Math 442/542 Optimization
Homework 2 Solutions

1. The Cauchy-Schwarz inequality

$$|X \bullet Y| \le \|X\| \|Y\|, \quad X, Y \in R^N,$$

is a basic and frequently used inequality for dot products. Derive this inequality as follows.

First treat the special case $\|X\| = 1 = \|Y\|$. For this case show that

$$\|X + tY\|^2 = \|X\|^2 + 2t[X \bullet Y] + t^2\|Y\|^2 = 1 + 2t[X \bullet Y] + t^2 \ge 0.$$

Find the value of t which minimizes this quadratic polynomial in t. Conclude that in the special case,

$$|X \bullet Y| \le 1.$$

Finish the general result by considering the cases:
(a) at least one of $\|X\|$, $\|Y\|$ is 0,
(b) both $\|X\| > 0$ and $\|Y\| > 0$.

Solution: By definition of the norm,

$$\|X + tY\|^2 = (X + tY) \bullet (X + tY) = X \bullet X + 2tX \bullet Y + t^2 Y \bullet Y$$

$$= \|X\|^2 + 2t[X \bullet Y] + t^2\|Y\|^2 = 1 + 2t[X \bullet Y] + t^2 \ge 0.$$

By calculus this function of t has its min when

$$2[X \bullet Y] + 2t = 0,$$

or

$$t = -X \bullet Y.$$

Inserting this value of t into the quadratic leads to

$$1 - 2[X \bullet Y]^2 + [X \bullet Y]^2 \ge 0,$$

or

$$1 \ge [X \bullet Y]^2,$$

which gives

$$|X \bullet Y| \le 1.$$

If at least one of $\|X\|$, $\|Y\|$ is 0, then both sides of the Cauchy-Schwarz inequality are 0. Otherwise start with the established case

$$|\frac{X}{\|X\|} \bullet \frac{Y}{\|Y\|}| \leq 1,$$

and multiply both sides by $\|X\| \ \|Y\|$.

2. The Cauchy-Schwarz inequality can be used to establish the triangle inequality

$$\|X - Y\| \leq \|X - Z\| + \|Y - Z\|, \quad X, Y, Z \in R^N.$$

To show this, start by squaring both sides. Then manipulate the identity

$$(X - Y) \bullet (X - Y) = ([X - Z] + [Z - Y]) \bullet ([X - Z] + [Z - Y])$$

and use the Cauchy-Schwarz inequality.

Solution: It suffices to show that

$$\|X - Y\|^2 \leq \Big(\|X - Z\| + \|Y - Z\|\Big)^2.$$

Starting on the left side, and using the Cauchy-Schwarz inequality, we have

$$\|X - Y\|^2 = (X - Y) \bullet (X - Y)$$

$$= ([X - Z] + [Z - Y]) \bullet ([X - Z] + [Z - Y])$$
$$= (X - Z) \bullet (X - Z) + 2(X - Z) \bullet (Y - Z) + (Y - Z) \bullet (Y - Z)$$
$$\leq (X - Z) \bullet (X - Z) + 2\|X - Z\|\|Y - Z\| + (Y - Z) \bullet (Y - Z).$$

On the right side,

$$[\|X - Z\| + \|Y - Z\|]^2 = \Big([(X - Z) \bullet (X - Z)]^{1/2} + [(Y - Z) \bullet (Y - Z)]^{1/2}\Big)^2$$

$$= (X - Z) \bullet (X - Z) + 2\|X - Z\|\|Y - Z\| + (Y - Z) \bullet (Y - Z).$$

Thus

$$\|X - Y\| \leq \|X - Z\| + \|Y - Z\|, \quad X, Y, Z \in R^N.$$

3. Use the triangle inequality to show that an open ball is open. That is, if

$$B(X^*, \delta) = \{X \in R^N \mid \|X - X^*\| < \delta\},$$

show that for every $Y \in B(X^*, \delta)$ there is an $\epsilon > 0$ such that

$$B(Y, \epsilon) \subset B(X^*, \delta).$$

It will help to draw a picture to determine how big ϵ should be.

Solution: Suppose that

$$\|Y - X^*\| = r < \delta.$$

Take

$$0 < \epsilon < \delta - r.$$

Then if

$$\|Y - Z\| < \epsilon$$

we have

$$\|Z - X^*\| \leq \|Z - Y\| + \|Y - X^*\|$$
$$\leq \epsilon + r < (\delta - r) + r = \delta.$$

Generalization

The dot product can be generalized to (real) inner products, which are functions

$$\langle X, Y \rangle : R^N \times R^N \to R$$

taking pairs of vectors to real numbers. To be an inner product this function must satisfy the following conditions for all $X, Y, Z \in R^N$ and $\alpha \in R$:

(a) $\langle X, Y \rangle = \langle Y, X \rangle$
(b) $\langle X, X \rangle \geq 0$ and $\langle X, X \rangle = 0$ if and only if $X = 0$,
(c) $\langle X, Y + Z \rangle = \langle X, Y \rangle + \langle X, Z \rangle$
(d) $\langle \alpha X, Y \rangle = \alpha \langle X, Y \rangle$.

4. Show that if $c_1, \ldots, c_N$ are all positive numbers, then

$$\langle X, Y \rangle = \sum_n c_n x_n y_n$$

is a (real) inner product.

Solution:

$$(a) \quad \langle X, Y \rangle = \sum_n c_n x_n y_n = \sum_n c_n y_n x_n = \langle Y, X \rangle.$$

(b) The sum $\sum_n c_n x_n x_n = \sum_n c_n x_n^2$ is nonnegative since $c_n > 0$. If $\sum_n c_n x_n^2 = 0$ then for each component $x_n = 0$, or $X = 0$.

$$(c) \quad \sum_n c_n x_n (y_n + z_n) = \sum_n (c_n x_n y_n + c_n x_n z_n) = \sum_n c_n x_n y_n + \sum_n c_n x_n z_n$$

$$(d) \quad \sum_n c_n (\alpha x_n) y_n = \alpha \sum_n c_n x_n y_n.$$

5. Extend the definition of norm to

$$\|X\| = \langle X, X \rangle^{1/2},$$

where the nonnegative root is taken. Show that the Cauchy-Schwarz inequality and the triangle inequality hold for any (real) inner product.

Solution: This is simply a matter of going through the earlier arguments, replacing dot products $X \bullet Y$ with inner products $\langle X, Y \rangle$.