Partial solutions for Homework 3

2. Classify the matrices as positive or negative (semi) definite.

$$2(a) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Positive definite.

$$2(b) \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Negative definite.

$$2.(f) \quad A = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

The principle minors are

$$\Delta_1 = 2, \quad \Delta_2 = 0, \quad \Delta_3 = 0$$

Since some of the minors are 0, the principle minors test does not apply.

Because of the block diagonal form of A, we see immediately that -3 is an eigenvalue (with eigenvector E_3). The other eigenvalues are the eigenvalues of the upper left 2×2 block, which are 0 and 10. This real symmetric matrix is indefinite since it has both positive and negative eigenvalues.

3. Find the quadratic forms.

$$(a) \quad A = \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix},$$

$$Q_A(X) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \bullet \begin{pmatrix} -1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -x_1^2 + 4x_1x_2 + 3x_2^2.$$

$$(c) \quad A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & 2 \\ 0 & 2 & 3 \end{pmatrix},$$

$$Q_A(X) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & 2 \\ 0 & 2 & 3 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 - 2x_1x_2 - 2x_2^2 + 4x_2x_3 + 3x_3^2.$$

4.. Find A so that the quadratic form is $X \bullet AX$.

$$(a) \quad Q_A(X) = 3x_1^2 - x_1x_2 + 2x_2^2 \quad A = \begin{pmatrix} 3 & -1/2 \\ -1/2 & 2 \end{pmatrix}.$$

$$(c) \quad Q_A(X) = 2x_1^2 - 4x_3^2 + x_1x_2 - x_2x_3, \quad A = \begin{pmatrix} 2 & 1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & -4 \end{pmatrix}$$

6. The matrix is not symmetric.

9. $f(x_1, x_2) = x_1^2 + x_2^3$ does not have a local minimum at $(0,0)$ since at points $(0,t)$ we have $f(0,t) = t^3$, which is negative whenever $t < 0$.

10. Find the global minimizers or maximizers if they exist.

$$(a) f(x_1, x_2) = x_1^2 - 4x_1 + 2x_2^2 + 7.$$

The partial derivatives vanish when

$$2x_1 - 4 = 0, \quad 4x_2 = 0,$$

so the only critical point is $x_1 = 2$, $x_2 = 0$. In this case the Hessian is the constant matrix

$$Hf = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

which is positive definite, so $(2,0)$ is a global minimum.

$$(b) \quad f(x_1, x_2) = e^{-(x_1^2+x_2^2)}.$$

In this case we can simply note that $e^0 = 1$ and $e^{-t} < 1$ if $0 < t < \infty$, so there is a global max at $(0,0)$. The partial derivatives vanish when

$$-x_1 e^{-(x_1^2+x_2^2)} = 0, \quad -x_2 e^{-(x_1^2+x_2^2)} = 0,$$

that is only if $x_1 = 0 = x_2$, so there is no global min.