Homework 7

1. *Show that the function $g = -x - y$ has no minimum when subjected to the constraints $x - y = 5$, $x \geq 0$, $y \geq 0$, and conclude that linear programs are not always solvable.*

Write the constraint $x - y = 5$ as $y = x - 5$. Along this line we can write $g(x, y) = G(x) = -x - (x - 5) = -2x + 5$. For $x \geq 0$ the function $-2x + 5$ has no minimum, so the linear programming problem to minimize $-x - y$ subject to the constraint $x - y = 5$ has no solution.

2. *Suppose $U_1 = (0, 0)$, $U_2 = (1, 0)$, and $U_3 = (0, 1)$. Show that the set K of convex combinations*

$$w_1 U_1 + w_2 U_2 + w_3 U_3, \quad w_j \geq 0, \quad w_1 + w_2 + w_3 = 1$$

is the same as the set

$$F = \{(x, y) \mid x \geq 0, y \geq 0, x + y \leq 1\}.$$

First, any point in K can be written as

$$w_1 U_1 + w_2 U_2 + w_3 U_3 = (w_2, w_3), \quad w_j \geq 0.$$

Since in addition $w_1 + w_2 + w_3 = 1$, we have $w_2 + w_3 \leq 1$. Thus $K \subset F$.

Next, suppose $(w_2, w_3) \in F$. Define $w_1 = 1 - w_2 - w_3$. Then $w_1 \geq 0$, $w_1 + w_2 + w_3 = 1$, and

$$(w_2, w_3) = w_1 U_1 + w_2 U_2 + w_3 U_3.$$

Thus $(w_2, w_3) \in K$, and the sets K and F are equal.

3. *Suppose that $\beta \in \mathbb{R}^N$, and $\Omega \subset \mathbb{R}^N$ is convex. Assume that*

$$\alpha = \max_{X \in \Omega} \langle \beta, X \rangle.$$

Show that if Z is an extreme point of

$$\Omega_\alpha = \Omega \cap \{X \in \Omega \mid \langle \beta, X \rangle = \alpha\},$$

then Z is an extreme point of Ω.

Try to write $Z = tU + (1 - t)V$ with $U, V \in \Omega$ and $0 < t < 1$. Then

$$\alpha = \langle \beta, Z \rangle = t\langle \beta, U \rangle + (1 - t)\langle \beta, V \rangle.$$

Since $0 < t < 1$ and

$$\langle \beta, U \rangle \le \alpha, \quad \langle \beta, V \rangle \le \alpha,$$

the only way to have $\alpha = \langle \beta, Z \rangle$ is to have

$$\langle \beta, U \rangle = \alpha, \quad \langle \beta, V \rangle = \alpha.$$

That means that $U, V \in \Omega_\alpha$, and since Z is extreme in Ω_α we have $U = V = Z$.

4. *Solve the following linear programming problem. Minimize* $g(x_1, \dots, x_4) = x_1 - x_2 + 3x_3 + 2x_4$ *subject to the constraints*

$$x_1 + x_2 + x_3 + x_4 = 4,$$

$$x_1 + x_2 - x_3 - x_4 = 3,$$

and all $x_k \ge 0$. *Use the idea that the maximum must occur at an extreme point, and the extreme points are feasible solutions* ($x_n \ge 0$) *of the constraint equations with at least* 2 *components equal to* 0.

First, the problem has a solution since the first constraint forces the feasible set to be bounded.

