A Concrete Introduction to Analysis

Robert Carlson
Chapter 1

Homework 6 solutions

Problem 3.1
Show that the series
\[ \sum_{k=0}^{\infty} \frac{2^{-k}}{1+k^2} \]
converges.

Since
\[ 0 \leq \frac{2^{-k}}{1+k^2} \leq 2^{-k}, \]
the series converges by comparison with the convergent geometric series \( \sum_k (1/2)^k \).

Problem 3.2
Suppose that \( c_k \geq 0 \) for \( k = 1, 2, 3, \ldots \), and that \( \sum_{k=1}^{\infty} c_k \) converges. Show that if \( 0 \leq b_k \leq M \) for some number \( M \), then \( \sum_{k=1}^{\infty} b_k c_k \) converges.

The sequence \( \{s_n\} \) of partial sums for the series \( \sum_{k=1}^{\infty} c_k \) is increasing and bounded above by \( S \). The sequence \( \{\sigma_n\} \) of partial sums for the series \( \sum_{k=1}^{\infty} b_k c_k \) is also increasing and bounded above by \( MS \). By the Bounded Monotone Sequence Property the series \( \sum_{k=1}^{\infty} b_k c_k \) converges.

Problem 3.3
Show that the series \( \sum_{k=1}^{\infty} k2^{-k} \) converges. Show that if \( m \) is a fixed positive integer then the series \( \sum_{k=1}^{\infty} k^m 2^{-k} \) converges.

We consider the general case using the ratio test.
\[ \frac{c_{k+1}}{c_k} = \frac{(k+1)^m 2^{-k-1}}{k^m 2^{-k}} = \frac{(k+1)^m 2^{-1}}{k^m} = 2^{-1}(1+1/k)^m, \]
so
\[ \lim_{k \to \infty} \frac{c_{k+1}}{c_k} = 2^{-1} < 1. \]
By the ratio test the series converges.
Problem 3.4  
Prove Lemma 3.1.1.  
By definition a series $\sum_{k=1}^{\infty} c_k$ converges when its sequence of partial sums has a (number) limit. Let 

$$s_n = \sum_{k=1}^{n} b_k, \quad \sigma_n = \sum_{k=1}^{n} c_k.$$ 

The lemma starts with the assumption that the following limits exist: 

$$\lim_{n \to \infty} s_n = S_1, \quad \lim_{n \to \infty} \sigma_n = S_2.$$ 

The partial sums for the combined series are 

$$\alpha_1 \sum_{k=1}^{n} b_k + \alpha_2 \sum_{k=1}^{n} c_k = \alpha_1 s_n + \alpha_2 \sigma_n.$$ 

By Theorem 2.1.3 

$$\lim_{n \to \infty} \alpha_1 s_n + \alpha_2 \sigma_n = \alpha_1 S_1 + \alpha_2 S_2,$$ 

so the limit of the partial sums for the combined series exists, and the series converges.

Problem 3.5  
  a) Show that the series 

$$\sum_{k=0}^{\infty} \frac{2^k}{k!}$$ 

converges.  

  b) Show that the series 

$$\sum_{k=0}^{\infty} \frac{k^k}{k!}$$ 

diverges.
a) Try the ratio test.

\[ \frac{c_{k+1}}{c_k} = \frac{2^{k+1}}{(k+1)!} \frac{k!}{2^k} = \frac{2}{k+1}. \]

In this case \( \lim_{k \to \infty} c_{k+1}/c_k = 0 \), so the series converges.

b) Notice that the terms

\[ \frac{k^k}{k!} = \frac{k}{k(k-1) \cdots 1} \]

are all at least 1. Since the terms do not have limit 0, the series can't converge.

It is also possible to use the ratio test, with the help of a limit formula you may have seen in calculus.

\[ \frac{c_{k+1}}{c_k} = \frac{(k+1)^{k+1}}{(k+1)!} \frac{k!}{k^k} = \frac{(k+1)(k+1)^k 1}{k+1} \frac{1}{k^k} = (1 + 1/k)^k \to e. \]

In this case \( \lim_{k \to \infty} c_{k+1}/c_k > 1 \), so the series diverges.

**Problem 3.6**

Suppose that \( c_k \geq 0 \) and

\[ \lim_{k \to \infty} c_k = r > 0. \]

Show that \( \sum c_k \) diverges.

By Lemma 3.3.1 the terms of a series must have limit zero if the series converges.

