, $t_3 = 6$, $t_4 = 8$.

First rewrite the equation in standard form,

$$\frac{dy}{dt} = f(t, y) = y + 1.$$

For Euler's method, take $y_0 = y(0) = 1$ and

$$y_{n+1} = y_n + h * f(t_n, y_n) = y_n + 2 * (y_n + 1).$$

We find

$$y_1 = 1 + 2 * 2 = 5, \quad y_2 = 5 + 2 * 6 = 17,$$

$$y_3 = 17 + 2 * 18 = 53, \quad y_4 = 53 + 2 * 54 = 161.$$

6. 17 points

A storage container has an internal temperature $u(t)$. If $v(t)$ is the external temperature, Newton's Law of Cooling says that

$$\frac{du}{dt} = -k[u(t) - v(t)]. \tag{1}$$

Suppose the external temperature varies daily according to the formula

$$v(t) = T_0 + T_1 \cos(\pi t), \tag{2}$$

where t has units of days.

(a) Assuming $v(t)$ as given in (2)*, find the general solution of* (1)*.* You may assume that

$$\int_0^t e^{ks} \cos(\pi s)\ ds = \frac{e^{kt}}{k^2 + \pi^2}[k\cos(\pi t) + \pi \sin(\pi t)] - \frac{k}{k^2 + \pi^2}.$$

Write (1) as

$$\frac{du}{dt} + ku = kv(t).$$

The integrating factor is $\mu(t) = e^{kt}$. We find

$$\frac{d}{dt} e^{kt} u = ke^{kt} v(t) = ke^{kt}[T_0 + T_1 \cos(\pi t)].$$

$$e^{kt} u(t) = C + \int_0^t ke^{ks}[T_0 + T_1 \cos(\pi s)]\ ds,$$

so that

$$u(t) = Ce^{-kt} + T_0[e^{kt} - 1]e^{-kt} + \frac{T_1}{k^2 + \pi^2}[k\cos(\pi t) + \pi \sin(\pi t)] - T_1 e^{-kt} \frac{k}{k^2 + \pi^2}.$$

(b) Describe what happens to $u(t)$ for very large times t. What is the long term influence of the initial internal temperature $u(0)$*. What is the average internal temperature over a long time period?*

The terms which are constants time the decaying exponential e^{-kt} will have limit zero as $t \to \infty$. Eventually the solution looks like

$$u(t) \simeq T_0 + \frac{T_1}{k^2 + \pi^2}[k\cos(\pi t) + \pi \sin(\pi t)].$$

Over long time periods the initial temperature, which determines C, has no effect. The average internal temperature is T_0 because $\sin(t)$ and $\cos(t)$ have averages approaching zero.