Math 340 Test 2 solutions Professor Carlson

SHOW ALL YOUR WORK

1. (20 pts) Consider the equation

$$y'' - 16y = 0. \tag{1}$$

a) *Find a pair of linearly independent solutions of the form e^{rt}. Show that your solutions are linearly independent.*

The characteristic equation is

$$r^2 - 16 = (r+4)(r-4) = 0.$$

The exponential solutions are

$$z_1 = e^{4t}, \quad z_2 = e^{-4t}.$$

To show independence we compute the Wronskian

$$W = y_1 y_2' - y_1' y_2 = -4e^{4t}e^{-4t} - 4e^{4t}e^{-4t} = -8.$$

Since the Wronskian is not zero, the solutions are independent.

b) *Describe the general solution of equation* (1)*.*

The general solution has the form

$$y(t) = c_1 e^{4t} + c_2 e^{-4t}.$$

c) *Find the solution $y(t)$ which satisfies the initial conditions*

$$y(0) = 1, \quad y'(0) = 0.$$

We want to find c_1 and c_2 such that

$$c_1 e^0 + c_2 e^0 = c_1 + c_2 = 1,$$

$$4c_1 e^0 - 4c_2 e^0 = 4c_1 - 4c_2 = 0.$$

The second equation says $c_1 = c_2$, so $c_1 = c_2 = \frac{1}{2}$ and

$$y(t) = \frac{1}{2}e^{4t} + \frac{1}{2}e^{-4t}.$$

2. *(20 pts) a) Find the exponential solutions* $y = e^{rt}$ *of the equation*

$$y'' + 4y' + 8y = 0.$$

The characteristic equation is

$$r^2 + 4r + 8 = 0.$$

The roots are

$$r = \frac{-4 \pm \sqrt{-16}}{2} = -2 \pm 2i.$$

The exponential solutions are

$$z_1 = e^{(-2+2i)t}, \quad z_2 = e^{(-2-2i)t}$$

b) Find two independent solutions of the form

$$y_1 = e^{\alpha t}\cos(\beta t), \quad y_2 = e^{\alpha t}\sin(\beta t),$$

where α *and* β *are real numbers.*

Use Euler's formula to get

$$z_1 = e^{-2t}[\cos(2t) + i\sin(2t)], z_2 = e^{-2t}[\cos(2t) - i\sin(2t)].$$

By combining these we get solutions

$$y_1 = e^{-2t}\cos(2t), \quad y_2 = e^{-2t}\sin(2t)$$

c) *Find the solution* y *of this equation satisfying*

$$y(0) = 0, \quad y'(0) = 1.$$

Write

$$y(t) = c_1 e^{-2t}\cos(2t) + c_2 e^{-2t}\sin(2t).$$

Since $y(0) = c_1 = 0$, the desired solution has the form

$$y(t) = c_2 e^{-2t}\sin(2t).$$

Differentiation gives

$$y'(t) = -2c_2 e^{-2t}\sin(2t) + 2c_2 e^{-2t}\cos(2t),$$

so the condition $y'(0) = 1$ leads to

$$y(t) = \frac{1}{2}e^{-2t}\sin(2t).$$

3. (20 pts) a)*Find the general solution of the equation*

$$y'' - 5y' + 6y = t.$$

Write the equation as

$$[D^2 - 5D + 6]y = t.$$

Apply D^2 to get

$$D^2[D^2 - 5D + 6]y = 0.$$

The roots of the characteristic equation are

$$r = 0, 0, 2, 3.$$

Look first for a solution $y = c_1 + c_2 t$. Putting this form into the equation yields

