Homework 6 Solutions

1.*a) Show that the function $f(x) = tan(x) - 2x$ has a root r with $0 < r < \pi/2$. (Hint: sketch a graph.)*

Your graph should show the following features. The function $f(x)$ has $f(0) = 0$ and $f'(0) = -1$. Thus $f(x) < 0$ for small positive x. But $\tan(x) \to \infty$ as $x \to \pi/2$, so there must be at least one root root r with $0 < r < \pi/2$.

b) Use a calculator or computer and the bisection method to find the root to some reasonable accuracy.

Here is a table of several computed values:

α_n	$f(\alpha_n)$	β_n	$f(\beta_n)$
1	-0.44	1.5	11.1
1	-0.44	1.25	0.51
1.125	-0.157	1.25	0.51
1.125	-0.157	1.187	0.101

c) Use a calculator or computer and Newton's method to find the root to some reasonable accuracy.

Newton's method generates the sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\tan(x_n) - 2x_n}{\sec^2(x_n) - 2}.$$

Here is a table of several computed values, starting with $x_0 = 1.0$.

x_n	$f(x_n)$
1.31	1.127
1.234	0.388
1.18	0.067

2. *Let*

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Define

$$f(X) = (X - Z) \bullet A(X - Z).$$

Starting with $X_0 = [0, 0]$, show that Newton's method finds the minimizer of f in one step.

First note that since A is real symmetric

$$f(X) = X \bullet AX - 2X \bullet AZ + Z \bullet AZ.$$

As on pages 89-90 of the text,

$$\nabla f(X) = -2AZ + 2AX, \quad Hf = 2A.$$

To minimize $f(X)$ we look for the critical points,

$$g(X) = 0, \quad g(X) = \nabla f(X) = -2AZ + 2AX.$$

Newton's method in this case is

$$X_{n+1} = X_n - [Hf]^{-1}(X_n)g(X_n).$$

Starting at $X = [0, 0]$ we get

$$X_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - [2A]^{-1}[-2AZ] = Z = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

which is obviously the minimizer.

3. *Attack problem two with the steepest descent method. Starting at X_0, use two iterations.*

We start by noting that

$$\nabla f(X_0) = -2AZ = \begin{pmatrix} -8 \\ -14 \end{pmatrix}.$$

Let

$$F(t) = f(X_0 + t\nabla f(X_0)) = f(-2tAZ) = \begin{pmatrix} -8t - 1 \\ -14t - 2 \end{pmatrix} \bullet A \begin{pmatrix} -8t - 1 \\ -14t - 2 \end{pmatrix}$$

$$= \begin{pmatrix} -8t - 1 \\ -14t - 2 \end{pmatrix} \bullet \begin{pmatrix} -30t - 4 \\ -50t - 7 \end{pmatrix} = 940t^2 + 260t + 18.$$

The minimum occurs at $t = -260/1880 = -13/94$, so

$$X_1 = \frac{13}{94} \begin{pmatrix} 8 \\ 14 \end{pmatrix} = \begin{pmatrix} 104/94 \\ 182/94 \end{pmatrix}.$$

4. Attack problem two with the Gauss-Seidel method. Starting at X_0, go through 4 steps, two for each of the two coordinate directions.

Let E_j denote the j-th standard basis vector. The first step considers

$$F(t) = f(X_0 + tE_1) = \begin{pmatrix} t-1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} t-1 \\ -2 \end{pmatrix} = 2t^2 - 8t + 18,$$

which is minimized at $t = 2$. This gives

$$X_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

The second step considers

$$F(t) = f(X_1 + tE_2) = \begin{pmatrix} 1 \\ t-2 \end{pmatrix} \bullet \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ t-2 \end{pmatrix} = 3t^2 - 10t + 10,$$

which is minimized at $t = 5/3$, giving

$$X_2 = \begin{pmatrix} 2 \\ 5/3 \end{pmatrix}.$$