Optimization Prof. Carlson

Homework 7

From the text, end of chapter 2 $\#1, 5, 7, 9$

1. Determine whether the given function is (strictly) convex or (strictly concave).

$$a)\ f(x) = \ln(x), \quad 0 < x < \infty.$$

In this case

$$f'(x) = 1/x, \quad f''(x) = -x^{-2}.$$

The second derivative is negative everywhere on the domain, so the function is strictly concave by Thm 2.3.7.

$$b)\ f(x) = e^{-x}, \quad -\infty < x < \infty.$$

Here $f''(x) = e^{-x} > 0$, so f is strictly convex.

$$c)\ f(x) = |x|, \quad -1 \le x \le 1.$$

Suppose that $0 \le t \le 1$. Then by the triangle inequality

$$f(tx + (1-t)y) = |tx + (1-t)y| \le |tx| + |(1-t)y| \le t|x| + (1-t)|y|.$$

Thus f is convex, but not strictly convex since the function is piecewise linear.

$$d)\ f(x) = |x^3|, \quad -\infty < xl < \infty.$$

We have

$$f'(x) = \left\{ \begin{matrix} 3x^2, & x \ge 0, \\ -3x^2, & x \le 0. \end{matrix} \right\}$$

The second derivative is

$$f''(x) = \left\{ \begin{matrix} 6x, & x \ge 0, \\ -6x, & x \le 0. \end{matrix} \right\}$$

The second derivative is nonnegative everywhere, so f is convex.

To see that f is strictly convex use Taylor's theorem in the form

$$f(y) = f(x) + f'(x)(y-x) + \int_x^y (y-t) f''(t)\ dt.$$

Since the derivative only vanishes at one point, and is otherwise positive, the integral term is positive if $y \neq x$. By 2.3.5 b) the function is strictly convex.

5. *Show that the convex hull of* $D \subset R^n$ *is the set* C *of all convex combinations from* D.

a) C is obviously convex, and contains D, so $co(D) \subset C$.

b) If B is convex and $D \subset B$ then B contains C by 2.1.3. Thus $C \subset co(D)$. Thus $C = co(D)$.

7. *Consider the function*

$$f(x_1, x_2, x_3) = (x_1)^{r_1} + (x_2)^{r_2} + (x_3)^{r_3},$$

defined on the set

$$D = \{(x_1, x_2, x_3) \mid x_1 > 0, x_2 > 0, x_3 > 0\}.$$

a) Show that f *is strictly convex if* $r_i > 1$.

The Hessian is

$$Hf = \begin{pmatrix} r_1(r_1 - 1) & 0 & 0 \\ 0 & r_2(r_2 - 1) & 0 \\ 0 & 0 & r_3(r_3 - 1) \end{pmatrix}.$$

Since the matrix is diagonal, the eigenvalues are the positive diagonal entries, so f is strictly convex.

b) Show that f *is strictly concave if* $0 < r_i < 1$.

For these values of r_i the eigenvalues are negative, so $-f$ is strictly convex.

2.9 *Suppose that* $f(X)$ *and* $g(X)$ *are convex functions defined on a convex set* C *in* R^N *and that*

$$h(X) = \max\{f(X), g(X)\}, \quad X \in C.$$

Show that $h(X)$ *is convex on* C.

Fix $X, Y \in C$ and t with $0 \leq t \leq 1$. Assume that $h((1-t)X + tY) = f((1-t)X + tY)$. Then by convexity of f we have

$$h((1-t)X + tY) = f((1-t)X + tY)$$

$$\leq (1-t)f(X) + tf(Y) \leq (1-t)h(X) + th(Y),$$

proving the convexity of h.

A. (i) Show that the intersection of any collection of convex sets is convex.

Suppose the sets U_α are convex and $V = \bigcap_\alpha U_\alpha$. If X_1 and X_2 are in V, then they are in each U_α. For $0 \le t \le 1$ the convex combination $tX_1+(1-t)X_2$ is in each U_α since each U_α is convex. Thus $tX_1+(1-t)X_2 \in V$.

(ii) Show by example that the union of convex sets may not be convex.

In $\mathbb{R}^2$ the sets $(x,0)$, $x \in \mathbb{R}$, and $(0,y)$, $y \in \mathbb{R}$, are convex, but the union is not.

B. Show that a strictly convex function $f : \mathbb{R} \to \mathbb{R}$ cannot have more than two zeros.

Suppose f had zeros at $x_1 < x_2 < x_3$. Write x_2 as a convex combination of x_1, x_3,

$$x_2 = \frac{x_2 - x_1}{x_3 - x_1} x_3 + \frac{x_3 - x_2}{x_3 - x_1} x_1.$$

By strict convexity

$$f(x_2) < \frac{x_2 - x_1}{x_3 - x_1} f(x_3) + \frac{x_3 - x_2}{x_3 - x_1} f(x_1) = 0,$$

so $f(x_2)$ can't be 0.

C. Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex. Show that the set Ω lying above the graph of f,

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid y \ge f(x)\}$$

is convex.

Suppose (x_1,y_1) and $(x_2,y_2) \in \Omega$, and let $0 \le t \le 1$. Since f is convex,

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2) \le ty_1 + (1-t)y_2.$$

Thus $(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \in \Omega$.

Suppose $f : \mathbb{R}^N \to \mathbb{R}$ is convex. Let $X \in \mathbb{R}^N$, $y \in \mathbb{R}$. Show that the set

$$\Omega = \{(X,y) \in \mathbb{R}^{N+1} \mid y \ge f(X)\}$$

is convex.

The proof is the same, although it is harder to draw a picture. Suppose (X_1,y_1) and $(X_2,y_2) \in \Omega$, and let $0 \le t \le 1$. Since f is convex,

$$f(tX_1 + (1-t)X_2) \le tf(X_1) + (1-t)f(X_2) \le ty_1 + (1-t)y_2.$$

Thus $(tX_1 + (1-t)X_2, ty_1 + (1-t)y_2) \in \Omega$.