**Problem 3.7**

Assume that \( c_k \geq 0 \) and \( \sum_{k=1}^{\infty} c_k \) converges. Suppose that there is a sequence \( \{a_k\} \), a positive integer \( N \), and a positive real number \( r \) such that \( 0 \leq a_k \leq rc_k \) for \( k \geq N \). Show that \( \sum_{k=1}^{\infty} a_k \) converges.

Define the partial sums

\[ s_n = \sum_{k=1}^{n} c_k, \quad \sigma_n = \sum_{k=1}^{n} a_k. \]

Since the series \( \sum c_k \) converges, the sequence of partial sums \( \{s_n\} \) is bounded, \( |s_n| \leq M \).

For \( n \geq N \) we have

\[ \sigma_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} a_k \leq \sum_{k=1}^{N-1} a_k + r \sum_{k=N}^{n} c_k \leq \sum_{k=1}^{N-1} a_k + rM. \]
Since the sequence of partial sums \( \{\sigma_n\} \) is increasing and bounded, it converges by the BMS Theorem.

**Problem 3.8**

*Assuming that \( k^2 + ak + b \neq 0 \) for \( k = 1, 2, 3, \ldots \), show that the series*

\[
\sum_{k=1}^{\infty} \frac{1}{k^2 + ak + b}
\]

*converges.*

First observe that

\[
k^2 + ak + b = k^2[1 + a/k + b/k^2],
\]

and \( \lim_{k \to \infty}[1 + a/k + b/k^2] = 1 \), so there is an \( N_1 \) such that

\[
k^2 + ak + b > 0, \quad \frac{1}{k^2 + ak + b} > 0, \quad k \geq N_1.
\]

Also, since \( \lim_{k \to \infty}[1 + a/k + b/k^2] = 1 \), there is an \( N_2 \) such that

\[
k^2 + ak + b \geq k^2/2, \quad \frac{1}{k^2 + bk + a} \leq \frac{2}{k^2}, \quad k \geq N_2.
\]

For \( N = \max(N_1, N_2) \), and any \( K > N \),

\[
\sum_{k=N+1}^{K} \frac{1}{k^2 + ak + b} \leq 2 \sum_{k=N+1}^{K} \frac{1}{k^2}.
\]

We know that \( \sum_{k=1}^{\infty} 1/k^2 \) converges, so by comparison

\[
S_1 = \lim_{K \to \infty} \sum_{k=N+1}^{K} \frac{1}{k^2 + ak + b}
\]

exists, and so does

\[
S = \lim_{K \to \infty} \sum_{k=1}^{K} \frac{1}{k^2 + ak + b}
\]

\[
= \sum_{k=1}^{N} \frac{1}{k^2 + ak + b} + \lim_{K \to \infty} \sum_{k=N+1}^{K} \frac{1}{k^2 + ak + b}.
\]
Problem 3.9

Find an example of a divergent series \( \sum_{k=1}^{\infty} c_k \) for which \( \lim_{k \to \infty} c_k = 0 \).
The harmonic series
\[
\sum_{k=1}^{\infty} \frac{1}{k}
\]
satisfies \( \lim_{k \to \infty} c_k = 0 \), but the series diverges.

Problem 3.10

(Root Test) Suppose that \( c_k \geq 0 \) for \( k = 1, 2, 3, \ldots \), and that
\[
\lim_{k \to \infty} c_k^{1/k} = L.
\]
Show that the series \( \sum_{k=1}^{\infty} c_k \) converges if \( L < 1 \) and diverges if \( L > 1 \).

The proof is similar to the proof of the ratio test. If \( 0 \leq L < 1 \), pick a number \( L_1 \) such that \( L < L_1 < 1 \). Since \( \lim_{k \to \infty} c_k^{1/k} = L \), there is an \( N \) such that when \( k \geq N \) we have
\[
0 \leq c_k^{1/k} < L_1, \quad c_k < L_1^k.
\]
For any \( M > N \) the sums \( \sum_{k=N}^{M} c_k \) are bounded by the sum of the convergent geometric series
\[
\sum_{k=N}^{\infty} L_1^k = L_1^N \frac{1}{1 - L_1}.
\]
Thus the increasing partial sums for the series \( \sum_{k=1}^{\infty} c_k \) are bounded by
\[
\sum_{k=1}^{N} c_k + L_1^N \frac{1}{1 - L_1},
\]
and the series converges by the Bounded Monotone Sequence Property.

In the other direction, if \( L > 1 \) simply pick \( L_1 \) with \( 1 < L_1 < L \). A comparison with the geometric series shows that
\[
\lim_{k \to \infty} c_k = \infty,
\]
so the series diverges.