$$-5c_2 + 6(c_1 + c_2 t) = t.$$

That is

$$c_2 = 1/6, \quad 6c_1 - 5c_2 = 0, \quad c_1 = 5/36.$$

The general solution is

$$y(t) = \frac{5}{36} + \frac{1}{6}t + c_3 e^{3t} + c_4 e^{2t}.$$

b) *Find the solution y of this equation satisfying*

$$y(0) = 0, \quad y'(0) = 0.$$

Evaluation gives

$$\frac{5}{36} + c_3 + c_4 = 0,$$

$$\frac{1}{6} + 3c_3 + 2c_4 = 0.$$

$$c_4 = -\frac{1}{4}, \quad c_3 = \frac{1}{9}.$$

The solution is

$$y(t) = \frac{5}{36} + \frac{1}{6}t + \frac{1}{9}e^{3t} - \frac{1}{4}e^{2t}.$$

4. (10 pts) a) *Show that an equation of the form*

$$t^2 y'' + \alpha t y' + \beta y = 0$$

has at least one solution (and usually two) of the form

$$y = t^r.$$

Putting t^r into the equation gives

$$t^2 r(r-1)t^{r-2} + \alpha t r t^{r-1} + \beta t^r = 0.$$

Divide by t^r and simplify to get

$$r^2 + (\alpha - 1)r + \beta = 0.$$

The quadratic formula gives at least one value of r.

b) *Find a solution* $y_1 = t^r$ *for the equation*

$$t^2 y'' - 3t y' + 4y = 0. \tag{2}$$

In this case we find

$$r^2 - 4r + 4 = (r-2)^2 = 0.$$

So $y = t^2$ is a solution.

5. (20) pts *a) Seek power series solutions*

$$y(x) = \sum_{k=0}^{\infty} c_k x^k$$

for the differential equation

$$y'' + xy' + 2y = 0.$$

Find the recurrence relation the coefficients c_k *satisfy.*

b) Find the first four coefficients c_k *of the solutions* $y_1(x), y_2(x)$ *if* $y_1(0) = 1, y_1'(0) = 0$ and $y_2(0) = 0, y_2'(0) = 1$.

a) First compute

$$y'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}, \quad y''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}.$$

Putting the series into the differential equation gives

$$\sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} + x \sum_{k=1}^{\infty} k c_k x^{k-1} + 2 \sum_{k=0}^{\infty} c_k x^k = 0.$$

Writing all series with powers x^k we get

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=0}^{\infty} k c_k x^k + 2 \sum_{k=0}^{\infty} c_k x^k = 0,$$

or

$$\sum_{k=0}^{\infty} [(k+2)(k+1) c_{k+2} + k c_k + 2 c_k] x^k = 0.$$

Since the coefficients must vanish, the recurrence relation is

$$c_{k+2} = -\frac{k c_k + 2 c_k}{(k+2)(k+1)} = -\frac{c_k}{k+1}.$$

b) For the solution $y_1(x)$ we have $c_0 = 1, c_1 = 0$. Thus $c_2 = -1/1 = -1$ and $c_3 = 0$.

For the solution $y_2(x)$ we have $c_0 = 0, c_1 = 1$. Thus $c_2 = 0$, and $c_3 = -1/2$.

6. (10 pts) a) *Find independent solutions y_1 and y_2 of*

$$y'' - 4y = 0.$$

$$y_1 = e^{-2t}, \quad y_2 = e^{2t}.$$

The Wronskian is

$$W = y_1 y_2' - y_1' y_2 = 4.$$

b) *The variation of parameters formula says that a particular solution of the equation*

$$y'' - 4y = g(t), \tag{3}$$

is given by

$$y_p(t) = -y_1(t)\int_0^t \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\ ds + y_2(t)\int_0^t \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\ ds.$$

Suppose that $g(t) = e^{2t}$ *in* (3). *Find* $y_p(t)$.

$$y_p(t) = -e^{-2t}\int_0^t \frac{e^{2s}e^{2s}}{4}\ ds + e^{2t}\int_0^t \frac{e^{-2s}e^{2s}}{4}\ ds = -e^{-2t}\int_0^t \frac{e^{4s}}{4}\ ds + e^{2t}\int_0^t \frac{1}{4}\ ds$$

$$= -\frac{1}{16}e^{-2t}[e^{4t}-1] + \frac{t}{4}e^{2t} = \frac{t}{4}e^{2t} - \frac{1}{16}e^{2t} + \frac{1}{16}e^{-2t}.$$