We check for solutions with at least two components equal to 0. Here are the solutions.

$$x_1 = 0, x_2 = 0, \quad \textit{no solutions},$$

$$x_1 = 0, x_3 = 0, \quad x_2 = 7/2, x_4 = 1/2,$$

$$x_1 = 0, x_4 = 0, \quad x_2 = 7/2, x_3 = 1/2,$$

$$x_2 = 0, x_3 = 0, \quad x_1 = 7/2, x_4 = 1/2,$$

$$x_2 = 0, x_4 = 0, \quad x_1 = 7/2, x_3 = 1/2,$$

$$x_3 = 0, x_4 = 0, \quad \textit{no solutions}.$$

The four cases with solutions are all feasible, so these are extreme points. Checking these cases we find

$$x_1 = 0, x_3 = 0, x_2 = 7/2, x_4 = 1/2, \quad g = -7/2 + 1 = -5/2,$$

$$x_1 = 0, x_4 = 0, x_2 = 7/2, x_3 = 1/2, \quad g = -7/2 + 3/2 = -2,$$

$$x_2 = 0, x_3 = 0, x_1 = 7/2, x_4 = 1/2, \quad g = 7/2 + 1 = 9/2,$$

$$x_2 = 0, x_4 = 0, x_1 = 7/2, x_3 = 1/2, \quad g = 7/2 + 3/2 = 5.$$

The minimum value is $-5/2$, which occurs at $x_1 = 0, x_3 = 0, x_2 = 7/2, x_4 = 1/2$.

5. *Start with the following linear program (from Meerschaert). Maximize $P = 400x_1 + 200x_2 + 250x_3$, subject to*

$$3x_1 + x_2 + 1.5x_3 \le 1000,$$

$$0.8x_1 + 0.2x_2 + 0.3x_3 \le 300,$$

$$x_1 + x_2 + x_3 \le 625,$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0.$$

(a) By introducing slack variables x_4, x_5, x_6, rewrite this problem as an equality linear program.

$$3x_1 + x_2 + 1.5x_3 + x_4 = 1000,$$

$$0.8x_1 + 0.2x_2 + 0.3x_3 + x_5 = 300,$$

$$x_1 + x_2 + x_3 + x_6 = 625,$$

$$x_1 \ge 0, \dots, x_6 \ge 0.$$

(b) *Show that the feasible set F for the equality program is bounded.*

Since all of the variables are nonnegative, the first constraint gives

$$x_1 \le 1000/3, x_2 \le 1000, x_3 \le 1000/1.5, x_4 \le 1000,$$

and the remaining constraints give

$$x_5 \le 300, x_6 \le 625.$$

Thus F is bounded.

(c) *Find one extreme point p_1 for this program. Say two extreme points are neighbors if there zero coordinates differ in one place. Find two extreme points p_2, p_3 which are neighbors of p_1. Compare the values of the profit at these 3 extreme points.*

Try $x_1 = x_2 = x_3 = 0$. This gives the feasible values

$$x_4 = 1000, x_5 = 300, x_6 = 625, \quad P = 0,$$

To find neighboring extreme points we try $x_2 = x_3 = x_4 = 0$ to get

$$x_1 = 1000/3, x_5 = 100/3, x_6 = 875/3, \quad P = \frac{400,000}{3},$$

which has the biggest profit of these three points, and $x_1 = x_3 = x_6 = 0$ to get

$$x_2 = 625, x_4 = 375, x_5 = 125, \quad P = 125,000.$$

6. *Suppose we have an equality linear program of the form minimize*

$$g(x_1, \ldots, x_N) = \sum_{n=1}^{N} a_n x_n,$$

subject to constraints $x_n \geq 0$ *for* $n = 1, \ldots, N$ *and*

$$a_{11}x_1 + \cdots + a_{1,N}x_N = b_1,$$

$$\vdots$$

$$a_{M1}x_1 + \cdots + a_{M,N}x_N = b_M.$$

(a) Show that the feasible set Ω *is bounded if* $a_{k,n} > 0$ *for some* k *and* $n = 1, \ldots, N$.

Because all $x_n \geq 0$, the condition $a_{kn} > 0$ for $n = 1, \ldots, N$ implies that

$$a_{kn}x_n = b_k - \sum_{j \neq n} a_{k,j}x_j \leq b_k,$$

and

$$x_n \leq b_k / a_{kn}, \quad n = 1, \ldots, N.$$

Thus all coordinates are bounded.