MATH 4420/5420 Optimization Test 1 Professor Carlson

SHOW ALL YOUR WORK

1.*Suppose that f is a real valued function defined on some subset $D \subset R^N$.*

(a) Define what it means for a point $X^* \in D$ to be a global minimizer, a strict global minimizer and a local minimizer for f on D.

(b) Suppose that $X^* \in D$ is an interior point which is a local minimizer of a function $f : D \to R$. Show that if f has partial derivatives at X^*, then

$$\frac{\partial f}{\partial x_n}(X^*) = 0, \quad n = 1, \dots, N, \quad X = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

Solution: See text.

2. *(a) What is the Hessian for a (sufficiently differentiable) function $f : R^N \to R$?*

(b) Compute the Hessian at $(x, y, z) = (0, 0, 0)$ for the function

$$f(x, y, z) = x + 2y^2 + yz + 3z^2.$$

(c) What condition on the Hessian ensures that a critical point of a function is a (i) local minimizer, (ii) global minimizer?

a) The Hessian Hf is the matrix of second partial derivatives,

$$Hf = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

b) In this case

$$Hf(x, y, z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 6 \end{pmatrix}.$$

c) If the Hessian is positive definite at a critical point, then the critical point is a local minimizer. If the Hessian is positive semidefinite everywhere, then any critical point is a global minimizer.

3. *Find the global minimizers of the function*

$$f(x, y) = \exp((x - 3)^2 + (y + 1)^2).$$

How do you know you have found the global minimizer?

Solution: The critical points of f satisfy

$$\frac{\partial f}{\partial x} = 2(x-3)\exp((x-3)^2+(y+1)^2) = 0,$$

$$\frac{\partial f}{\partial y} = 2(y+1)\exp((x-3)^2+(y+1)^2) = 0,$$

Since the exponential function is never zero, the only critical point is $(3, -1)$.

There are two possible ways of seeing that the criitcal point is a global minimizer. First, you can note that the exponential function e^t is at least 1 if $t \geq 0$. Since $(x-3)^2+(y+1)^2 \geq 0$, the place where it is 0 is a minimizer.

Second, the Hessian $Hf(x,y)$ is

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \exp((x-3)^2+(y+1)^2)\begin{pmatrix} 4(x-3)^2+2 & 4(x-3)(y+1) \\ 4(x-3)(y+1) & 4(y+1)^2+2 \end{pmatrix}.$$

By the principle minors test the Hessian is positive definite everywhere, so the critical point in a global minimum.

4.*(a) What does it mean for a real symmetric matrix to be positive definite?*

(b) What condition on the eigenvalues of a real symmetric matrix is equivalent to positive definiteness?

(c) Is the matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

positive definite?

a) The $n \times n$ real symmetric matrix A is positive definite if the quadratic form

$$Q_A(X) = X \bullet AX$$

is positive for all nonzero vectors $X \in R^n$.

b) A real symmetric matrix A is positive definite if and only if all the eigenvalues of A are positive.

c) The principal minors of A are 3, 6 and 3, so by the principal minors test A is positive definite.

5.*Suppose that* $\mathbf{A}$ *is a real symmetric matrix* $N \times N$ *with eigenvalues* $\lambda_1 > \lambda_2 > \cdots > \lambda_N \geq 0$*, and corresponding orthonormal eigenvectors* $X_1, \ldots, X_N$*.*

By expressing a vector X as a linear combination of the eigenvectors, find the maximum value of the quadratic form $Q_{\mathbf{A}}(X) = X \bullet \mathbf{A}X$ if $\|X\| = 1$. What vectors X maximize $Q_{\mathbf{A}}(X)$?

Since there is an orthonormal basis of eigenvectors we can write

$$X = \sum_{n=1}^{N} c_n X_n,$$

with

$$\|X\|^2 = 1 = \sum_{n=1}^{N} |c_n|^2.$$

Since $AX_n = \lambda_n X_n$, the quadratic form is

$$Q_{\mathbf{A}}(X) = X \bullet \mathbf{A}X = \sum_{n=1}^{N} c_n X_n \bullet \sum_{n=1}^{N} \lambda_n c_n X_n = \sum_{n=1}^{N} \lambda_n |c_n|^2.$$

The maximum value of the quadratic form is obtained by taking $|c_1| = 1$, which gives $Q_{\mathbf{A}}(X) = \lambda_1$. The vectors $\pm X_1$ maximize $Q_{\mathbf{A}}(X)$.