Partial solutions for Homework 4

12.a The first example,

$$f(x, y, z) = x^3 + y^3 + z^3 - xy$$

is not coercive, since if we restrict (x, y, z) to the form

$$x = 0, \quad y = 0, \quad z = t,$$

we see that

$$f(0, 0, t) = t^3,$$

and $f \to -\infty$ as $t \to -\infty$.

b. Show that

$$f(x, y, z) = x^4 + y^4 + z^2 - 3xy - z$$

is coercive.

Suppose $x^2+y^2+z^2 = R^2$. Break the analysis into two cases: (i) $x^2+y^2 \geq R^{3/2}$, and (ii) $x^2 + y^2 \leq R^{3/2}$.

Case (i). At least one of x^2 or y^2 is at least $R^{3/2}/2$. Then

$$x^4 + y^4 \geq R^3/4, \quad |x| \leq R, \quad |y| \leq R.$$

and

$$f(x, y, z) = x^4+y^4+z^2-3xy-z \geq R^3/4-3R^2-R = R^3[1/4-3/R-1/R^2].$$

Thus $f \to \infty$ as $R \to \infty$.

Case (ii). For R large enough, $z^2 \geq R^2/2$ and

$$|x| \leq R^{3/4}, \quad |y| \leq R^{3/4}.$$

Then

$$f(x, y, z) = x^4+y^4+z^2-3xy-z \geq R^2/2-3R^{3/2}-R = R^2[1/2-3R^{-1/2}-1/R].$$

Thus $f \to \infty$ as $R \to \infty$.

c.

$$f(x, y, z) = x^4 + y^4 + z^2 - 7xyz^2.$$

Take $x = y = z = R$. Then

$$f(x, y, z) = R^4 + R^4 + R^2 - 7R^4 \to -\infty.$$

This function is not coercive.

13. Let

$$f(x, y) = x^4 + y^4 - 32y^2.$$

a. Find a point where Hf is indefinite.

The Hessian is

$$Hf = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 - 64 \end{pmatrix}.$$

If $x = y = 1$ then

$$Hf = \begin{pmatrix} 12 & 0 \\ 0 & -52 \end{pmatrix},$$

which is indefinite.

b. Show that f is coercive.

If $x^2 + y^2 = R^2$, then at least one of x^2 or y^2 is at least $R^2/2$, and as $R \to \infty$

$$f(x, y) = x^4 + y^4 - 32y^2 \geq R^4/4 - 32R^2 \to \infty.$$

c. Minimize f

The critical points satisfy

$$4x^3 = 0, \quad 4y^3 - 64y = 0,$$

so $x = 0$, $y = 0, \pm 4$.

Plugging in these values we get

$$f(0, 0) = 0, \quad f(0, 4) = -256 = f(0, -4).$$

There are two minima at $(0, \pm 4)$.

14. Define

$$f(X) = A \bullet X.$$

a. Show that f is not coercive.

If $A = 0$ then $f = 0$. If $A \neq 0$ let $X = -RA$. Then $f(-RA) = -R(A \bullet A) \to -\infty$.

b. Show that for any $\epsilon > 0$,

$$f(X) = A \bullet X + \epsilon \|X\|^2$$

is coercive.

By the Cauchy-Schwarz inequality, $|A \bullet X| \leq \|A\|\|X\|$. If $\|X\|^2 = R^2$, then

$$f(X) = A \bullet X + \epsilon \|X\|^2 \geq \epsilon R^2 - \|A\| R \to \infty.$$

18. Choose a and b to minimize

$$f(a,b) = \sum_{i=1}^{n} (a + bx_i - y_i)^2.$$

We define

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \overline{y} = \frac{1}{n} \sum_{i=1}^{n} y_i.$$

We will first try to find any critical points for $f(a,b)$. Computing

$$\frac{\partial f}{\partial a} = 2 \sum_{i=1}^{n} (a + bx_i - y_i)$$

and

$$\frac{\partial f}{\partial b} = 2 \sum_{i=1}^{n} (a + bx_i - y_i) x_i$$

we see that the critical point equations are

$$0 = 2 \sum_{i=1}^{n} (a + bx_i - y_i) \tag{A}$$

and

$$0 = 2 \sum_{i=1}^{n} (a + bx_i - y_i) x_i. \tag{B}$$

Solving the first equation (A) for a gives

$$a = \frac{1}{n} \sum_{i=1}^{n} y_i - \frac{b}{n} \sum_{i=1}^{n} x_i = \overline{y} - b\overline{x}.$$

Solving equation (B) for b gives

$$b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n}(y_i - a)x_i = \sum_{i=1}^{n} y_i x_i - an\overline{x}.$$

Plugging in the expression for a gives

$$b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i x_i - n[\overline{y} - b\overline{x}]\overline{x}.$$

Thus

$$b\Big[\sum_{i=1}^{n} x_i^2 - n[\overline{x}]^2\Big] = \sum_{i=1}^{n} y_i x_i - n\overline{y}\overline{x},$$

which is the same as the answer given in the text.

So far we've shown that there is only one critical point, but we don't know for sure that this is a minimum. We can examine the Hessian, which is

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial a^2} & \frac{\partial^2 f}{\partial a \partial b} \\ \frac{\partial^2 f}{\partial a \partial b} & \frac{\partial^2 f}{\partial b^2} \end{pmatrix} = 2\begin{pmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{pmatrix}.$$

To see if this matrix is positive definite, we check the quadratic form

$$Q = nx^2 + 2[\sum_{i=1}^{n} x_i]xy + [\sum_{i=1}^{n} x_i^2]y^2.$$

By the Cauchy-Schwarz inequality

$$|\sum_{i=1}^{n} x_i| = |\sum_{i=1}^{n} 1 \cdot x_i| \leq n^{1/2}[\sum_{i=1}^{n} x_i^2]^{1/2}$$

so that

$$Q \geq nx^2 - 2n^{1/2}[\sum_{i=1}^{n} x_i^2]^{1/2}xy + [\sum_{i=1}^{n} x_i^2]y^2 = (\sqrt{n}x - [\sum_{i=1}^{n} x_i^2]^{1/2}y)^2 \geq 0.$$

Thus the Hessian is positive semidefinite, and the critical point is a global minimum.