Optimization - Homework 1

1. Textbook p. 31 number 1 - Find maxima and minima

$$a) \quad f(x) = x^2 + 2x, \quad f'(x) = 2x + 2, \quad f''(x) = 2.$$

The only critical point is $x = -1$. Since $f''(x)$ is positive everywhere, $x = -1$ is a global minimizer.

$$b) \quad f(x) = x^2 e^{-x^2}, \quad f'(x) = 2x(1 - x^2)e^{-x^2},$$
$$f''(x) = (2 - 10x^2 + 4x^4)e^{-x^2}.$$

The critical points are $x = 0, \pm 1$. We have $f''(0) = 2$, $f''(1) = -4$, $f''(-1) = -4$. Since f is everywhere nonnegative, $x = 0$ is a global minimizer. Since $\lim_{|x| \to \infty} f(x) = 0$ and f is continuous, there must be a global maximizer, and there are two at $x = \pm 1$.

$$c) \quad f(x) = x^4 + 4x^3 + 6x^2 + 4x,$$
$$f'(x) = 4x^3 + 12x^2 + 12x + 4 = 4(x+1)^3, \quad f''(x) = 12(x+1)^2.$$

The critical point is $x = -1$. We have $f''(-1) = 0$, and $f''(x) \geq 0$. Thus $x = -1$ is a global minimizer.

$$d) \quad f(x) = x + \sin(x), \quad f'(x) = 1 + \cos(x), \quad f''(x) = -\sin(x).$$

The critical points are $x = [2n+1]\pi$. We have $f''([2n+1]\pi) = 0$, with f'' changing sign at every critical point. Taylor's theorem becomes

$$f(x) = f(x^*) + \frac{f''(z)}{2}(x - x^*)^2, \quad x^* = [2n+1]\pi,$$

and because f'' changes sign at each critical point, none of the critical points are maximizers or minimizers.

2. Suppose the derivative of a differentiable function f is never 0 on $[0, 1]$. Show that $f(0) \neq f(1)$.

If $f(0) = f(1) = c$ then the function $g(t) = f(t) - c$ would vanish at $x = 0, 1$. By Rolle's theorem we would have $g'(z) = f'(z) = 0$ somewhere in the interval $(0, 1)$. By assumption this doesn't happen, so $f(0) \neq f(1)$.

3. Suppose that f is differentiable for $a \le x \le b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

The Mean Value Theorem says that for some $z \in (a, b)$,

$$f'(z) = \frac{f(b) - f(a)}{b - a} < 0$$

since $b > a$ and $f(b) < f(a)$.

4. Show that if $f' > 0$ throughout an interval $[a, b]$, then f has at most one zero in $[a, b]$.

The argument is basically the same as 4. If f has more than one root, then f' must vanish somewhere by Rolle's Theorem.

5. Suppose an auto manufacturer realizes a \$ 1500 profit on the sale of a certain model. If a rebate is offered, sales go up but profit per sale goes down. It is observed that sales increase by 15 % for every \$ 100 of rebate. Find the amount of rebate that will maximize profit.

Suppose that n is the number of sales with no rebate, and r is the rebate in dollars. By assumption the sales as a function of rebate has the form

$$S(r) = n + .0015rn = (1 + .0015r)n.$$

The profit based on these sales is

$$P(r) = (1500 - r)S(r) = (1500 - r)(1 + .0015r)n.$$

Solving for $P'(r) = 0$ gives

$$0 = P'(r) = -(1 + .0015r)n + (1500 - r)(.0015)n,$$

or

$$0 = -1 - .0015r + 2.25 - .0015r,$$

$$r = 1250/3.$$

6. In simple population biology models the growth rate dx/dt of a population is modelled by the function

$$rx(1 - x/K),$$

where x is the current population, r is a growth rate parameter, and K is the maximum sustainable population. Suppose that for a certain species of fish we currently have

$$r = 0.08, \quad K = 4 \times 10^5, \quad x = 7 \times 10^4.$$

Suppose that the number of fish harvested per year is $10^{-5}Ex$, where E is the level of fishing effort in boat days. Assume that for a fixed level of effort the fish population will eventually stablize at the level where growth rate equals harvest rate.

What level of effort will maximize the sustained harvest rate?

Let $e = 10^{-5}E$. Since population eventually stabilizes when growth rate equals harvest rate we consider

$$rx(1 - x/K) = ex.$$

Solve for a nonzero population to get

$$x = K\frac{(r-e)}{r}.$$

We want to maximize ex, which is

$$eK\frac{(r-e)}{r}.$$

Setting the derivative to 0 gives

$$K\frac{(r-e)}{r} - eK/r = 0,$$

or

$$e = r/2.$$

The corresponding value of E is

$$E = 4,000 \textit{ boat days}.$$